ON COMPLETING LATIN SQUARES

HOLLIE L. BUCHANAN II
MICHAEL N. FERENCAK

Department of Mathematics
West Virginia University
Morgantown, WV 26506
email: hb@math.wvu.edu, feren@andrew.cmu.edu

ABSTRACT. This paper characterizes a particular scheme of partially filled Latin squares and when they can be completed to full Latin squares. In particular given an $n \times n$ array with the first s rows and the first d cells of row $s+1$ filled with n distinct symbols in such a way that no symbol occurs more than once in any row or column, necessary and sufficient conditions are found for when this array can be completed to a full Latin square.

This is a brief note to characterize a particular scheme of partially filled Latin squares and when they can be completed to full Latin squares. As this is a consequence of a result on regular bipartite graphs, it follows from a theorem of Stuart Allen [1], but written herein in the form of [3], to wit,

Theorem (Buchanan). *Let $G = (A, B)$ be a bipartite simple graph and let E and F be disjoint subsets of $\mathcal{E}(G)$ with E independent. Then G admits a matching of A into B which contains E and is disjoint from F iff there are no partitions $A_1 \dot\cup A_2 = A$ and $B_1 \dot\cup B_2 = B$ having $E \subseteq \mathcal{E}(A_2, B_1)$ and $\mathcal{E}(A_1, B_2) \subseteq F$, and such that $|A_1| > |B_1| - |E|$.*

Theorem 1. *Let the first s rows and the first d cells of row $s+1$ of an $n \times n$ array be filled with the symbols $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ in such a way that no symbol occurs more than once in any row or column. Then the remaining cells can be filled so as to form a Latin square iff there is no collection, C, of columns and collection, Σ, of symbols such that*

(1) *All the symbols used in the d assigned cells of row $s+1$ are included in Σ,*
(2) *None of the first d columns is in C,*
(3) *Every symbol not in Σ occurs in the first s rows of each column from C, and*
(4) $|C| > |\Sigma| - d$.

JCMCC 34 (2000), pp. 129-132

That this requirement is necessary can be seen by noting that in row $s+1$ the $|C|$ cells in columns from C must be filled by symbols from Σ (from condition (3)), of which d are already used in the first d cells. So there are more cells to fill than there are appropriate symbols with which to fill them, and so row $s+1$ cannot be completed.

Sufficiency is established by first noting that the array can be completed to a Latin square iff row $s+1$ can be filled without repeating any symbol in any row or column (see [4]). Then construct a complete bipartite graph, G, having as its vertices the set of columns and the set of symbols in the array, with the d cells from row $s+1$ defining the edge set E and the cells in rows 1 through s defining the edge set F. Filling row $s+1$ is now equivalent to find a perfect matching in G which includes E and is disjoint from F. We see this is possible iff there is no subset, C, of columns and subset, Σ, of symbols filling the roles of A_1 and B_1, respectively. Conditions (1) and (2) establish that $E \subseteq \mathcal{E}(\bar{C}, \Sigma)$, condition (3) establishes that $\mathcal{E}(C, \bar{\Sigma}) \subseteq F$, and condition (4) is the fatal inequality from above.

Theorem 1 implies a result of Brualdi and Csima [2], which states that such a partially filled array can always be completed to form a Latin square if $2s+d \leq n$. This follows since C and Σ satisfying conditions (1) through (3) must have $|C| \leq s$ (each symbol not in Σ occurs $|C|$ times in the first s rows) and $|\Sigma| \geq n-s$ (all symbols not in Σ appear in the s rows of each column of C, so $n - |\Sigma| \leq s$), forcing $|C| \leq s \leq n-s-d \leq |\Sigma| - d$, violating (4).

Notice that Theorem 1 essentially ignores the first d cells of the first s rows, suggesting the following result.

Theorem 2. *Let an $n \times n$ array have the last $n-d$ cells of each of the first s rows and the first d cells of row $s+1$ filled so that no symbol occurs more than once in any row or column. Then this array can be completed to form a Latin square iff*

1 *There are no C and Σ that satisfy each of the conditions* (1)-(4) *from Theorem 1, above,*
2 *Each symbol occurs at least $s-d$ times in the first s rows, and*
3 *No symbol occurring exactly $s-d$ times in the first s rows also occurs in the first d cells of row $s+1$.*

This is really just Theorem 1 combined with Ryser's Theorem [5], since **1** determines when row $s+1$ can be extended and conditions **2** and **3** verify that Ryser's condition is satisfied by the block formed from the last $n-d$ columns of the first $s+1$ rows, after row $s+1$ has been filled (noting that the symbols in the first d cells are exactly those which would not be added to that block). Alternately, **2** and **3** could have been combined as

4 Each symbol occurs at least $[s-d+$ *the number of occurrences of that symbol in row* $(s+1)]$ times in the first s rows.

In particular, if $2s+d \leq n$ and $s-d+1 \leq 0$ (forcing $3s \leq n-1$) then such an array can always be completed to form a Latin square.

Finally, at the suggestion of A.J.W. Hilton, we show that Theorem 1 implies the following result conjectured by John Goldwasser[1].

Theorem 3. *Let the first s rows and the first d cells of row $s+1$ of an $n \times n$ array R be filled in such a way that no symbol occurs more than once in any row or column. Then this array can be completed to form a Latin square iff there are no permutations of the columns and symbols of R that yield an $(s+1)\times(n-a)$ array R^* in the the first $(s+1)$ rows and $(n-a)$ columns of R with the last $n-a-d$ cells in row $s+1$ unfilled having the property that these cells cannot be filled in a manner satisfying Ryser's condition (that each distinct symbol σ occurs at least $(s+1)+(n-a)-n = s+1-a$ times in R^*).*

Necessity follows directly from Ryser's Theorem [5] since regardless of how the appropriate columns and rows are permuted, R^* must satisfy the Ryser condition.

Now suppose that we cannot complete the array to obtain an $n \times n$ Latin square. Let the first d cells of row $s+1$ be filled with the symbols $\{\sigma_1,\ldots,\sigma_d\}$. Then by Theorem 1, there is a collection C, of $|C| = a$ columns and a collection Σ of symbols that allow us, by permuting columns $d+1,\ldots,n$ and symbols $\sigma_{|\Sigma|+1},\ldots,\sigma_n$, to rearrange R so that

- (*i*) The members of C occur as the last a columns of the structure,
- (*ii*) All of the symbols not in Σ, (without loss of generality let those symbols be $\{\sigma_{|\Sigma|+1},\ldots,\sigma_n\}$), occur in each column of C, and
- (*iii*) $a > |\Sigma| - d$.

Let R^* be the array obtained by using the first $n-a$ columns and $s+1$ rows of this rearrangement with the last $n-a-d$ cells of row $s+1$ unfilled. If, after filling the last $n-a-d$ cells of row $s+1$, the resulting array is to satisfy the Ryser condition, then each symbol must occur at least $s+1-a$ times in that array. From our construction, each of the symbols $\sigma_{|\Sigma|+1},\ldots,\sigma_n$ occurs exactly $s-a$ times in the first s rows of R^* and so each must be placed once among the $n-a-d$ empty cells of row $s+1$. But (*iii*) implies that $n-|\Sigma| > n-a-d$ and thus there are more symbols to place in the unfilled cells than there are cells to receive them.

[1]Personal communication

References

1. S. M. Allen, *Bipartite multigraphs with 1-factors containing specified edges*, J. Graph Theory **20, No. 3** (1995), 323-326.
2. R. A. Brualdi and J. Csima, *Extending subpermutation matrices in regular classes of matrices8*, Discrete Math. **62** (1986), 99–101.
3. H. L. Buchanan II, *Matchings including or excluding certain edge sets in bipartite graphs*, Bull. London Math. Soc. **29** (May 1997), no. 138, 303–306 (to appear).
4. Marshall Hall, *An existence theorem for latin squares*, Bull. Amer. Math. Soc. **51** (1945), 387–388.
5. H. J. Ryser, *A combinatorial theorem with an application to latin rectangles*, Proc. Amer. Math. Soc. **2** (1951), 550–552.