Counterexamples to the Theorems of Integrity of Prisms and Ladders

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ABSTRACT. We give counterexamples for two theorems given for the integrity of prisms and ladders in [2] (Theorem 2.17 and Theorem 2.18 in [1]). We also compute the integrity of several special graphs.

1 Introduction

In an analysis of the vulnerability of a communication network to disruption, two quantities (there may be others) that come to mind are (i) the number of elements that are not functioning and (ii) the size of the largest remaining group within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the two quantities can be made to be simultaneously small.

The concept of integrity was introduced as a measure of a graph vulnerability in this sense. Formally, the *vertex-integrity* (frequently called just *integrity*) is

$$I(G) = \min_{X \subset V(G)} \{|X| + m(G - X)\},$$

where $m(H)$ denotes the order of a largest component of H. This concept was introduced by Barefoot, Entringer and Swart [4], who discovered many of the early results on the subject. In his thesis [8], Goddard added many results and developed some generalizations. There is also nice survey given by Bagga *et al.* [1].

A few further comments on notation are appropriate here. The order of a graph G (that is, the number of vertices) will generally be denoted by n, but may also be denoted by $|G|$. As usual, V and E will denote respectively the set of vertices and edges of G, and X will denote a proper subset of V. As noted earlier, $m(G)$ equals the largest order among the components of G. $K_{1,n}$ denotes star graph with $n+1$ vertices, P_n denotes path with n vertices. C_n denotes cycle of order n. Most of the notations used in this paper are same as in [5].

In the Section 2, we review known results on the subject. Section 3 gives two counterexamples for the theorems in [2] (Theorem 2.17 and Theorem 2.18 in [1]). In the Section 4, we compute the integrity of some special graphs.

2 Some of the Results on Integrity

In this section we will review some of the known results. In the following theorems, the integrity of families of graphs are given

Theorem 2.1. [1,3] *The integrity of*

(a) *the complete graph* K_n *is* n;

(b) *the null graph* $\overline{K}_n$ *is* 1;

(c) *the star* $K_{1,n}$ *is* 2;

(d) *the path* P_n *is* $\lceil 2\sqrt{n+1}\rceil - 2$;

(e) *the cycle* C_n *is* $\lceil 2\sqrt{n}\rceil - 1$;

(f) *the complete bipartite graph* $K_{n,m}$ *is* $1+\min\{m,n\}$;

(g) *any complete multipartite graph of order* n *and largest partite set of order* r *is* $n-r+1$;

(h) *the comet* $C_{n-r,r}$ *is* $I(P_n)$, *if* $r \le \sqrt{n+1} - \frac{5}{4}$; $\lceil 2\sqrt{n-r}\rceil - 1$, *otherwise, where comet is a special graph (see e.g., [1]).*

The following two theorems [1] (also from [2]) give the integrity of ladders and prisms.

Theorem 2.2. [1,2] *For* $n \ge 2$, *if* $n = r^2 + k$ *with* $0 \le k \le 2r$, *then*

$$I(K_2 \times P_n) = \begin{cases} 2I(P_n) - 1, & \text{if } 0 \le k < \frac{r}{2} \text{ or } r \le k < \frac{3r}{2}, \\ 2I(P_n), & \text{otherwise.} \end{cases}$$

Theorem 2.3. [1,2]

(a) *For* $n = 3$ *or* 4, $I(K_2 \times C_n) = 2I(C_n) - 1 = 5$.

(b) *For* $n \geq 5$, *if* $n = r^2 + k$ *with* $0 \leq k \leq 2r$, *then*

$$I(K_2 \times C_n) = \begin{cases} 2I(C_n) - 1, & \text{if } 1 \leq k \leq \frac{r}{2} \text{ or } r < k \leq \frac{3r}{2}, \\ 2I(C_n), & \text{otherwise.} \end{cases}$$

Another successful effort yield the integrity of products of stars [7]

Theorem 2.4. [7] *If* $r \leq s$, *then*

$$I(K_{1,r} \times K_{1,s}) = \begin{cases} 2r - 1, & \text{if } r = s, \\ 2r, & \text{otherwise.} \end{cases}$$

Even the integrity of the product of complete graphs is complicated. The expression in the next formulation in fact involves the solution of an integer optimization problem.

Theorem 2.5. [1] *Let* $2 \leq m \leq n$. *Then*

$$I(K_m \times K_n) = mn - \max_{1 \leq j < m} j \lfloor \frac{n(m-j)}{m} \rfloor.$$

Corollary 2.1. [1]

(a) $I(K_{2m} \times K_{2n}) = 3mn$.

(b) *if* $2m + 1 \leq n^2$, *then* $I(K_{2n} \times K_{2m+1}) = 3mn + 2n$.

(c) *Let* $m = 2r + 1$, $n \geq r^2$, *and* $n \equiv t \pmod{m}$ *with* $0 \leq t < m$. *Then*

$$I(K_m \times K_n) = \begin{cases} mn - (r+1)\lfloor \frac{rn}{m} \rfloor, & \text{if } t \text{ is odd,} \\ mn - r\lfloor \frac{(r+1)n}{m} \rfloor, & \text{if } t \text{ is even.} \end{cases}$$

3 Counterexamples

In this section we give two counterexamples for Theorem 2.2 and Theorem 2.3, respectively.

Example 1. Consider the ladder $K_2 \times P_{10}$.

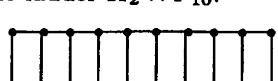

Figure 1

Claim:
$$I(K_2 \times P_{10}) > 9.$$

Proof of Claim: Suppose $I(K_2 \times P_{10}) = 9$. Hence there is a subset S of $V(K_2 \times P_{10})$ such that $|S| + m(K_2 \times P_{10} - S) = 9$. Let $|S| = s$ and $m(K_2 \times P_{10} - S) = r$. In order to have this equality, we have the following cases:

(a) $s = 8$ and $r = 1$,

(b) $s = 7$ and $r = 2$,

(c) $s = 6$ and $r = 3$,

(d) $s = 5$ and $r = 4$,

(e) $s = 4$ and $r = 5$,

(f) $s = 3$ and $r = 6$,

(g) $s = 2$ and $r = 7$,

(h) $s = 1$ and $r = 8$.

- It is obvious that cases (a), (f), (g), and (h) are not possible, that is, there is no such subset S that is satisfying any of (a), (f), (g), or (h).
- If $s = 7$. Removing seven vertices from $K_2 \times P_{10}$ can produce seven, six, five, four, three, two, or one components. If we have seven components, observe that at least five of these components will each have exactly one vertices. Therefore one of the remaining components will have more than two vertices. Similarly if there are six or five components, then there is at least one component that has more than two vertices. If we have four components, then it can be observed that at most three components will each have exactly two vertices. Hence one of the remaining component will have more that two vertices. If we have three, or two, or one component, it is obvious that condition (b) cannot be true. Hence there is no such S that $s = 7$ and $r = 2$ can be satisfied.
- Similar argument can be done for other cases.

Hence $I(K_2 \times P_{10}) > 9$. It is easy to find a $S \subset V(K_2 \times P_{10})$ such that $|S| = 4$ and $m(K_2 \times P_{10} - S) = 6$. Therefore $I(K_2 \times P_{10}) = 10$. On the other hand $10 = 3^2 + 1$ and $1 < \frac{3}{2}$. Therefore according to Theorem 2.2 $I(K_2 \times P_{10}) = 9$. But this is not true as we see above. □

We now we give the second example.

Example 2. Consider the prism $K_2 \times C_6$.

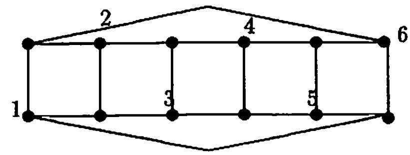

Figure 2

According to Theorem 2.3, we have $I(K_2 \times C_6) = 8$ since $n = 6 = 2^2 + 2$. But we can easily find a set $S \subset V(K_2 \times C_6)$ such that $|S| = 6$ and $m(K_2 \times C_6 - S) = 1$. Hence $|S| + m(K_2 \times C_6 - S) = 7$. This contradicts with Theorem 2.3. Such a set is $S = \{1, 2, 3, 4, 5, 6\}$. □

We restate Theorem 2.2 and Theorem 2.3.

Theorem 3.1. *Let $n \geq 3$ be an integer and let $a = \lfloor \sqrt{n+1} \rfloor$ and $b = \lceil 2\sqrt{n+1} \rceil$ be two integers. Then*

$$I(K_2 \times P_n) = \begin{cases} 2I(P_n) - 1, & \text{if } n+1 \leq a(b-a-\frac{1}{2}), \\ 2I(P_n), & \text{otherwise.} \end{cases}$$

Proof: Suppose $S \subset V(K_2 \times P_n)$ such that $I(K_2 \times P_n) = |S| + m(K_2 \times P_n - S)$. Let r be number of components after removing $|S|$ vertices. With an argument given for $I(P_n)$, we can conclude that $r \leq \sqrt{n+1} + 1$. In order to have r components we have to remove at least r and at most $2(r-1)$ vertices. Therefore if we remove $r + k$ vertices then there are $r - (k+2)$ components that have exactly one vertex and $k + 2$ components that have more than one vertices, where $0 \leq k \leq r - 2$. Hence

$$I(K_2 \times P_n) = \min_{k,r}\{r + k + \frac{2(n-r+1)}{k+2}\}.$$

Claim. For any $k = 0, 1, 2, \ldots, r-3$ we have

$$r + k + \frac{2(n-r+1)}{k+2} \geq 2r - 2 + \frac{2(n-r+1)}{r}.$$

Proof of Claim:

$$r + k + \frac{2(n-r+1)}{k+2} \geq 2r - 2 + \frac{2(n-r+1)}{r} \Leftrightarrow$$
$$2(n-r+1) \geq r(k+2) \Leftrightarrow$$
$$r(k+2) \leq r^2 \leq (\sqrt{n+1}+1)^2 \leq 2(n - \sqrt{n+1}) \leq 2(n-r+1) \Leftrightarrow$$
$$n \geq 21.$$

For remaining $n = 3, 4, \ldots, 20$ one can show that claim is true. Hence

$$I(K_2 \times P_n) = f(r) = \min_{0 \le r \le \sqrt{n+1}+1} \{2r - 2 + \frac{2(n-r+1)}{r}\}.$$

$f(r)$ takes its minimum value at $r = \sqrt{n+1}$ and $f(\sqrt{n+1}) = 4(\sqrt{n+1} + 1)$. One can easily show $2I(P_n) - 1 = 2\lceil 2\sqrt{n+1}\rceil - 5 \le 4(\sqrt{n+1} - 1)$ for all $n \ge 3$. Therefore lower bound of $I(K_2 \times P_n)$ is $2I(P_n) - 1$. It is obvious that upper bound is $2I(P_n)$.

Now remove $2(a-1)$ vertices in such way that each of the $a-1$ components will have $2(b-a-1)-1$ vertices. The last component has $2n-(a-1)(2(b-a-1)+1)$ vertices. If $2n-(a-1)(2(b-a-1)+1) \le 2(b-a-1)-1$, that is, $n+1 \le a(b-a-\frac{1}{2})$ then the integrity of $K_2 \times P_n$ is $2(a-1)+2(b-a-1)-1 = 2(b-2) = 2I(P_n) - 1$. This proves the theorem. □

Theorem 3.2. *Let $n \ge 3$ be an integer and let $a = \lfloor\sqrt{n}\rfloor$ and $b = \lceil 2\sqrt{n}\rceil$ be two integers. Then*

(i) *for $n = 3$ or $n = 4$, $I(K_2 \times C_n) = 2I(C_n) - 1 = 5$.*

(ii) *for $n \ge 5$,*

$$I(K_2 \times C_n) = \begin{cases} 2I(C_n) - 1, & \text{if } n+1 \le a(b-a-\frac{1}{2}), \\ 2I(C_n), & \text{otherwise.} \end{cases}$$

Proof: With a similar argument given in the previous proof we can show the following

$$2I(C_n) - 1 \le I(K_2 \times C_n) \le 2I(C_n).$$

Observe that $I(K_2 \times C_n) = 2 + I(K_2 \times P_{n-1})$. Use the fact $1 + I(P_{n-1}) = I(C_n)$ to conclude the proof. □

4 Integrity of Special Graphs

In this section we give integrity of some special graphs. These graphs are binomial tree B_n, full *k-array* tree H_n^k with $\frac{k^n-1}{k-1}$ vertices and full binary tree H_n^2. The binomial tree B_n is defined recursively. As shown in Figure 3(a), the binomial tree B_0 consists of single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other. Figure 3(b) shows binomial trees B_0 through B_3. Figure 3(c) shows another way of representing the binomial tree B_n. We call the vertex u *top vertex* of B_n. For more information one can refer to [6, pages 96 and 401].

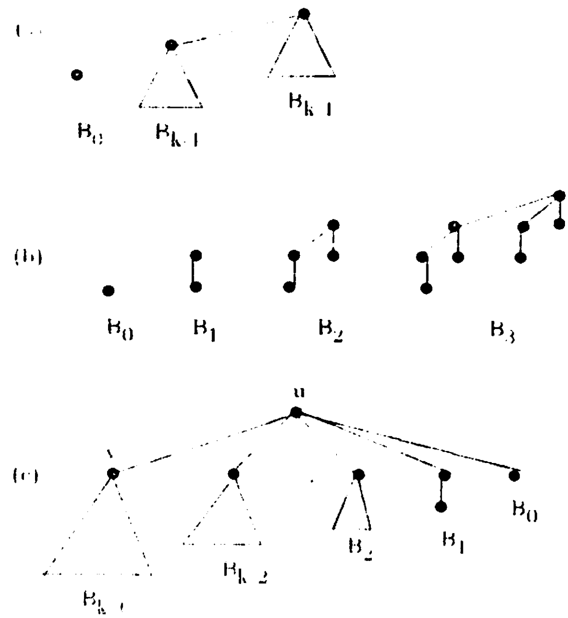

Figure 3

We first give the integrity of binomial tree B_n.

Theorem 4.1. *Let n be a positive integer. Then*

$$I(B_n) = \begin{cases} 2^{\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \\ 3 \times 2^{\frac{n}{2}-1}, & \text{if } n \text{ is even.} \end{cases}$$

Proof:

Step 1: If we remove any vertex other than top vertex u of B_n, then remaining graph will have a component with at least $2^{n-1}+1$ vertices. When we remove the top vertex then $B_{n-1}, B_{n-2}, \ldots, B_0$ are the components. Then the largest component has 2^{n-1} vertices.

Step 2: Removing the top vertices of B_{n-1} will produce two of the each binomial tree B_k as components where $k = 0, 1, 2, \ldots, n-2$. Hence the largest component will have 2^{n-2} vertices. But removing any vertex other than the top vertex will produce a component that has at least $2^{n-2}+1$ vertices.

Step 3: Similarly removing the top vertex of each B_{n-2} leaves four of the each binomial tree B_k as components, where $k = 0, 1, \ldots, n-3$. So the number of vertices of largest component is 2^{n-3}.

Step 4: Iteratively at each step we remove top vertex.

If we remove r top vertices where $2^i \le r < 2^{i+1}$ and $0 \le i \le n-1$, then one of the remaining connected components which is a binomial tree has $2^{n-(i+1)}$ vertices. Therefore

$$I(B_n) = \min_{0 \le r < n} \{2^r + 2^{n-(r+1)}\}.$$

Function $2^r + 2^{n-(r+1)}$ takes its minimum value at $r = \frac{n-1}{2}$ when n is odd and $r = \frac{n}{2}$ when n is even. □

In view of proof given above, we can state the following theorems.

Theorem 4.2. *Let n and $k \ge 2$ be two integers. Then*

$$I(H_n^k) = \begin{cases} \frac{k^{\frac{n}{2}}(k+1)-2}{k-1}, & \text{if } n \text{ is even,} \\ \frac{k^{\frac{n+1}{2}}+k^{\frac{n-1}{2}}-2}{k-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 4.1. *Let n be an integer. Then*

$$I(H_n^2) = \begin{cases} 3 \times 2^{\frac{n}{2}} - 2, & \text{if } n \text{ is even,} \\ 2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 2, & \text{if } n \text{ is odd.} \end{cases}$$

Remaining theorems are about Cartesian product of some special graphs.

Theorem 4.3. *Let m and $n \ge 5$ be two integers. Then*

(a) $I(P_m \times B_n) = mI(B_n) - \lfloor \frac{m}{2} \rfloor$;

(b) $I(K_{1,m} \times B_n) = (1+m)I(B_n) - \lfloor \frac{1+m}{2} \rfloor$;

(c) $I(C_m \times B_n) = mI(B_n) - \lfloor \frac{m}{2} \rfloor$;

(d) $I(W_m \times B_n) = (1+m)I(B_n) - \lfloor \frac{1+m}{2} \rfloor$,

where W_m is a graph that contains an m-cycle and one additional vertex that is adjacent to all the vertices of the cycle.

Theorem 4.4. *Let m and $n \ge 4$ be two integers. Then*

$$I(P_m \times H_n^2) = \begin{cases} m(2^{\frac{n-1}{2}} - 1) + m(2^{\frac{n+1}{2}} - 1), & \text{if } n \text{ is odd,} \\ 2m(2^{\frac{n}{2}} - 1), & \text{if } n \text{ is even.} \end{cases}$$

Corollary 4.2. *Let n be an positive integer. Then*

$$I(P_2 \times H_n^2) = \begin{cases} 3 \times 2^{\frac{n+1}{2}} - 4, & \text{if } n \text{ is odd,} \\ 4(2^{\frac{n}{2}} - 1), & \text{if } n \text{ is even.} \end{cases}$$

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