On the spectrum of Steiner (v, k, t) trades (I)

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Abstract: A (v, k, t) trade $T = T_1 - T_2$ of volume m consists of two disjoint collections T_1 and T_2 each containing m blocks (k-subsets) such that every t-subset is contained in the same number of blocks in T_1 and T_2. If each t-subset occurs at most once in T_1, then T is called a Steiner (k, t) trade. In this paper the spectrum (that is, the set of allowable volumes) of Steiner trades is discussed, with particular reference to the case $t = 2$. It is shown that the volume of a Steiner $(k, 2)$ trade is at least $2k - 2$ and cannot equal $2k - 1$. We show how to construct a Steiner $(k, 2)$ trade of volume m when $m \geq 3k - 3$, or m is even and $2k - 2 \leq m \leq 3k - 4$. For $k = 5$ or 6, the non-existence of Steiner $(k, 2)$ trades of volume $2k + 1$ is demonstrated, and for $k = 7$, we exhibit a Steiner $(k, 2)$ trade of volume $2k + 1$. In addition, the structure of Steiner $(k, 2)$ trades of volumes $2k - 2$ and $2k$ ($k \neq 3, 4$) is shown to be unique. A generalisation of our constructions to trades with blocks based on arbitrary simple graphs is also presented.

1 Introduction

A (v, k, t) **trade** T of **volume** m consists of two disjoint collections T_1 and T_2, each containing m k-subsets (**blocks**) of some set V, such that every t-subset of V is contained in the same number of blocks in T_1 and T_2. The single collection T_1 is often referred to as a trade. The pair T_1 and T_2 is sometimes said to be **mutually t-balanced.** When $m = 0$, T is called the **null** trade. Note that there may exist elements of V which occur in no block of T. The set of elements of V contained in a set of blocks X is called the **foundation** of X, denoted by $F(X)$. Let $m(T) = m$ and $f(T) = |F(T)|$;

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whence $f(T) \leq v \leq |V|$. The trade T is often written as $T_1 - T_2$, where the following example illustrates this notation.

Example 1.1 $T = T_1 - T_2 = +x12+x34+y13+y24-x13-x24-y12-y34$ is a $(6,3,2)$ trade, with $F(T) = \{1,2,3,4,x,y\}$, $f(T) = 6$ and $m(T) = 4$.

Trades have many uses in the theory of designs. They can be used to construct t-designs with different support sizes [3], and are related to the problem of finding defining sets of designs [5] and to the design intersection problem [1]. As we are not concerned in this paper with the value of v, we write (k,t) trade instead of (v,k,t) trade. Throughout, it is assumed that k and t satisfy the conditions $k > t > 0$.

Definition 1.2 *A (k,t) trade $T = T_1 - T_2$ with any t-subset occurring at most once in T_1 is said to be a* **Steiner (k,t) trade.**

Notation For a collection A of s-subsets and an element $x \notin F(A)$, xA denotes the set of $(s+1)$-subsets formed by adjoining x to each of the s-subsets in A.

Example 1.3 Let $A^1 = \{12, 34\}$, $A^2 = \{13, 24\}$ and x, y be distinct elements not equal to $1, 2, 3$ or 4. Then the trade $T = T_1 - T_2$ in Example 1.1 could be written as $T = xA^1 + yA^2 - xA^2 - yA^1$. Note that T is a Steiner $(3,2)$ trade as no pair of elements occurs more than once in the blocks of T_1.

Definition 1.4 *The* **spectrum** *$S(k,t)$ of Steiner (k,t) trades is*

$$S(k,t) = \{m \mid \textit{there exists a Steiner } (k,t) \textit{ trade of volume } m\}.$$

The problem of determining $S(k,t)$ is intimately related to the intersection problem for Steiner systems. Adapting the notation used by Billington [1], let $I(v,k,t)$ denote the set of integers y for which there exist two $t-(v,k,1)$ Steiner systems $(V,\mathcal{B}_1)$ and $(V,\mathcal{B}_2)$ with $|\mathcal{B}_1 \cap \mathcal{B}_2| = y$. The *expected value* of $\bigcup_v \{|\mathcal{B}| - i \mid i \in I(v,k,t)\}$ is $S(k,t)$. It is well known [1] that $S(3,2) = \{0,4,6,7,8,\ldots\}$, $S(4,2) = \{0,6,8,9,10,\ldots\}$ and $S(4,3) = \{0,8,12,14,15,16,\ldots\}$. It is the primary purpose of this paper to prove the following result concerning $S(k,2)$.

Theorem 1.5
(1) *If $0 < m < 2k-2$ or $m = 2k-1$, then $m \notin S(k,2)$;*
(2) *If $m \geq 3k-3$, or m is even and $2k-2 \leq m \leq 3k-4$, then $m \in S(k,2)$;*

(3) *A Steiner* $(k, 2)$ *trade with* $m = 2k - 2$, *or* $m = 2k$ $(k \neq 3, 4)$, *has a unique structure.*

We fully determine the spectrum $S(k, 2)$ for $k = 5, 6$, and leave one unresolved case for $k = 7$. A generalisation of these results to trades with blocks based on arbitrary simple graphs is presented. Finally, the case $t > 2$ is briefly discussed.

2 Preliminary Results

For completeness, we determine $S(k, 1)$.

Theorem 2.1 $S(k, 1) = \{0, 2, 3, 4, \ldots\}$. *Further, if* $T = T_1 - T_2$ *is a* $(k, 1)$ *Steiner trade of volume* m, *then* T_1 *consists of* m *disjoint* k*-subsets.*

Proof Clearly, the volume of a (k, t) trade cannot equal one. If T_1 is a Steiner $(k, 1)$ trade, then the blocks of T_1 must be disjoint or else a 1-subset is repeated. Let T_1 be any collection of m (≥ 2) mutually disjoint k-subsets $\{A_1, A_2, \ldots, A_m\}$ and choose $x_i \in A_i$ for $i = 1, 2, \ldots, m$. Let $B_i = A_i \setminus \{x_i\}$ for $i = 1, 2, \ldots, m$. Then $T_1 = \{x_1B_1, x_2B_2, \ldots, x_mB_m\}$ trades with $T_2 = \{x_mB_1, x_1B_2, x_2B_3, \ldots, x_{m-1}B_m\}$ and $T_1 - T_2$ is a Steiner $(k, 1)$ trade of volume m. □

As the spectrum problem for $t = 1$ is completely solved, for the remainder of this paper, we assume that $t > 1$.

Lemma 2.2 *Suppose* $T^a = T_1^a - T_2^a$ *and* $T^b = T_1^b - T_2^b$ *are* (k, t) *trades. Then* $T = T^a + T^b = T_1^a + T_1^b - T_2^a - T_2^b$ *is a* (k, t) *trade (not necessarily Steiner) of volume*

$$m(T^a) + m(T^b) - |T_1^a \cap T_2^b| - |T_2^a \cap T_1^b|.$$

Proof That T is a trade is easily proved, or see, for instance, Hwang [4]. The volume of T equals $m(T^a) + m(T^b)$ minus the number of blocks in $T_1^a \cap T_2^b$ and $T_2^a \cap T_1^b$. □

Lemma 2.3 *Suppose* $m_1, m_2 \in S(k, t)$, *where* m_1 *and* m_2 *may be equal. Then:*

(1) $m_1 + m_2 - 1 \in S(k, t)$;

(2) $m_1 + m_2 \in S(k, t)$. *That is,* $S(k, t)$ *is closed under addition.*

Proof Let $T^a = T_1^a - T_2^a$ and $T^b = T_1^b - T_2^b$ be Steiner $(k, 2)$ trades such that $m(T^a) = m_1$ and $m(T^b) = m_2$.

(1) We first show that $m_1 + m_2 - 1 \in S(k,t)$. Choose a block B of T_1^a. Relabel the elements of any block C of T_2^b so that $C = B$. Now relabel the elements of $F(T^b) \setminus B$ so that $F(T^a) \cap (F(T^b) \setminus B) = \emptyset$. To show that $T^a + T^b$ is Steiner we need only consider t-subsets of B because of the choice of $F(T^b)$. However, since T^a and T^b are Steiner, any t-subset S of B occurs precisely once in each of T_1^a, T_2^a, T_1^b and T_2^b. Since B is common to T_1^a and T_2^b, $B \notin T_1^a - T_2^b$ and so S occurs precisely once in $T^a + T^b$. In fact, B is the only block common to T^a and T^b and hence $m(T^a + T^b) = m_1 + m_2 - 1$.

(2) To show that $m_1 + m_2 \in S(k,t)$, relabel the elements of $F(T^b)$ so that $F(T^a) \cap F(T^b) = \emptyset$. Then $T^a + T^b$ is a Steiner (k,t) trade of volume $m_1 + m_2$. □

We illustrate the construction method of Lemma 2.3(1) with the following example.

Example 2.4 Two Steiner $(3,2)$ trades of volume six are

$$\begin{aligned} T^a &= T_1^a - T_2^a \\ &= +134 + 156 + 178 + 235 + 247 + 268 \\ &\quad -135 - 147 - 168 - 234 - 256 - 278, \end{aligned}$$

and

$$\begin{aligned} T^b &= T_1^b - T_2^b \\ &= +134 + 156 + 235 + 246 + 036 + 045 \\ &\quad -135 - 146 - 236 - 245 - 034 - 056. \end{aligned}$$

We show how to construct a Steiner $(3,2)$ trade of volume eleven. Let $B = +134 \in T_1^a$ and $C = -135 \in T_2^b$. We first transpose the elements $4, 5 \in F(T^b)$ so that $B = C$. Next relabel the elements of $F(T^b) \setminus B$ to $\{\tilde{0}, \tilde{2}, \tilde{5}, \tilde{6}\}$ so that $F(T^a) \cap \{\tilde{0}, \tilde{2}, \tilde{5}, \tilde{6}\} = \emptyset$. Now,

$$\begin{aligned} T^b &= T_1^b - T_2^b \\ &= +13\tilde{5} + 14\tilde{6} + \tilde{2}34 + \tilde{2}\tilde{5}\tilde{6} + \tilde{0}3\tilde{6} + \tilde{0}4\tilde{5} \\ &\quad -134 - 1\tilde{5}\tilde{6} - \tilde{2}3\tilde{6} - \tilde{2}4\tilde{5} - \tilde{0}3\tilde{5} - \tilde{0}4\tilde{6}. \end{aligned}$$

It is simple to check that $T^a + T^b$ is a Steiner $(3,2)$ trade of volume eleven which completes the example.

If a single non-null Steiner (k,t) trade exists, we can obtain a surprisingly strong result concerning $S(k,t)$. Let $\overline{S}(k,t) = \{n \mid n \geq 0 \text{ and } n \notin S(k,t)\}$.

Theorem 2.5 *Exactly one of the following is true:*
(1) $S(k,t) = \{0\}$;
(2) $\overline{S}(k,t)$ *is finite.*

Proof If $S(k,t) = \{0\}$, then (1) is true and (2) is obviously false. If $S(k,t) \neq \{0\}$, then $m \in S(k,t)$ for some $m > 0$. By Part (1) of Lemma 2.3, $2m-1 \in S(k,t)$. As m and $2m-1$ are coprime and $S(k,t)$ is closed under addition, $\overline{S}(k,t)$ is finite. □

Corollary 2.6 $\overline{S}(t+1,t)$ *is finite.*

Proof It is easily seen that the $(t+1,t)$ trades of volume 2^t constructed in Theorem 2 of [4] are Steiner. □

3 The spectrum of Steiner $(k,2)$ trades

In this section, we determine $S(k,2)$ for all k, except for a finite number of values for each k. Our main tool for constructing Steiner $(k,2)$ trades is solely 1-balanced families.

Definition 3.1 *Let A^1 and A^2 be collections of s-subsets of V. If $A^1 - A^2$ is a Steiner $(s,1)$ trade and A^1 and A^2 contain no common 2-subset, then A^1 and A^2 are said to be* **solely** *1-balanced. A set $\{A^1,\ldots,A^n\}$ of collections A^i of s-subsets such that A^i and A^j are solely 1-balanced for each $i \neq j$ is said to be a solely 1-balanced family.*

The construction of the trade in Example 1.3 illustrates the use of solely 1-balanced families and generalises to give the following lemma.

Lemma 3.2 *Let A^1 and A^2 be collections of $(k-1)$-subsets such that $\{A^1, A^2\}$ is a solely 1-balanced family. Choose distinct $x, y \notin F(A^1)$, and let $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$. Then $T = T_1 - T_2$ is a Steiner $(k,2)$ trade.*

Proof Clearly, there are no k-subsets common to T_1 and T_2, and T is a $(k,2)$ trade. It remains to show that T is Steiner. Let $\{\alpha,\beta\}$ be a pair of elements in a block of T_1. It suffices to show that $\{\alpha,\beta\}$ is contained in precisely one k-subset in each of T_1 and T_2. There are two cases to consider.

Case 1: Suppose $\{\alpha,\beta\} \cap \{x,y\} = \emptyset$. Without loss of generality, suppose $\{\alpha,\beta\}$ is contained in A^1. As $A^1 - A^2$ is a Steiner $(k-1,1)$ trade, α and $\{\alpha,\beta\}$ are contained precisely once in the $(k-1)$-subsets of A^1. As A^1 and A^2 are solely 1-balanced, $\{\alpha,\beta\}$ does not occur in A^2. Hence $\{\alpha,\beta\}$ is contained exactly once in each of T_1 and T_2.

Case 2: Suppose $\{\alpha,\beta\} \cap \{x,y\} \neq \emptyset$. In this case, without loss of generality, suppose that $\alpha = x$ and $y \neq \beta \in F(A^1)$. But β is contained precisely once in each of A^1 and A^2 as $A^1 - A^2$ is a Steiner $(k-1,1)$ trade. Thus $\{\alpha,\beta\}$ is contained precisely once in each of T_1 and T_2. □

To prove our non-existence results, we will need some technical lemmas regarding the multiplicity of a set of elements in the blocks of a trade.

Definition 3.3 *For an s-subset S and trade $T = T_1 - T_2$, let $r_S(T_1)$ equal the number of blocks in T_1 which contain S. We say that S has* **multiplicity** $r_S(T_1)$ *in T_1 and note that if $s \leq t$, then $r_S(T_1) = r_S(T_2)$ $(= r_S(T))$. If $S = \{x\}$, we write r_x for $r_{\{x\}}(T_1)$. Define*

$$r(T) = \min\{r_x \mid x \in F(T_1)\}.$$

Lemma 3.4 *If S is an s-subset, $1 \leq s < t$, and T is a (k,t) trade, then*

$$r_S(T) \neq 1, m(T) - 1.$$

Proof See, for example, the proof by Hwang of Lemma 3 in [4]. □

Corollary 3.5 *If T is a (k,t) trade and $t > 1$, then $r(T) \geq 2$.* □

Lemma 3.6 *If T is a Steiner $(k,2)$ trade and $x \in F(T)$, then*

$$2 \leq r_x \leq \frac{k\,m(T)}{(2k-1)}.$$

Proof If x has multiplicity r_x, then the $r_x(k-1)$ elements occurring with x in T_1 must all be distinct, since the trade is Steiner. Since $r(T) \geq 2$ by Corollary 3.5, each of these elements must appear at least once more. Thus, $k\,m(T) \geq r_x + 2r_x(k-1)$ and so $r_x \leq k\,m(T)/(2k-1)$. □

Lemma 3.7 *If $T = T_1 - T_2$ is a Steiner $(k,2)$ trade with $r(T) > 2$, then $m(T) \geq 2k+1$.*

Proof If $a_1 a_2 \dots a_k$ is a block of T_1, then the elements $a_1, a_2 \dots, a_k$ each occur at least two more times in the blocks of T_1 since $r(T) > 2$. As no pair of these elements can occur in more than one block of T_1, there are at least $2k+1$ blocks in T_1. □

Lemma 3.8 *Suppose $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade, with $r_\alpha = 2$ for some $\alpha \in F(T_1)$ and either (1) $m(T) < 4k-10$; or (2) $m(T) \leq 2k-1$. If B_1 and B_2 are the two blocks of T_1 containing α, then there exist (distinct) elements $x \in B_1$ and $y \in B_2$ such that at least $k-1$ blocks of T_1 contain x but not y, and at least $k-1$ blocks of T_1 contain y but not x.*

Proof For $\alpha \in F(T)$ with $r_\alpha = 2$, without loss of generality, let $B_1 = \alpha a_1 \dots a_{k-1}$ and $B_2 = \alpha b_1 \dots b_{k-1}$ represent the two distinct blocks in T_1 containing α. Let C_1 and C_2 represent the two blocks in T_2 also containing α. As T_1 and T_2 contain precisely the same pairs involving α, each of $a_1, \dots, a_{k-1}, b_1, \dots, b_{k-1}$ is contained in exactly one of C_1, C_2. Let $j_a = |(B_1 \cap C_1) \setminus \{\alpha\}|$, so $1 \leq j_a \leq k-2$. Relabel C_1, C_2 and the elements of B_1 and B_2 as necessary so that, without loss of generality,

$$C_1 = \alpha a_1 \dots a_{j_a} b_1 \dots b_{j_b}, \qquad C_2 = \alpha a_{j_a+1} \dots a_{k-1} b_{j_b+1} \dots b_{k-1},$$

where $j_a + j_b = k-1$ and $j_b \geq j_a \geq 1$. Note that, subject to these constraints, the minimum of the product $j_a j_b$ occurs when $(j_a, j_b) = (1, k-2)$.

Let $i_1 \in \{1, \dots, j_a\}$, $i_2 \in \{1, \dots j_b\}$, $i_3 \in \{j_a+1, \dots, k-1\}$, $i_4 \in \{j_b+1, \dots, k-1\}$. The pairs $\{a_{i_1}, b_{i_2}\}$ and $\{a_{i_3}, b_{i_4}\}$ occur in T_2 and thus must also occur in T_1. Moreover, each of these pairs must occur in a separate block of T_1 as T is a Steiner trade. This implies $m(T) \geq 2 + j_a j_b + (k-1-j_a)(k-1-j_b) = 2 + 2j_a j_b$.

If $(j_a, j_b) \neq (1, k-2)$, then $j_a j_b \geq 2(k-3)$ which implies that $m(T) \geq 2 + 4(k-3) = 4k-10$. So if (1) is true, then $(j_a, j_b) = (1, k-2)$. If (2) is true (and (1) is not), then $4k-10 \leq m \leq 2k-1$ which implies $k \leq 4$. For $k = 3, 4$, j_a and j_b are uniquely determined as 1 and $k-2$ respectively. Thus $(j_a, j_b) = (1, k-2)$ in all cases and the element a_1 occurs in blocks with $b_1, b_2, \dots, b_{k-2}$. As $B_2 \in T_1$ and T is Steiner, each of these $k-2$ blocks in T_1 is distinct and cannot contain b_{k-1}. However block B_1 also contains a_1 and does not contain b_{k-1}, and thus there are at least $k-1$ blocks in T_1 containing a_1 and not b_{k-1}. By symmetry, there are at least $k-1$ blocks containing b_{k-1} and not a_1 in T_1. Letting $x = a_1$ and $y = b_{k-1}$ yields the result. □

Lemma 3.9 *Suppose $T = T_1 - T_2$ is a Steiner $(k,2)$ trade and there exist distinct elements $x, y \in F(T)$ such that either*
(1) $k = 3$, $r_x + r_y = m(T)$ and $r_{\{x,y\}} = 0$; or
(2) $k > 3$, $r_x + r_y \geq m(T)$.
Then $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$ where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$. Thus $r_x = r_y = m(T)/2$.

Proof The proof of (1) when $k = 3$ is simple and thus omitted. Assume that $k > 3$. We first show that $r_{\{x,y\}} = 0$. Suppose not; then $r_{\{x,y\}} = 1$ as T is Steiner and B, say, is the unique block in T_1 containing both x and y. There is at most one block in T_1 which contains neither x nor y. Thus any two elements distinct from x and y which occur in B cannot both occur again in T_1 without repeating a pair. Hence $r_{\{x,y\}} = 0$.

It is now obvious that $r_x + r_y = m(T)$ and the blocks of T_1 can be written as $\{xA^1, yA^2\}$ for some collections A^1, A^2 of disjoint $(k-1)$-subsets. However, as each element in A^1 must occur at least twice in the blocks of T_1 by Lemma 3.4, it follows that $F(A^1) = F(A^2)$ and $\{A^1, A^2\}$ is a solely 1-balanced family.

It is easy but tedious to show that any block ($(k-1)$-subset) S of A^1 must also occur in T_2. It then follows that $T_2 = xA^2 + yA^1$ as claimed. □

Lemma 3.8 allows us to prove the following two non-existence results for the volumes of Steiner $(k,2)$ trades. Additionally, using Lemma 3.9, the structure of Steiner $(k,2)$ trades of certain volumes can be determined.

Theorem 3.10 *If $T = T_1 - T_2$ is a Steiner $(k,2)$ trade, then $m(T) \geq 2k-2$. If $m(T) = 2k-2$, then $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$ where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$.*

Proof If $m(T) < 2k - 2$, then by Lemma 3.7 $r(T) = 2$. By Lemma 3.8, there exist at least $k-1$ blocks which contain x but not y and at least $k-1$ blocks which contain y but not x. Thus $m(T) \geq 2(k-1)$, a contradiction. When $m(T) = 2(k-1)$, then the structure of T follows from Lemmas 3.7, 3.8 and 3.9. □

Note how, in the previous theorem, the transposition (xy) applied to T_1 yields T_2. We call a trade with this property a **transposition** trade. Many of the trades constructed in this section will be transposition trades, or variants thereof.

Theorem 3.11 *If $T = T_1 - T_2$ is a Steiner $(k, 2)$ trade, then $m(T) \neq 2k-1$.*

Proof Assume T is a Steiner $(k, 2)$ trade and $m(T) = 2k - 1$. Then $r(T) = 2$ by Lemma 3.7. By Lemma 3.8 there exists two elements a and b each occurring in at least $(k-1)$ blocks of T_1 in which the other element does not occur. We show that a and b are contained in precisely $k-1$ blocks each. Suppose there is a block in T_1 that contains both a and b. Any other element of this block must occur in at least one other block by Lemma 3.4 but all the remaining blocks contain either a or b which contradicts the fact that T is Steiner. Also, one of a or b is not contained in k blocks by Lemma 3.9. Thus there is a block $B \in T_1$ containing neither a nor b, and a and b are each contained in precisely $k - 1$ blocks.

Consider any element α other than a or b; we show that for $k > 4$, if α occurs in a block with a, then it must also occur in a block with b. For if not, then $r_\alpha = 2$ and $\alpha \in B$. By Lemma 3.8 there exists element c contained in B such that c is contained in at least $(k - 1)$ blocks. We have shown that c cannot equal a or b and so there are three distinct elements a, b, c each occurring in at least $k-1$ blocks. Further, there is no block containing both a and b and so the number of blocks in T_1 is at least $3(k-1)-2 > 2k-1$ for $k > 4$. This is a contradiction and so we conclude that any element occurring in a block with a occurs in a block with b and conversely.

Now, consider the k elements of the block B. Each of these elements occurs in at least one more block of T_1 and so each of these elements occurs in a block with a. But a is contained in exactly $k - 1$ blocks and so one of the pairs of elements in B is repeated in T_1 contradicting the fact that T_1 is a Steiner $(k, 2)$ trade.

That the result holds when $k = 3$ or 4 is known [1] which completes the proof. □

Lemma 3.12 *Suppose $k = 3, 4$ or 5 and let $T = T_1 - T_2$ be a Steiner $(k, 2)$ trade with $m(T) = 2k$.*
(1) If $k = 3$, then either $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$, or $T_1 = xA^1 + yA^2 + zA^3$ and $T_2 = xA^2 + yA^3 + zA^1$, where A^1 and A^2 (respectively A^1, A^2 and A^3) are solely 1-balanced and $x, y, z \notin F(A^1)$;
(2) If $k = 4$, then there are at least two structures for T. One of these has $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$, where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$;

(3) *If* $k=5$*, then* $T_1 = xA^1 + yA^2$ *and* $T_2 = xA^2 + yA^1$*, where* A^1 *and* A^2 *are solely* 1*-balanced and* $x, y \notin F(A^1)$*.*

Proof If $k = 3$ and $m(T) = 6$, then all the elements of $F(T)$ have multiplicity two or three, by Lemma 3.6, and it is straightforward to check that T must be as claimed. The trades T^a and T^b in Example 2.4 illustrate these two structures.

For $k = 4$, it is easy to construct the trade based on solely 1-balanced families. For another structure, consider the fact that no element in

$$\begin{aligned} T &= +045a + 069b + 167a + 158b + 289a + 247b + 3468 + 3579 \\ &\quad -046a - 059b - 158a - 167b - 279a - 248b - 3457 - 3689 \end{aligned}$$

has multiplicity four.

See Appendix A for a proof of the case where $k = 5$. □

Theorem 3.13 *Suppose* $k > 5$*,* $T = T_1 - T_2$ *is a Steiner* $(k,2)$ *trade, and* $m(T) = 2k$*. Then* $T_1 = xA^1 + yA^2$ *and* $T_2 = xA^2 + yA^1$*, where* A^1 *and* A^2 *are solely* 1*-balanced and* $x, y \notin F(A^1)$*.*

Proof If $m(T) = 2k$ then, by Lemma 3.7, $r(T) = 2$. For $k > 5$, $m(T) = 2k < 4k - 10$ and by Lemma 3.8 there exist at least $k-1$ blocks in T_1 which contain x but not y and at least $k-1$ blocks which contain y but not x. Suppose there is a block B in T_1 which contains both x and y. The remaining $k-2$ elements of B must occur at least twice in T_1. All except possibly one block of T_1 contain either x or y and so the remaining $k-2$ elements cannot occur again in T_1 without repeating a pair. Thus x and y do not occur in a block together.

Suppose that B is a block in T_1 which contains neither x nor y. We show that the multiplicity of any element c which is contained in B is at least three. Suppose $r_c = 2$. Then by Lemma 3.8, there exists a distinct element, say z, such that z is in B and z is contained in at least $k-1$ blocks of T_1. This would imply that $m(T) \geq 3(k-1) - 2 = 3k - 5$. Thus $m(T) > 2k$ for $k > 5$ which is a contradiction, so $r_c \geq 3$ (*).

The only possible values for (r_x, r_y) are $(k-1, k-1)$, $(k, k-1)$, $(k-1, k)$ or (k, k), and we consider each of these cases separately.

Case $(k-1, k-1)$: Consider the two blocks C and D which contain neither x nor y. One of the k elements of C, say a, is not contained in any of the

$k-1$ blocks in which y occurs (or else a pair is repeated in the blocks of T_1). But $r_a \geq 3$ by (*). This implies that $a \in D$ and in a block containing x. Similarly, one of the elements of C, say b, is not contained in any of the $k-1$ blocks in which x occurs. But also $r_b \geq 3$ which implies $b \in D$. The case $a = b$ is disallowed by our construction and the pair $\{a, b\}$ is repeated in the blocks C and D which is a contradiction.

Case $(k, k-1)$ *or* $(k-1, k)$: Without loss of generality, we show only that the case $(k, k-1)$ is impossible. Consider the block C which contains neither x nor y. One of the k elements of C, say a, is not contained in any of the $k-1$ blocks in which y occurs (or else a pair is repeated in the blocks of T_1). However, by (*), $r_a \geq 3$. This would imply that a is contained in two blocks which contain x and the pair $\{a, x\}$ is repeated, a contradiction.

Case (k, k)*:* This case must hold. Now $r_x + r_y = m(T)$ and the structure of T follows from Lemma 3.9. □

In the remainder of this section, we construct Steiner $(k, 2)$ trades using solely 1-balanced families.

Notation Let $A(k)$ equal the $(k-1) \times (k-1)$ array with entries (positions)

$$a_{ij} = (i-1)(k-1) + j, \qquad \text{for } i, j = 1, 2, \ldots, (k-1).$$

Write R, C and F for the collections of elements of each of the rows, columns and forward diagonals of $A(k)$ respectively, suppressing k, since it will be fixed. Define $\tilde{A}(k, r)$ to be the $(k-1) \times (k-1)$ array with each of the elements a_{ij} of the first r rows of $A(k)$ replaced by $\tilde{a}_{ij}$, where

$$\tilde{a}_{ij} \neq a_{kl} \text{ for } 1 \leq i, k \leq r, \ 1 \leq j, l \leq k-1.$$

$\tilde{R}_r$ and $\tilde{C}_r$ are the sets of elements of each of the rows and columns of $\tilde{A}(k, r)$ respectively. It is easy to see that $\{\tilde{R}_r, \tilde{C}_r\}$ and $\{R, C, F\}$ are solely 1-balanced families.

Example 3.14 Let $k = 4$. Then,

$$\begin{array}{rclrcl} R &=& \{123, 456, 789\}, & C &=& \{147, 258, 369\}, \\ \tilde{R}_1 &=& \{\tilde{1}\tilde{2}\tilde{3}, 456, 789\}, & \tilde{C}_1 &=& \{\tilde{1}47, \tilde{2}58, \tilde{3}69\}, \\ F &=& \{159, 267, 348\}. & & & \end{array}$$

Now let

$$\begin{array}{rcl} T^a &=& +xR + yF - xF - yR, \\ T^b &=& -x\tilde{R}_1 - z\tilde{C}_1 + x\tilde{C}_1 + z\tilde{R}_1, \end{array}$$

where $x, y, z \notin F(R \cup \tilde{R}_1)$. Then

$$\begin{aligned} T^a + T^b &= +x123 + y159 + y267 + y348 \\ &\quad +x\tilde{1}47 + x\tilde{2}58 + x\tilde{3}69 + z\tilde{1}\tilde{2}\tilde{3} + z456 + z789 \\ &\quad -x159 - x267 - x348 - y123 - y456 - y789 \\ &\quad -x\tilde{1}\tilde{2}\tilde{3} - z\tilde{1}47 - z\tilde{2}58 - z\tilde{3}69, \end{aligned}$$

is a Steiner $(4,2)$ trade of volume ten. Note how $+x456$ and $+x789$ are in T^a and $-x456$ and $-x789$ are in T^b and that these blocks cancel in $T^a + T^b$.

The straightforward construction of Example 3.14 will be modified to provide our existence results. In Lemmas 3.15 and 3.16 we choose the foundations of T^a and T^b so that $T^a + T^b$ is a Steiner trade of the required volume.

Lemma 3.15 *There exists a Steiner* $(k,2)$ *trade of volume* $2(k-1) + 2r$ *for each* $r = 0, 1, \ldots k-1$.

Proof Let $T^a = T_1^a - T_2^a = +xR + yF - xF - yR$ and $T^b = T_1^b - T_2^b = +x\tilde{C}_r + y\tilde{R}_r - x\tilde{R}_r - y\tilde{C}_r$ with distinct $x, y \notin F(R \cup \tilde{R}_r)$. That T^a, T^b and $T^a + T^b$ are trades follows from Lemmas 2.2 and 3.2. It remains to show that $T^a + T^b$ is a Steiner trade of the required volume.

Any pair of elements in xR that occurs more than once in $T_1^a + T_1^b$ must occur in $y\tilde{R}_r$ or $x\tilde{C}_r$. Any pair of elements in yF that occurs more than once in $T_1^a + T_1^b$ must occur in $y\tilde{R}_r$. However, the blocks containing such pairs cancel in the addition of T^a and T^b; either in the $k-1-r$ blocks in xR which cancel with $x\tilde{R}_r$ or in the $k-1-r$ blocks in $y\tilde{R}_r$ which cancel with yR. Thus $T^a + T^b$ is a Steiner $(k,2)$ trade of volume

$$\begin{aligned} m(T^a + T^b) &= 4(k-1) - |xR \cap x\tilde{R}_r| - |yR \cap y\tilde{R}_r| \\ &= 4(k-1) - 2(k-1-r) \\ &= 2(k-1) + 2r, \end{aligned}$$

as required. □

Lemma 3.16 *There exists a Steiner* $(k,2)$ *trade of volume* $3(k-1) + r$ *for each* $r = 0, 1, \ldots k-1$.

Proof Let $T^a = +xR + yF - xF - yR$ and $T_b = +x\tilde{C}_r + z\tilde{R}_r - x\tilde{R}_r - z\tilde{C}_r$ with distinct $x, y, z \notin F(R \cup \tilde{R}_r)$. Then, as in the proof of Lemma 3.15, $T^a + T^b$ is a Steiner $(k,2)$ trade of volume $4(k-1)-|xR\cap x\tilde{R}_r| = 4(k-1)-(k-1-r) = 3(k-1)+r$. □

Lemma 3.17 *There exists a Steiner* $(k,2)$ *trade of volume* $4(k-1)+r$ *for each* $r = 0, 1, \ldots k-1$.

Proof First note that by Lemma 3.15, $2(k-1) \in S(k,2)$, as are even values between $2(k-1)$ and $3(k-1)$ inclusive. Thus all values between $4(k-1)$ and $5(k-1)$ inclusive are in $S(k,2)$ by Lemma 2.3. This completes the proof. □

Theorem 3.18 *If* $m \geq 3(k-1)$, *then there exists a Steiner* $(k,2)$ *trade of volume* m.

Proof We have shown that $2(k-1) \in S(k,2)$ and that $\{3k-3, 3k-2, \ldots, 5k-5\} \subseteq S(k,2)$. As $S(k,2)$ is closed under addition, the result follows. □

It only remains to determine whether the odd integers between $2k+1$ and $3k-4$ inclusive are in $S(k,2)$, for $k \geq 5$. We complete the cases $k = 5$ and 6, and leave one volume unresolved for the case $k = 7$.

Theorem 3.19 $S(5,2) = \{0, 8, 10, 12, 13, 14, \ldots\}$.

Proof By the results of this section, the only unresolved volume for $S(5,2)$ is eleven. A proof that $11 \notin S(5,2)$ is given in Appendix A. □

Theorem 3.20 $S(6,2) = \{0, 10, 12, 14, 15, 16, \ldots\}$.

Proof By the results of this section, the only unresolved volume for $S(6,2)$ is thirteen. A computer-assisted proof that $13 \notin S(6,2)$ is given in Appendix A. □

Theorem 3.21 $S(7,2) \supseteq \{0, 12, 14, 15, 16, 18, 19, 20, \ldots\}$ *and* $\overline{S}(7,2) \supseteq \{1, 2, \ldots, 11, 13\}$. *The existence of a Steiner* $(7,2)$ *trade of volume seventeen is unresolved.*

Proof By the results of this section, the only unresolved volumes for $S(7,2)$ are fifteen and seventeen. A Steiner $(7,2)$ trade $T = T_1 - T_2$ of volume fifteen is given in Figure 1. T_1 and T_2 are isomorphic to the dual of the $2-(15,3,1)$ design constructed from the points and lines of $PG(3,2)$. □

T_1								T_2						
0	1	2	3	4	5	6		0	1	2	7	8	13	14
0	7	8	9	10	11	12		0	3	4	9	10	15	16
0	13	14	15	16	17	18		0	5	6	11	12	17	18
1	7	13	19	20	21	22		1	3	5	19	20	23	24
1	8	14	23	24	25	26		1	4	6	21	22	25	26
2	7	14	27	28	29	30		2	3	6	27	28	31	32
2	8	13	31	32	33	34		2	4	5	29	30	33	34
3	9	15	19	23	27	31		7	9	11	19	21	27	29
3	10	16	20	24	28	32		7	10	12	20	22	28	30
4	9	16	21	25	29	33		8	9	12	23	25	31	33
4	10	15	22	26	30	34		8	10	11	24	26	32	34
5	11	17	19	24	29	34		13	15	17	19	22	31	34
5	12	18	20	23	30	33		13	16	18	20	21	32	33
6	11	18	21	26	27	32		14	15	18	23	26	27	30
6	12	17	22	25	28	31		14	16	17	24	25	28	29

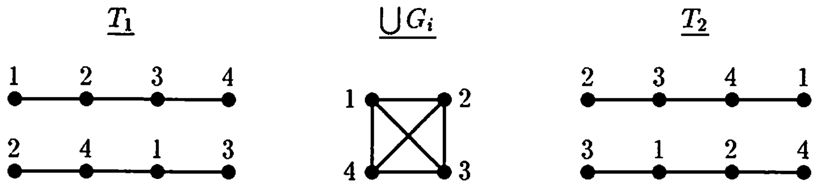
Figure 1: A Steiner $(7, 2)$ trade of volume 15

4 G-trades

A block of size k in a trade can be viewed as a *complete graph* of order k. Billington and Hoffman [2] generalise the trade spectrum problem to trades based on arbitary simple graphs. We keep this section as brief as possible and the reader is advised to consult [2] for a more detailed discussion of G-trades. As far as possible, our notation is consistent with that in [2]. Let G be a simple (*cf.*, Steiner property) graph with $\nu(G)$ vertices. We call a graph isomorphic to G a **block based** on G.

Definition 4.1 *Let* $T_1 = \{G_1, G_2, \ldots, G_m\}$ *and* $T_2 = \{G'_1, G'_2, \ldots, G'_m\}$ *each be collections of* m *blocks based on* G. *If* $\bigcup_{i=1}^m G_i = \bigcup_{i=1}^m G'_i$ *is a simple graph and* $G_i \neq G'_j$, $1 \leq i, j \leq m$, *then* $T = T_1 - T_2$ *is a* G*-trade of volume* m.

Example 4.2 *In this example,* G *is a path of length three and* T_1 *and* T_2 *are disjoint decompositions of* K_4 *into blocks based on* G.

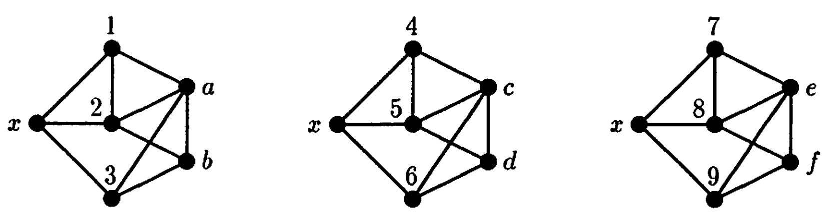


The construction method of Lemma 2.3(1) fails for general G-trades. For instance, in Example 4.2, relabelling all the elements of any block of T_1 relabels all the elements of the trade.

For the remainder of this section, the minimum degree of G, $\delta(G)$, is at least two. Let $\delta(G) = k-1$. For this value of k, the collections R, C, F and $\tilde{R}_r, \tilde{C}_r$ are defined as in Section 3 with the understanding that the rows ($(k-1)$-subsets) will be used to label vertices of blocks based on G.

We use triples of the form (x, N, D) to represent $k-1$ blocks based on G constructed as follows. Here, x represents a (fixed) vertex of minimum degree $k-1$, $N \in \{R, C, F, \tilde{R}_r, \tilde{C}_r\}$ and D is a collection of $k-1$ disjoint sets (rows) each of $\nu(G) - k$ vertices. The $k-1$ blocks based on G are formed by placing a copy of G on x and the (ordered) ith rows of N and D in a consistent manner; ensuring, for instance, that the neighbourhood of x equals the ith row of N for each $i = 1, \ldots, k-1$, and two vertices from the same positions of N and D in different blocks are either always adjacent or non-adjacent depending on how the first copy of G was placed.

Example 4.3 *In this example, our notation is illustrated for a graph G with $\delta(G) = 3$. So $k = 4$ and $R = \{\{123\}, \{456\}, \{789\}\}$. Choose $D = \{\{a, b\}, \{c, d\}, \{e, f\}\}$ so that the three blocks based on G shown can be written as (x, R, D).*



We will now show how our constructions in Section 3 can be applied to G-trades. The following result is analogous to Lemma 3.16. Recall that $\tilde{A}_r$

represents A with the elements a_{ij} of the first r rows of A relabelled to $\tilde{a}_{ij}$. $\tilde{D}_r$ is defined similarly.

Lemma 4.4 *There exists a G-trade of volume* $3(k-1)+r$ *for each* $r = 0, 1, \ldots, k-1$.

Proof Let

$$\begin{aligned} T^a &= +(x, R, D^1) + (y, F, D^2) - (x, F, D^2) - (y, R, D^1), \\ T^b &= +(x, \tilde{C}_r, D^3) + (z, \tilde{R}_r, \tilde{D}_r^1) - (z, \tilde{C}_r, D^3) - (x, \tilde{R}_r, \tilde{D}_r^1), \end{aligned}$$

where the $F(D^i)$ are mutually disjoint, $F(D^i \cup \tilde{D}_r^i) \cap F(R \cup \tilde{R}_r) = \emptyset$ for $i = 1, 2, 3$ and $x, y, z \notin F(D^1 \cup \tilde{D}_r^1 \cup D^2 \cup D^3 \cup R \cup \tilde{R}_r)$.

Then $T^a + T^b$ is a G-trade if each edge in the positive half of $T^a + T^b$ occurs precisely once. Suppose that there exists an edge e in the positive half of $T^a + T^b$ occurring twice. It is immediate that e is not incident with y, z or any vertices of D^2 or D^3. Additionally e is not in F. Therefore, e is necessarily in (x, R, D^1) and $(x, \tilde{C}_r, D^3) \cup (z, \tilde{R}_r, \tilde{D}_r^1)$. However, this implies that e is in one of the blocks of (x, R, D^1) that cancels. Hence $T^a + T^b$ is a G-trade of the required volume. □

Lemmas for G-trades analogous to Lemmas 3.15 and Lemma 2.3(2) are similarly proved. Although there is no general analogue to Lemma 2.3(1), our choice for blocks based on G utilising the solely 1-balanced families and mutually disjoint D^i allows us to apply a restricted version of Lemma 2.3(1). So we can also prove an analogue to Lemma 3.17. Combining these results we obtain the following.

Theorem 4.5 *Let G be a simple graph with minimum degree* $\delta(G) \geq 2$. *If* $s \geq 3\delta(G)$, *or* $s \geq 2\delta(G)$ *and s is even, then there exists a G-trade of volume s.*

5 Conclusions

Given two distinct 2-$(v, k, 1)$ designs, $D_1 = (V, \mathcal{B}_1)$ and $D_2 = (V, \mathcal{B}_2)$, let $\mathcal{I} = \mathcal{B}_1 \cap \mathcal{B}_2$. Then $T = (\mathcal{B}_1 \setminus \mathcal{I}) - (\mathcal{B}_2 \setminus \mathcal{I})$ is a Steiner $(k, 2)$ trade. Non-exhaustive computer searches based on this observation, and using distinct 2-$(v, k, 1)$ designs with $v < 100$ and $7 \leq k \leq 10$, failed to find any Steiner $(k, 2)$ trades of odd volumes less than $3k - 3$. We suspect that determining the remaining values of $S(k, 2)$ will be difficult. For instance,

$2k+1 \in S(k,2)$ for $k = 3,4,7$ but $2k+1 \notin S(k,2)$ for $k = 5,6$. Observe also the high degree of structure of the Steiner $(7,2)$ trade of volume fifteen in Figure 1; for instance, each pair of blocks in T_1 intersects in exactly one point. We intend to address *linked* trades and their relationship to (r,λ)-designs in a sequel to this paper.

Finally, the problem of determining $S(k,t)$ for $t > 2$ appears even more difficult. Hwang has shown that the volume of a (k,t) trade is at least 2^t. However, the volume of a smallest Steiner (k,t) trade grows rapidly with k, as the following result demonstrates.

Theorem 5.1 *If $T = T_1 - T_2$ is a Steiner (k,t) trade, $t > 1$, then*

$$m(T) \geq 1 + \binom{k}{t-1}.$$

Proof Let B be a block in T_1. By Lemma 3.4, each $(t-1)$-subset of B must occur at least twice in T_1. If any two of these $(t-1)$-subsets occur together in a block other than B, then a t-subset is repeated in T_1 contradicting the fact that T is a Steiner trade. The number of $(t-1)$-subsets in B is $\binom{k}{t-1}$ and the result follows. □

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A Appendix

A.1 The swap matrix technique

The proofs in this appendix use what we term the *swap matrix* technique. Suppose that an element x has multiplicity r_x in a Steiner $(k,2)$ trade $T = T_1 - T_2$. The idea is to list the possible ways elements differ in the blocks containing x in T_1 and T_2.

Recall that the $r_x(k-1)$ elements with which x occurs must all be distinct. Since $T_1 \cap T_2 = \emptyset$, the $2r_x$ blocks in $T_1 \cup T_2$ containing x must all be distinct. Label the blocks of each of T_1 and T_2 containing x with $1, 2, \ldots, r_x$. Let

$$S = [s_{ij}],\ 1 \leq i,j \leq r_x,$$

be an $r_x \times r_x$ matrix where s_{ij} is the number of elements, other than x, from block j of T_1 that appear in block i of T_2. S is called a **swap matrix** and it is simple to see that $0 \leq s_{ij} \leq k-2$ and the row and column sums of S equal $k-1$. Thus, each row and column of S is a partition of $k-1$ into r_x parts, at least two of which are non-zero.

Any swap matrix can occur in many distinct but equivalent forms. Permutations of rows and columns lead to equivalent swap matrices. Permuting rows is equivalent to reordering the blocks of T_2 and permuting columns to reordering the blocks of T_1. Transposing S is equivalent to exchanging T_1 and T_2. We will always use a form where $s_{11} \geq s_{ij}$ and $s_{12} \geq s_{21}$, and where the first row and the first column are non-increasing. Note that this is *not* sufficient to eliminate all equivalences but suffices for our purposes.

A pair of elements that occur together with x in a block of T_1 but do not occur together in any block with x in T_2 is called a **broken pair.** A swap matrix is said to be **feasible** if all the broken pairs can be contained, as partial blocks, in the $m(T) - r_x$ remaining blocks of T_2 without violating the Steiner property. Given a feasible swap matrix, an apportioning of the broken pairs to the remaining blocks of T_2 is called a **viable template.** Note that a feasible swap matrix may yield more than one viable template, and that a viable template is not necessarily extendable to a Steiner trade.

We will determine possible swap matrices for Steiner $(k,2)$ trades of volume m for specific values of k and m. In the next two sections of this appendix, these are used to prove non-existence for the cases where $k=5$, $m=11$ and $k=6$, $m=13$. In the final section, they are used to show structural uniqueness for the case where $k=5$, $m=10$.

A.2 Steiner $(5,2)$ trades of volume 11

For Steiner (5,2) trades, the only volume where existence is not settled is eleven. We will assume that such a trade, $T = T_1 - T_2$, exists and obtain a contradiction. Including repetitions, there are $11 \times 5 = 55$ elements in T_1 and hence there must be an element in $F(T)$ with odd multiplicity. Since each element must occur at least twice and no more than six times, by Lemma 3.6, there must be an element with multiplicity three or five. We will show that both cases are impossible.

Lemma A.1 *If T is a Steiner $(5,2)$ trade with $m(T) < 12$, then there is no element with multiplicity three.*

Swap Matrix	No.	T_1	T_2	Broken Pairs
3 1 0	1	x 1 2 3 4	x 1 2 3 5	14, 24, 34*
1 2 1		x 5 6 7 8	x 4 6 7 9	56, 57, 58, 68, 78
0 1 3		x 9 a b c	x 8 a b c	$9a, 9b, 9c$*
3 1 0	2	x 1 2 3 4	x 1 2 3 5	14, 24, 34*
1 1 2		x 5 6 7 8	x 4 6 9 a	56, 57, 58, 67, 68
0 2 2		x 9 a b c	x 7 8 b c	$9b, 9c, ab, ac$*
3 1 0	3	x 1 2 3 4	x 1 2 3 5	14, 24, 34*
1 0 3		x 5 6 7 8	x 4 9 c b	56, 57, 58*
0 3 1		x 9 a b c	x 6 7 8 c	$9c, ac, bc$*
2 2 0	4	x 1 2 3 4	x 1 2 5 6	13, 14, 23, 24*
2 0 2		x 5 6 7 8	x 3 4 9 a	57, 58, 67, 68*
0 2 2		x 9 a b c	x 7 8 b c	$9b, 9c, ab, ac$*
2 2 0	5	x 1 2 3 4	x 1 2 5 6	13, 14, 23, 24, 34
1 1 2		x 5 6 7 8	x 3 7 9 a	57, 58, 67, 68, 78
1 1 2		x 9 a b c	x 4 8 b c	$9b, 9c, ab, ac$*
2 1 1	6	x 1 2 3 4	x 1 2 5 9	13, 14, 23, 24, 34
1 2 1		x 5 6 7 8	x 3 6 7 a	56, 57, 58, 68, 78
1 1 2		x 9 a b c	x 4 8 b c	$9a, 9b, 9c, ab, ac$

Table 1: The swap matrices for Steiner (5,2) trades with $r_x = 3$

Proof Let $x \in F(T)$ and $r_x = 3$. The possible partitions in a swap matrix for x are {3,1,0}, {2,2,0} and {2,1,1}. It is easy to see that, up to equivalence, there are only six possible swap matrices. These are shown in the first column of Table 1. Let the elements occurring with x be $1, 2, \ldots, 9, a, b, c$. The column headed T_2 is obtained by applying the swap matrix to T_1. The broken pairs are listed in the final column. Each row of broken pairs comes from the corresponding block of T_1. We will show that none of these swap matrices is feasible in less than 12 blocks.

For a given swap matrix, any two broken pairs from two different rows contain four distinct elements. Since there are only three rows containing x, the two broken pairs cannot appear together in a block in T_2 without duplicating a pair already in the blocks of T_2.

The rows of broken pairs labelled with a star are generated by columns of the swap matrix with a partition of {3,1,0} or {2,2,0}. These rows have the property that no two broken pairs in the same row can occur in the same block of T_2 without violating the Steiner propery. Thus we see that the third and fourth swap matrices imply that there are at least twelve and fifteen blocks in T_2 respectively. Similarly, to generate a trade in less than twelve sets from the second swap matrix would require that all of the pairs in the second row occurred in the same set, which would repeat the pair 78 in T_2.

The remaining three swap matrices all contain rows of five broken pairs derived from the partition {2,1,1}. All these sets of five broken pairs are isomorphic to the set {56, 57, 58, 68, 78}. The pair 67 already appears in T_2, so the pairs 56 and 57 must occur in different blocks in T_2, as must the pairs 68 and 78. If we used the two partial blocks 568 and 578 to cover all broken pairs, the pair 58 would occur twice. Thus each row of broken pairs without a star requires at least three blocks in T_2.

The first, fifth and sixth swap matrices would require an additional nine, ten and nine blocks respectively in T_2. Thus the total number of blocks in T_2 is greater than eleven. □

Lemma A.2 *If T is a Steiner $(5,2)$ trade of volume eleven, then there is no element with multiplicity six.*

Proof Suppose $y \in F(T)$ and $r_y = 6$. Then y is paired with $6(k-1) = 24$ distinct elements. Each of these elements must occur at least once in the five blocks not containing y, leaving exactly one position of these blocks unaccounted for. This cannot be a new element since it would have multiplicity one. Thus one of the elements occurring with y must have multiplicity three, contradicting Lemma A.1. □

As the number of elements with odd multiplicity is necessarily odd and there is no element with multiplicity three, there must be an odd number of elements with multiplicity five. It is easy to see that three or more elements with multiplicity five cannot be contained in eleven blocks without duplicating a pair. Thus there must be exactly one element with multiplicity five.

For $2 \leq i \leq 6$, let n_i denote the number of elements of $F(T)$ that have multiplicity i. We have shown that $n_3 = n_6 = 0$ and that $n_5 = 1$. Thus $2n_2 + 4n_4 + 5 = 55$; that is, $n_2 + 2n_4 = 25$. Given that $n_5 = 1$, it is straightforward to check that there are at most four elements with multiplicity four; that is, $n_4 \leq 4$.

To complete our analysis, we will need the following lemma analogous to Lemma 3.8.

Lemma A.3 *Let $T = T_1 - T_2$ be a Steiner $(5,2)$ trade with $r_\alpha = 2$ for some $\alpha \in F(T)$, and suppose that $\sum_{i \geq 3} n_i \leq 7$. If B_1 and B_2 are the two blocks of T_1 containing α, then there exist (distinct) elements $x \in B_1$ and $y \in B_2$ such that at least four blocks of T_1 contain x but not y and at least four blocks contain y but not x.*

Proof The possible partitions are {3,1} and {2,2} and there are only two inequivalent swap matrices for α,

$$S_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Without loss of generality, let $B_1 = \alpha 1234$ and $B_2 = \alpha 5678$. We first prove that S_2 is not feasible.

If S_2 is feasible, then without loss of generality $\alpha 1256$ and $\alpha 3478$ are in T_2. This yields the set of broken pairs {13, 14, 23, 24, 57, 58, 67, 68}. Now, no two of these pairs can occur together in T_2, since the trade is Steiner. Thus each of the elements $1, \ldots, 8$ must occur at least three times in T_2, contradicting $\sum_{i \geq 3} n_i \leq 7$.

We now show S_1 is feasible and yields a single viable template. We can assume, without loss of generality, that $\alpha 1235$ and $\alpha 4678$ are in T_2. This yields the set of broken pairs {14, 24, 34, 56, 57, 58}. Now, no two of these pairs can occur together in T_2, since the trade is Steiner. Thus both $x = 4$ and $y = 5$ occur in at least four blocks that do not contain the other element. □

Theorem A.4 *If T is a Steiner $(5,2)$ trade, then $m(T) \neq 11$.*

Proof Assume that $T = T_1 - T_2$ is a Steiner $(5,2)$ trade of volume eleven. We will derive a contradiction.

The only integral solutions (n_2, n_4) to $n_2 + 2n_4 = 25$ satisfying the necessary condition $n_4 \leq 4$ are (17,4), (19,3), (21,2), (23,1) and (25,0). The case (25,0) is eliminated immediately by Lemma A.3. Recall that $n_5 = 1$ and note that $n_4 > 0$ in all remaining cases. Let x be the element with multiplicity five. Since $n_4 \leq 4$, there must be at least one block, say B, which contains x and none of the multiplicity four elements.

First, suppose that each of the elements of multiplicity four occurs in some (but not necessarily the same) block in which x occurs. The four elements that occur with x in B must be multiplicity two elements. Thus the elements of multiplicity at least four that these elements are paired with must be x and one of the multiplicity four elements. But all multiplicity four elements are paired with x, contradicting Lemma A.3.

Thus there exists an element, say y, with multiplicity four which does not occur in a block with x. Let U and V be the two blocks containing neither x nor y. Consider the $5(k-1) = 20$ distinct elements contained in a block with x. Since $n_4 \leq 4$, at least 17 of these have multiplicity two. Now at most $4(k-1) = 16$ of these can occur in blocks containing y. Thus one of them, say z, occurs in one of U or V, say U. By Lemma A.3, z occurs with two elements, say α and β, each of which occurs in four blocks not containing the other. Now, at least one of α or β, say α, must have multiplicity four and be contained in U. The other three occurrences of α must be in V, in a block with y, and a block with x (not that containing z). Now consider β. We cannot have $\beta = x$, since we already have the pair αx, contradicting Lemma A.3. Thus $r_\beta = 4$, and it must occur in the block containing xz, in a block with y, and in both U and V. But the pair $\alpha\beta$ is repeated and this completes the proof. □

A.3 Steiner $(6,2)$ trades of volume 13

The only volume for Steiner (6,2) trades whose existence is unresolved is thirteen. Such a trade would have a total of 78 elements in T_1, with possible element multiplicities of $r = 2, \ldots, 7$. Exhaustive searches by computer programmes based on the swap matrix method yielded a single equivalence class of feasible swap matrices for each of $r = 2$, 5 and 6, and showed that none existed for $r = 3$ or 4. Each feasible swap matrix yielded a single viable template. The matrices, with T_1 and the templates for T_2, are shown in Table 2. Note that all three of the swap matrices are symmetric, so the templates for T_1 are isomorphic to those for T_2. Using these templates, we will show that it is not possible to construct a Steiner $(6,2)$ trades of volume thirteen.

Lemma A.5 *If $T = T_1 - T_2$ is a Steiner $(6,2)$ trade of volume thirteen, then there is no element with multiplicity seven.*

Proof Suppose there exists an element of multiplicity seven in T_1. Such an element is paired with $7(k-1) = 35$ distinct other elements. Each of these 35 elements must occur at least once more which leaves one position of the trade unaccounted for. This remaining position must be filled by an element of multiplicity one or three. However, multiplicity one is not

r	Swap Matrix	T_1	T_2
2	4 1	x 1 2 3 4 5	x 1 2 3 4 6
	1 4	x 6 7 8 9 A	x 5 7 8 9 A
			1 5 * * * *
			2 5 * * * *
			3 5 * * * *
			4 5 * * * *
			6 7 * * * *
			6 8 * * * *
			6 9 * * * *
			6 A * * * *
			* * * * * *
			* * * * * *
			* * * * * *
5	1 1 1 1 1	x 1 2 3 4 5	x 1 6 B G L
	1 1 1 1 1	x 6 7 8 9 A	x 2 7 C H M
	1 1 1 1 1	x B C D E F	x 3 8 D I N
	1 1 1 1 1	x G H I J K	x 4 9 E J O
	1 1 1 1 1	x L M N O P	x 5 A F K P
			1 2 3 4 5 *
			6 7 8 9 A *
			B C D E F *
			G H I J K *
			L M N O P *
			* * * * * *
			* * * * * *
			* * * * * *
6	1 1 1 1 1 0	x 1 2 3 4 5	x 1 6 B G L
	1 1 1 1 0 1	x 6 7 8 9 A	x 2 7 C H Q
	1 1 1 0 1 1	x B C D E F	x 3 8 D M R
	1 1 0 1 1 1	x G H I J K	x 4 9 I N S
	1 0 1 1 1 1	x L M N O P	x 5 E J O T
	0 1 1 1 1 1	x Q R S T U	x A F K P U
			1 2 3 4 5 *
			6 7 8 9 A *
			B C D E F *
			G H I J K *
			L M N O P *
			Q R S T U *
			* * * * * *

Table 2: The swap matrices and templates for Steiner (6,2) trades

possible and there are no viable templates for $r = 3$. This completes the proof. □

Theorem A.6 *If T is a Steiner $(6,2)$ trade, then $m(T) \neq 13$.*

Proof Let n_2, n_5 and n_6 denote the number of elements of $F(T)$ that have multiplicity 2, 5 and 6 respectively. Then $2n_2 + 5n_5 + 6n_6 = 78$. Now it is easy to see that $n_5 \leq 3$ and $n_6 \leq 2$, else the Steiner property is violated. Further, the feasible template for $r = 2$ shows that, if $n_2 \geq 1$, then $n_5 + n_6 \geq 2$. Considering all these equations and inequalities, and noting that n_5 must be even, we see that the only integral solutions (n_2, n_5, n_6) are (28,2,2), (31,2,1), (33,0,2) and (34,2,0). Note that $f(T) = n_2 + n_5 + n_6$, and that the templates use 11, 26 and 31 distinct elements respectively. The last three, three and one blocks of these templates respectively are said to be **wholly undetermined.**

Case (28,2,2): Clearly this contradicts the Steiner property.

Case (31,2,1): Three distinct new elements must be added to the multiplicity six template to reach the final foundation size of 34. At most two of the existing elements could occur again (as multiplicity five elements). There are thus at most five distinct elements unplaced to fill the six places of the wholly undetermined block in the template.

Case (33,0,2): Four distinct new elements must be added to the multiplicity six template to reach the final foundation size of 35. At most one of the existing elements could occur again (as a multiplicity six element). There are thus at most five distinct elements unplaced to fill the six places of the wholly undetermined block in the template.

Case (34,2,0): The template for multiplicity two shows that the two elements that occur five times each do not occur together. Thus, in the template for multiplicity five the element other than x of multiplicity five, say y, is distinct from $1, \ldots, P$. Now at least two of the five occurrences of y must be in the partially filled sets of T_2. By the symmetry of the swap matrix, it is easy to see that the second occurrences of the five elements distinct from y of such a set are in separate sets in T_1. Thus, to balance pairs, all five occurrences of y in T_1 and in T_2 must be with the partial sets in the templates for T_1 and T_2. But now the ten sets containing x or y in T_1 and in T_2 form a Steiner $(6,2)$-trade of volume ten. Thus the three remaining sets in T_1 and in T_2 must form a Steiner $(6,2)$-trade of volume $3 < 2(k-1) = 10$, which is impossible by Theorem 3.10. □

A.4 Steiner $(5,2)$ trades of volume 10

Throughout this section of the appendix, suppose that $T = T_1 - T_2$ is a Steiner $(5,2)$ trade of volume ten. We will prove that T has a uniquely determined structure as in Lemma 3.12(3). By Lemma 3.6, $2 \leq r_x \leq 5$ for all $x \in F(T)$. For $2 \leq i \leq 5$, let n_i denote the number of elements in $F(T)$ that have multiplicity i. Then $2n_2 + 3n_3 + 4n_4 + 5n_5 = 50$. By Lemma A.1, $n_3 = 0$, and it is easy to see that $n_5 \geq 3$ is not possible, since T is Steiner. So, since n_5 must be even, we need only consider the cases $n_5 = 0$ or 2.

Lemma A.7 *If $n_5 = 2$, then $T_1 = xA^1 + yA^2$ and $T_2 = xA^2 + yA^1$, where A^1 and A^2 are solely 1-balanced and $x, y \notin F(A^1)$.*

Proof This follows immediately from Lemma 3.9. □

Lemma A.8 *If $n_5 = 0$, then T does not exist.*

Proof Since $n_3 = n_5 = 0$, then $n_2 + 2n_4 = 25$. It is straightforward to check that $n_4 \leq 5$, and that $n_4 = 4$ and $n_4 = 5$ both require that any two multiplicity four elements occur together in a block. Since $n_2 > 0$ in all the remaining cases, Lemma A.3 implies that $n_4 \geq 2$ and that the $n_4 = 4$ or 5 cases are not possible. So $n_4 = 2$ or 3. We now use the unique multiplicity two template for T_2 given in the proof of Lemma A.3, with foundation size nine and elements 4 and 5 of multiplicity four. Since $m(T) = 10$, there are two blocks, say U and V, which do not yet have any determined elements.

Case $n_2 = 21$, $n_4 = 2$: Here $f(T) = 23$, so fourteen new elements are to be added to the multiplicity two template. By Lemma A.3, each of these multiplicity two elements has to occur with both of the multiplicity four elements, which is impossible.

Case $n_2 = 19$, $n_4 = 3$: Here $f(T) = 22$, so thirteen new elements are required. If each of these has multiplicity two, then the argument of the previous case applies. So one of these elements, say α, has multiplicity four. Now α must occur in both U and V, in a block with 4, and a block with 5. Suppose β is a new multiplicity two element that occurs in the block U. One of the multiplicity four elements that β occurs with must be α. But α occurs with both of the other multiplicity four elements, contradicting Lemma A.3. □

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