The spectrum problem for λ-fold Petersen graph designs

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ABSTRACT: Necessary and sufficient conditions for the existence of a decomposition of λK_v into edge-disjoint copies of the Petersen graph are proved.

1 Introduction

A decomposition of a graph H into edge-disjoint copies of a given graph G is called a G-design of H. A G-design of λK_v (the multigraph with v vertices and λ edges between every pair of distinct vertices) is called a G-design of order v and index λ. The problem of determining all values of v for which there is a G-design of order v and index λ is called the spectrum problem for λ-fold G-designs.

The spectrum problem for G-designs has been considered for many graphs G. For example, if G is a complete graph on k vertices, then a G-design of order v and index λ is a (v, k, λ)-BIBD. A great deal of work has been done on the spectrum problem for G-designs in the case where G is an m-cycle (see [12]). Other graphs G, for which the spectrum problem has been considered include all graphs on 5 vertices or less (see [4] and [5]), cubes (see [3] and [6]) and Platonic graphs (see [2]). For a survey of results see [10].

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Figure 1). We denote the Petersen graph P with vertices and edges as shown in Figure 1 by $(v_1, v_2, v_3, \ldots, v_{10})$.

It is straightforward to check that the following conditions are necessary for the existence of a P-design of λK_v:

(1) $v \geq 10$;

(2) if $\lambda \equiv 1, 2, 4, 7, 8, 11, 13, 14 \pmod{15}$, then $v \equiv 1, 10 \pmod{15}$;

(3) if $\lambda \equiv 3, 6, 9, 12 \pmod{15}$, then $v \equiv 0, 1 \pmod{5}$; and

(4) if $\lambda \equiv 5, 10 \pmod{15}$, then $v \equiv 1 \pmod{3}$.

Adams and Bryant in [1] prove the following:

Theorem 1 *For all $v \equiv 1$ or 10 (mod 15), $v \neq 10$, there exists a P-design of K_v.*

We make use of *group divisible designs* and *pairwise balanced designs*. A group divisible design, $(K, \lambda, M; v)$GDD, is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, where the v-set is partitioned into disjoint subsets (called groups) of size $m \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in λ blocks. If $M = \{m\}$ and $K = \{k\}$ we write $(k, \lambda, m; v)$GDD. A pairwise balanced design, (v, K, λ)PBD, is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, such that every pair of distinct elements of the v-set is contained in exactly λ blocks. We will use the notation k^* or m^* to specify that there is precisely one block of size k or precisely one group of size m in a GDD or PBD.

We will need some notation. We denote by $K_{r(s)}$ the complete multipartite graph with r parts each of size s. The complete graph of order v with a hole of size u (that is, the graph with vertex set V and edge set $\{ab : a, b \in V \setminus U, a \neq b\} \cup \{ab : a \in V \setminus U \text{ and } b \in U\}$ where $|V| = v$, $|U| = u$ and $U \subseteq V$) is denoted by $K_v \setminus K_u$. The vertices in U are said to be *the vertices in the hole.* For a given graph G we denote by λG the multigraph with vertex set $V(G)$ and λ edges between the vertices x and y for every edge xy in G.

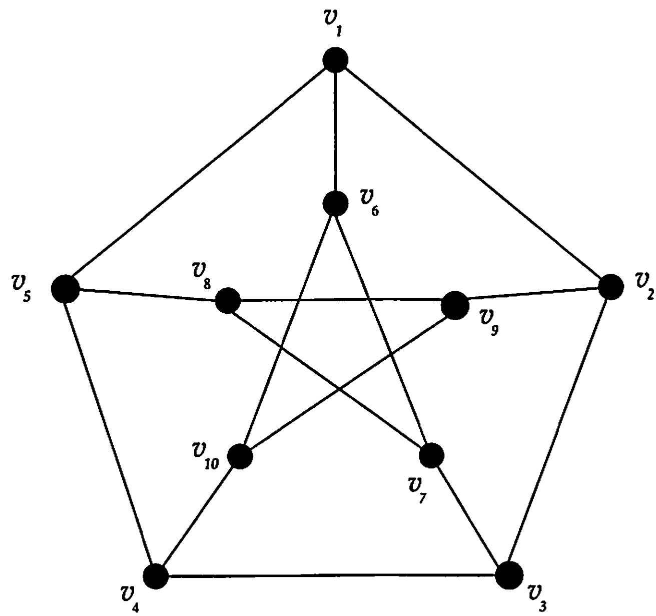

Figure 1: The Petersen graph

2 Constructions

We will make use of the following two well-known lemmas. Lemma 2 is essentially a variant of Wilson's fundamental construction for GDD's [13].

Lemma 2 *Suppose there exists a* $(K, 1, M; v)$*GDD and let* $m^* \in M$*. If there exists*

(1) a P-design of $\lambda K_{r(s)}$ *for each* $r \in K$*;*

(2) a P-design of $\lambda(K_{sm+h} \setminus K_h)$ *for each* $m \in M \setminus \{m^*\}$*; and*

(3) a P-design of λK_{sm^*+h}*,*

then there exists a P-design of λK_{sv+h}*.*

Lemma 3 *If there exists a P-design of* $\lambda_1 H$ *and a P-design of* $\lambda_2 H$*, then there exists a P-design of* $(m\lambda_1 + n\lambda_2)H$ *for any non-negative integers* m *and* n*.*

2.1 The case $\lambda \in \{2, 4, 7, 8, 11, 13, 14\}$

From necessary conditions we have $v \equiv 1$ or $10 \pmod{15}$.

Theorem 4 *Let $\lambda \in \{2,4,7,8,11,13,14\}$. Then there exists a P-design of λK_v for all $v \equiv 1$ or 10 (mod 15).*

Proof: Applying Lemma 3 and Theorem 1 ($m = \lambda, \lambda_1 = 1, \lambda_2 = 0$) leaves us with the case $v = 10$. For a P-design of λK_{10}, by Lemma 3, we only need a P-design of $2K_{10}$ and of $3K_{10}$. See the Appendix for these two P-designs.

2.2 The case $\lambda \in \{3,6,9,12\}$

From necessary conditions we have $v \equiv 0$ or 1 (mod 5). First we consider the case $\lambda = 3$. The other cases then follow by Lemma 3.

Lemma 5 *There exists a P-design of $3K_v$ for all $v \equiv 0$ or 1 (mod 5), $v \geq 10$.*

Proof: For $v \equiv 1$ or 10 (mod 15), $v \neq 10$, we apply Lemma 3 and Theorem 1. A P-design of $3K_{10}$ is given in the Appendix. For the other values of v we apply Lemma 2 with GDDs and P-designs as shown in Table 1. The P-designs used in this table, except a P-design of $3K_{4(5)}$ and of $3K_{16}$, and the other cases are given in the Appendix. For a P-design of $3K_{4(5)}$ and of $3K_{16}$ we apply Lemma 3 with a P-design of $K_{4(5)}$ and of K_{16} which exist (see [1]).

v	GDDs used	P-designs of	other cases
$30x$	$(3,1,2;6x)$GDD $x \geq 1$ (see [8])	$3K_{3(5)}$, $3K_{10}$	none
$30x+15$	$(3,1,3;6x+3)$GDD $x \geq 1$ (see [8])	$3K_{3(5)}$, $3K_{15}$	none
$15x+5$	$(\{3,4\},1,\{3,4\};3x+1)$GDD $x \geq 4$ (see [11])	$3K_{3(5)}$, $3K_{4(5)}$, $3K_{15}$, $3K_{20}$	$3K_{35}$, $3K_{50}$
$15x+6$	$(\{3,4\},1,\{3,4\};3x+1)$GDD $x \geq 4$	$3K_{3(5)}$, $3K_{4(5)}$, $3K_{16}$, $3K_{21}$	$3K_{36}$, $3K_{51}$
$30x+11$	$(3,1,2;6x+2)$GDD, $x \geq 1$	$3K_{3(5)}$, $3K_{11}$	none
$30x+26$	$(3,1,\{3,5^*\};6x+5)$GDD $x \geq 2$ (see [8])	$3K_{3(5)}$, $3K_{16}$, $3K_{26}$	$3K_{56}$

Table 1

Now applying Lemma 3 leads to the following result.

Theorem 6 *Let $\lambda \in \{3,6,9,12\}$. Then there exists a P-design of λK_v for all $v \equiv 0$ or 1 (mod 5), $v \geq 10$.*

2.3 The case $\lambda \in \{5, 10\}$

From the necessary conditions if $\lambda = 5$ or 10, then $v \equiv 1 \pmod 3$. First we consider the case $\lambda = 5$. The case $\lambda = 10$ then follows by Lemma 3.

Lemma 7 *There exists a* $(\{4,5\}, 1, \{m, n^*\}; 4m+n)$*GDD for* $m \neq 2, 3, 6, 10$ *and* $0 \leq n \leq m$.

Proof: Take a $(5, 1, m; 5m)$GDD which exists for $m \neq 2, 3, 6, 10$ (see [7]). Then remove $m - n$ elements from the last group. Also remove these elements from the blocks. The result is a $(\{4,5\}, 1, \{m, n^*\}; 4m+n)$GDD.

Lemma 8 *Let* $v \equiv 1 \pmod 3$, $10 \leq v \leq 139$. *Then there exists a P-design of* $5K_v$.

Proof: For $v = 10$ we take the union of a P-design of $2K_{10}$ and a P-design of $3K_{10}$. For $v \in \{16, 25, 31, 40, 46, 55, 91\}$ we take five copies of a P-design of K_v. For $v \in \{64, 67, 79, 82, 139\}$ we apply Lemma 2 with the ingredients in Table 2. The P-designs used in this table, except a P-design of $5K_{16}$, are in the Appendix. For a P-design of $5K_{16}$ we apply Lemma 3 with a P-design of K_{16} which exists by Theorem 1. To obtain the $(\{4,5\}, 1, \{5, 6^*\}; 46)$GDD used in Table 2, take a $(53, \{5, 13^*\}, 1)$PBD which exists by [9]. Then delete seven elements of the block of size 13 and remove these elements from the other blocks. For $v = 88$ use Lemma 2 (with a (4,1,1;4) GDD) together with a P-design of $5K_{4(22)}$ and a P-design of $5K_{22}$ (see the Appendix). For $v \in \{13, 19, 22, 28, 34, 37, 43, 49, 52\}$ see the Appendix. For the remaining values of v we apply Lemma 2 with a $(\{4,5\}, 1, \{m, n^*\}; 4m+n)$GDD, where $4m+n \geq 16$, $0 \leq n \leq m$ and $n \neq 1$ or 2 and P-designs of $5K_{4(3)}$, $5K_{5(3)}$, $5K_{3m+1}$ and $5K_{3n+1}$.

v	GDDs used	P-designs of
64	$(4,1,4;16)$GDD	$5K_{4(4)}$ and $5K_{16}$
67	$(5,1,4;20)$GDD	$5K_{5(3)}$, $5(K_{19} \setminus K_7)$ and $5K_{19}$
79	$(5,1,5;25)$GDD	$5K_{5(3)}$, $5(K_{19} \setminus K_4)$ and $5K_{19}$
82	$(5,1,5;25)$GDD	$5K_{5(3)}$, $5(K_{22} \setminus K_7)$ and $5K_{22}$
139	$(\{4,5\},1,\{4,6^*\};46)$GDD	$5K_{4(3)}$, $5K_{5(3)}$, $5K_{13}$ and $5K_{19}$

Table 2

Lemma 9 *If there exists a P-design of* $5K_v$ *for all* $v \equiv 1 \pmod 3$, $10 \leq v \leq 139$ *then there exists a P-design of* $5K_v$ *for all* $v \equiv 1 \pmod 3$, $v \geq 10$.

Proof: Let $v = 3u + 1$ and $u \geq 47$. Then there are m and n such that $m \geq 11$, $0 \leq n \leq m$, $n \neq 1$ or 2 and $u = 4m+n$. Now apply Lemma 2 with a $(\{4,5\},1,\{m,n^*\};u)$ GDD, $0 \leq n \leq m$ and $n \neq 1$ or 2, and P-designs of $5K_{4(3)}$, $5K_{5(3)}$, $5K_{3m+1}$ and $5K_{3n+1}$.

Now applying Lemma 3 leads to the following result.

Theorem 10 *Let $\lambda \in \{5,10\}$. Then there exists a P-design of λK_v for all $v \equiv 1$ (mod 3), $v \geq 10$.*

2.4 The case $\lambda = 15$

Theorem 11 *There exists a P-design of $15K_v$ for all $v \geq 10$.*

Proof: Using Lemma 3 and Theorems 6 and 10 one can find a P-design of $15K_v$ for $v \equiv 0,1,4,5,6,7,10,11,13$ (mod 15), $v \geq 10$. For the other values of v, namely $v \equiv 2,3,8,9,12,14$ (mod 15), we apply Lemma 2 with GDDs and P-designs as shown in Table 3. The P-designs used in this table, except a P-design of $15K_{4(5)}$, and the other cases are either in the Appendix or can be obtained by Lemma 3 and P-designs with smaller λ given in the Appendix. For a P-design of $15K_{4(5)}$ we apply Lemma 3 with a P-design of $K_{4(5)}$ which exists (see [1]).

3 The Main Result

Now we are ready to prove the main result of this paper.

Main theorem There exists a P-design of λK_v if and only if $v \geq 10$ and

(1) $\lambda \equiv 1,2,4,7,8,11,13,14$ (mod 15) and $v \equiv 1,10$ (mod 15), $(\lambda, v) \neq (1,10)$; or

(2) $\lambda \equiv 3,6,9,12$ (mod 15) and $v \equiv 0,1,5,6,10,11$ (mod 15); or

(3) $\lambda \equiv 5$ or 10 (mod 15) and $v \equiv 1,4,7,10,13$ (mod 15); or

(4) $\lambda \equiv 0$ (mod 15).

Proof: It is straightforward to check that the above conditions are necessary. We prove that these conditions are also sufficient. Let $\lambda = 15x + \lambda_0$ with $1 \leq \lambda_0 \leq 15$. If $\lambda_0 \neq 1$ or $\lambda_0 = 1$ and $v \neq 10$, then there exists a P-design of $\lambda_0 K_v$ by Theorems 4, 6, 10 and 11. Hence we apply Lemma 3 with a P-design of $\lambda_0 K_v$ and a P-design of $15K_v$. If $\lambda_0 = 1$ and $v = 10$, then we apply Lemma 3 with a P-design of $15K_{10}$ and P-designs of $8K_{10}$.

v	GDDs used	P-designs of	other cases
$30x+2$	$(3,1,2;6x)$GDD $x \geq 1$	$15K_{3(5)}$, $15K_{12}$, $15(K_{12} \setminus K_2)$	none
$30x+17$	$(3,1,3;6x+3)$GDD $x \geq 1$	$15K_{3(5)}$, $15K_{17}$, $15(K_{17} \setminus K_2)$	none
$30x+3$	$(3,1,2;6x)$GDD $x \geq 1$	$15K_{3(5)}$, $15K_{13}$, $15(K_{13} \setminus K_3)$	none
$30x+18$	$(3,1,3;6x+3)$GDD $x \geq 1$	$15K_{3(5)}$, $15K_{18}$, $15(K_{18} \setminus K_3)$	none
$15x+8$	$(\{3,4\},1,\{3,4\};$ $3x+1)$GDD, $x \geq 4$	$15K_{3(5)}$, $15K_{4(5)}$, $15K_{23}$, $15(K_{18} \setminus K_3)$, $15(K_{23} \setminus K_3)$	$15K_{38}$, $15K_{53}$
$15x+9$	$(\{3,4\},1,\{3,4\};$ $3x+1)$GDD, $x \geq 4$	$15K_{3(5)}$, $15K_{4(5)}$, $15K_{24}$, $15(K_{19} \setminus K_4)$, $15(K_{24} \setminus K_4)$	$15K_{39}$, $15K_{54}$
$30x+12$	$(3,1,2;6x+2)$GDD $x \geq 1$	$15K_{3(5)}$, $15K_{12}$, $15(K_{12} \setminus K_2)$	none
$30x+27$	$(3,1,\{3,5^*\};6x+5)$GDD $x \geq 2$	$15K_{3(5)}$, $15K_{27}$, $15(K_{17} \setminus K_2)$	$15K_{57}$
$30x+14$	$(3,1,2;6x+2)$GDD $x \geq 1$	$15K_{3(5)}$, $15K_{14}$, $15(K_{14} \setminus K_4)$	none
$30x+29$	$(3,1,\{3,5^*\};6x+5)$GDD $x \geq 2$	$15K_{3(5)}$, $15K_{29}$, $15(K_{19} \setminus K_4)$	$15K_{59}$

Table 3

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4 Appendix

P-designs with $\lambda = 2$

$\boxed{2K_{10}}$ $V = \{i_j \mid 0 \le i \le 2;\ j = 1,2,3\} \cup \{\infty\}$.

P-design follows, cycled modulo 3_-:

$(0_1, 1_1, 2_1, 0_2, 1_2, 0_3, 1_3, 2_2, 2_3, \infty)$, $(0_1, 0_2, 1_3, 2_1, 0_3, 2_2, 2_3, 1_2, 1_1, \infty)$.

P-designs with $\lambda = 3$

$\boxed{3K_{10}}$ $V = \mathbb{Z}_9 \cup \{\infty\}$. P-design follows, cycled modulo 9:

$(0, 1, 2, 3, 6, 5, 7, 4, 8, \infty)$.

$\boxed{3K_{11}}$ $V = \mathbb{Z}_{11}$. P-design follows, cycled modulo 11:

$(0, 1, 2, 3, 5, 6, 9, 7, 4, 10)$.

$\boxed{3K_{15}}$ $V = \{i_j \mid 0 \le i \le 6;\ j = 1,2\} \cup \{\infty\}$.

P-design follows, cycled modulo 7_-:

$(0_1, 1_1, 2_1, 3_1, 5_1, 4_1, 6_1, 0_2, 1_2, 2_2)$, $(0_1, 3_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 5_2, \infty)$,
$(0_1, 3_2, 0_2, 3_1, 6_2, 5_2, 4_2, 2_2, 6_1, \infty)$.

$\boxed{3K_{20}}$ $V = \mathbb{Z}_{19} \cup \{\infty\}$. P-design follows, cycled modulo 19:

$(0, 1, 2, 3, 5, 4, 6, 8, 11, 14)$, $(0, 4, 9, 1, 7, 8, 15, 2, 11, \infty)$.

$\boxed{3K_{21}}$ $V = \mathbb{Z}_{21}$. P-design follows, cycled modulo 21:

$(0, 1, 2, 3, 5, 4, 6, 8, 11, 14)$, $(0, 4, 9, 2, 14, 8, 15, 6, 12, 17)$.

$\boxed{3K_{26}}$ $V = \{i_j \mid 0 \le i \le 12;\ j = 1,2\}$. P-design follows, cycled modulo 13_-:

$(7_2, 9_2, 0_2, 5_2, 2_1, 4_2, 5_1, 8_2, 1_2, 10_1)$, $(1_1, 10_1, 2_2, 7_1, 12_1, 3_2, 0_2, 9_1, 10_2, 8_2)$,
$(8_1, 9_1, 6_1, 10_1, 12_2, 9_2, 6_2, 5_2, 11_1, 3_2)$, $(0_1, 1_1, 2_1, 4_1, 7_1, 5_1, 9_1, 3_2, 6_1, 2_2)$,
$(0_1, 6_1, 2_2, 4_1, 3_2, 10_2, 11_2, 5_1, 5_2, 6_2)$.

$\boxed{3K_{35}}$ $V = \{i_j \mid 0 \le i \le 16;\ j = 1,2\} \cup \{\infty\}$. P-design follows, cycled mod 17_-:

$(0_1, 1_1, 2_1, 3_1, 5_1, 4_1, 6_1, 8_1, 11_1, 14_1)$, $(0_1, 4_1, 9_1, 1_1, 6_1, 7_1, 15_1, 0_2, 12_1, 1_2)$,
$(0_1, 0_2, 1_1, 1_2, 2_2, 3_2, 4_2, 3_1, 5_2, 2_1)$, $(0_1, 1_2, 3_1, 0_2, 4_2, 5_2, 7_2, 9_1, 2_2, 7_1)$,

$(0_1,4_2,7_1,0_2,5_2,6_2,1_2,9_1,15_2,2_2)$, $(0_1,7_2,10_1,0_2,8_2,9_2,1_2,10_2,4_2,\infty)$,
$(0_1,7_2,11_1,2_2,9_2,13_2,6_2,3_2,0_2,\infty)$.

$3K_{36}$ $V = \{i_j \mid 0 \le i \le 8;\ j = 1,2,3,4\}$. *P*-design follows, cycled modulo 9_-:

$(3_1,2_2,6_4,7_4,4_4,0_2,7_3,8_4,5_2,1_4)$, $(8_1,1_2,0_2,3_3,5_2,5_3,6_4,8_4,7_4,1_1)$,
$(8_1,6_4,0_2,5_3,7_1,4_3,2_3,1_2,2_4,5_1)$, $(1_3,7_3,5_4,2_2,3_3,5_2,3_2,5_1,8_2,6_1)$,
$(4_2,8_4,0_3,8_1,5_2,7_3,4_1,0_4,3_4,6_1)$, $(2_4,2_1,1_1,2_3,7_1,8_3,6_3,3_4,1_2,6_2)$,
$(5_2,1_2,8_2,6_2,2_2,3_1,8_1,3_4,4_2,3_3)$, $(3_1,2_3,5_1,5_4,5_3,4_4,2_1,7_2,6_3,6_1)$,
$(8_1,0_4,2_3,4_1,3_1,1_3,7_1,6_4,6_3,4_4)$, $(1_4,6_1,1_1,7_1,6_3,3_4,4_1,2_2,8_1,3_2)$,
$(5_2,2_3,8_1,6_2,4_4,5_3,2_4,7_3,6_3,6_4)$, $(0_1,2_1,2_2,1_1,1_2,4_2,4_3,0_2,5_4,3_4)$,
$(0_1,1_2,2_3,2_2,2_4,4_2,8_3,3_4,7_3,0_4)$, $(0_1,6_3,2_3,3_3,5_4,7_4,8_3,0_3,2_4,0_2)$.

$3K_{50}$ $V = \mathbb{Z}_{49} \cup \{\infty\}$. *P*-design follows, cycled modulo 49:

$(3,37,7,29,9,27,46,47,6,48)$, $(0,12,22,42,33,6,11,9,14,26)$,
$(1,33,28,36,23,17,35,19,1,11)$, $(0,1,2,5,9,3,14,26,4,20)$,
$(0,6,12,3,28,23,38,15,42,\infty)$.

$3K_{51}$ $V = \mathbb{Z}_{51}$. *P*-design follows, cycled modulo 51:

$(2,50,23,32,1,28,30,7,40,24)$, $(7,50,14,20,45,4,10,22,19,15)$,
$(3,4,37,5,20,21,45,50,9,29)$, $(0,1,2,4,11,7,16,25,12,26)$,
$(0,10,22,5,34,12,47,13,48,27)$.

$3K_{56}$ $V = \{i_j \mid 0 \le i \le 6;\ 1 \le j \le 8\}$. *P*-design follows, cycled modulo 7_-:

$(1_4,5_4,5_3,4_4,3_4,6_1,6_2,6_8,5_6,4_6)$, $(4_1,1_4,5_2,3_7,2_7,1_5,1_1,4_7,4_8,3_8)$,
$(2_6,2_8,5_3,6_3,5_1,3_2,3_7,0_5,2_3,4_3)$, $(6_5,5_5,3_6,5_8,4_1,4_3,1_2,5_7,1_5,0_5)$,
$(5_2,2_2,2_3,6_1,3_8,1_1,3_6,1_3,6_3,0_7)$, $(3_4,6_1,4_6,2_4,2_2,4_4,1_3,4_7,0_3,5_4)$,
$(3_7,0_3,5_8,2_2,4_4,3_4,4_1,5_2,6_3,0_8)$, $(6_5,3_4,6_4,4_5,1_3,5_8,5_6,4_6,5_1,0_4)$,
$(3_8,2_4,6_3,5_5,6_7,4_7,2_6,1_2,0_2,0_6)$, $(4_5,2_5,4_6,1_7,1_8,0_7,6_4,5_3,1_1,4_8)$,
$(5_2,3_4,6_3,2_7,3_6,3_2,5_3,6_2,0_3,5_6)$, $(1_6,5_5,5_3,6_8,2_1,6_3,5_4,1_8,4_2,0_6)$,
$(6_6,2_6,4_8,0_7,5_4,1_1,3_7,2_7,3_6,5_8)$, $(1_8,3_1,2_6,3_5,4_3,4_4,5_7,5_4,2_8,1_2)$,
$(1_6,0_2,4_1,6_1,3_4,1_1,4_7,2_6,0_6,5_7)$, $(3_6,0_2,4_5,4_2,0_8,3_1,2_3,2_4,2_1,0_7)$,
$(2_7,0_6,4_1,1_5,6_4,1_1,3_4,3_2,0_2,5_7)$, $(0_7,1_3,2_4,5_2,2_7,6_5,4_6,0_8,4_2,4_5)$,
$(3_1,3_8,4_2,1_4,1_8,6_6,1_5,2_8,4_6,2_7)$, $(0_3,5_2,2_1,2_2,2_4,4_6,3_8,5_6,3_2,3_5)$,
$(6_2,3_3,4_8,2_5,1_8,0_4,4_5,3_6,6_8,5_2)$, $(4_7,3_3,1_4,6_8,3_6,6_1,2_8,4_3,3_1,1_8)$,
$(4_2,5_6,2_2,6_1,4_1,3_5,5_2,4_4,6_7,0_6)$, $(4_5,0_4,6_8,2_6,3_8,5_5,6_4,1_4,1_8,2_7)$,
$(4_8,4_1,1_6,3_4,3_6,5_1,6_4,1_1,4_5,2_2)$, $(0_2,5_2,4_2,2_4,4_7,5_1,2_6,4_6,0_3,2_1)$,
$(6_3,5_7,3_7,0_5,4_8,5_8,1_2,4_3,2_4,3_5)$, $(5_7,2_2,3_7,3_3,3_5,1_4,3_1,2_7,3_2,2_6)$,
$(3_7,2_3,4_1,0_4,0_7,6_8,6_6,1_2,2_5,6_1)$, $(4_2,2_4,3_2,2_8,2_5,1_6,1_3,3_6,1_5,4_5)$,
$(3_5,6_1,2_5,3_8,5_7,1_8,4_2,3_1,6_7,0_5)$, $(4_2,2_1,3_1,5_5,6_1,4_3,4_1,0_6,0_5,1_6)$,
$(5_4,1_6,4_2,0_5,0_4,4_7,2_1,4_1,2_2,5_5)$, $(6_7,0_3,5_1,6_2,1_8,2_8,2_3,4_8,2_1,6_3)$,
$(2_8,1_2,3_5,3_8,5_8,5_6,5_3,3_7,6_3,1_4)$, $(6_6,1_3,3_7,1_2,5_4,2_5,6_3,1_5,1_7,2_7)$,
$(6_2,5_3,2_3,4_2,4_8,5_7,2_7,1_5,3_1,6_8)$, $(0_6,5_6,0_3,6_8,4_4,4_6,4_1,4_5,6_7,0_4)$,

$(0_1, 1_1, 4_1, 3_2, 6_3, 3_1, 3_3, 5_4, 2_6, 2_5)$, $(0_1, 6_2, 2_5, 1_3, 5_5, 1_5, 0_4, 3_7, 6_7, 4_3)$,
$(0_1, 6_2, 0_7, 2_3, 6_4, 3_3, 2_6, 6_5, 6_7, 3_6)$, $(0_1, 2_3, 6_3, 3_4, 1_5, 2_8, 3_8, 4_3, 3_5, 1_7)$,
$(0_1, 1_4, 3_5, 0_4, 6_5, 3_4, 2_6, 6_6, 6_7, 1_5)$, $(0_2, 3_7, 1_3, 2_5, 6_8, 0_8, 3_8, 4_7, 6_4, 5_8)$.

$3(K_{23} \setminus K_3)$ $V = \{i_j \mid 0 \leq i \leq 4;\ j = 1,2,3,4\} \cup \{A, B, C\}$; hole on $\{A, B, C\}$.

P-design follows, cycled modulo 5_-:

$(2_1, 1_2, 1_3, 0_3, 1_4, 2_2, 4_1, A, 3_3, B)$, $(2_3, 3_1, 3_2, 2_2, 4_1, 0_4, 1_4, A, 2_1, B)$,
$(0_1, 1_3, 4_3, 4_1, 3_3, 0_2, 2_3, A, 4_2, C)$, $(0_1, 3_4, 4_3, 3_1, 1_4, 1_1, 0_2, A, 1_2, C)$,
$(0_3, 0_2, 4_1, 4_4, 2_2, 0_4, 3_4, B, 2_4, C)$, $(0_2, 0_3, 0_1, 1_3, 2_1, 4_3, 4_1, B, 2_3, C)$,
$(0_1, 3_1, 1_1, 4_1, 1_2, 2_2, 3_2, 2_1, 0_3, 1_3)$, $(0_2, 1_2, 4_2, 0_3, 0_4, 3_3, 2_4, 0_1, 1_4, 2_2)$,
$(0_2, 3_2, 4_3, 0_4, 3_4, 1_4, 2_4, 0_1, 4_4, 4_1)$, $(0_2, 2_2, 1_4, 4_3, 0_4, 4_4, 2_3, 3_3, 3_4, 2_4)$.

$3(K_{24} \setminus K_4)$ $V = \{i_j \mid 0 \leq i \leq 4;\ j = 1,2,3,4\} \cup \{A, B, C, D\}$;

hole on $\{A, B, C, D\}$. P-design follows, cycled modulo 5_-:

$(1_4, 0_4, 4_2, 2_2, 4_1, 0_1, 1_2, A, 3_2, B)$, $(3_1, 2_2, 3_2, 4_4, 4_1, 2_4, 3_4, A, 1_2, B)$,
$(4_1, 3_4, 3_1, 1_1, 3_3, 4_3, 0_4, A, 1_4, B)$, $(0_4, 1_2, 2_3, 3_3, 1_3, 0_3, 4_3, A, 0_1, B)$,
$(1_1, 4_4, 1_4, 3_1, 2_2, 2_4, 4_1, C, 1_3, D)$, $(0_2, 0_3, 0_4, 1_3, 2_4, 0_1, 2_3, C, 1_4, D)$,
$(0_3, 0_1, 1_4, 0_2, 2_4, 1_2, 2_1, C, 1_1, D)$, $(0_1, 2_1, 0_3, 2_4, 3_3, 1_3, 3_2, C, 0_2, D)$,
$(0_1, 0_2, 1_3, 2_1, 3_3, 0_3, 4_1, 1_2, 4_4, 4_2)$, $(0_1, 3_2, 0_3, 1_2, 4_3, 2_2, 2_4, 4_2, 2_3, 0_4)$,

together with the following four uncycled Petersen graphs:

$(0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 2_2, 4_2, 1_2, 3_2)$, $(0_2, 1_2, 2_2, 3_2, 4_2, 0_3, 2_3, 4_3, 1_3, 3_3)$,
$(0_3, 1_3, 2_3, 3_3, 4_3, 0_4, 2_4, 4_4, 1_4, 3_4)$, $(0_4, 1_4, 2_4, 3_4, 4_4, 0_1, 2_1, 4_1, 1_1, 3_1)$.

$3K_{5,5,5}$ $V = \{i_j \mid 0 \leq i \leq 4;\ j = 1,2,3\}$. P-design follows, cycled modulo 5_-:

$(0_1, 0_2, 1_1, 1_2, 0_3, 2_2, 1_3, 2_1, 2_3, 3_1)$, $(0_1, 0_2, 1_1, 2_2, 2_3, 1_2, 0_3, 4_2, 2_1, 3_3)$,
$(0_1, 1_2, 1_3, 0_2, 2_3, 3_3, 3_2, 1_1, 4_3, 2_1)$.

P-designs with $\lambda = 5$

$5K_{13}$ $V = \mathbb{Z}_{13}$. P-design follows, cycled modulo 13:

$(0, 1, 2, 3, 4, 5, 6, 8, 10, 7)$, $(0, 2, 4, 1, 6, 5, 10, 3, 8, 11)$.

$5K_{19}$ $V = \mathbb{Z}_{19}$. P-design follows, cycled modulo 19:

$(0, 1, 2, 3, 4, 5, 6, 8, 10, 7)$, $(0, 2, 4, 1, 5, 3, 9, 14, 7, 13)$,
$(0, 3, 9, 1, 11, 10, 17, 4, 15, 7)$.

$5K_{22}$ $V = \{i_j \mid 0 \leq i \leq 10;\ j = 1,2\}$. P-design follows, cycled modulo 11_-:

$(0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 8_1, 10_1, 7_1)$, $(0_1, 2_1, 5_1, 1_1, 6_1, 3_1, 8_1, 0_2, 7_1, 1_2)$,
$(0_1, 3_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 4_2, 5_2)$, $(0_1, 0_2, 1_1, 1_2, 2_2, 3_2, 4_2, 3_1, 5_2, 2_1)$,

$(0_1,2_2,3_1,0_2,3_2,4_2,1_2,5_1,7_2,5_2)$, $(0_1,4_2,6_1,0_2,7_2,6_2,1_2,3_2,7_1,2_2)$,
$(0_1,4_2,7_1,1_2,8_2,7_2,2_2,5_2,9_1,3_2)$.

$5K_{28}$ $V=\{i_j \mid 0\le i\le 6;\ j=1,2,3,4\}$. P-design follows, cycled modulo 7_-:

$(3_3,6_3,1_4,2_4,5_3,5_2,3_1,4_4,1_2,0_4)$, $(4_3,3_4,4_4,6_4,6_3,5_3,0_2,1_2,1_3,4_2)$,
$(1_1,2_4,6_3,0_4,6_1,4_4,4_2,6_4,2_2,0_3)$, $(6_3,3_4,4_4,0_4,2_4,0_3,4_3,0_2,6_1,1_3)$,
$(3_2,4_3,4_2,3_3,4_4,6_1,6_3,5_2,0_2,3_1)$, $(0_2,6_2,1_4,5_2,2_2,6_1,0_4,5_3,4_4,0_1)$,
$(5_1,2_4,3_2,6_3,2_2,0_4,4_1,2_3,3_4,4_2)$, $(3_2,4_2,3_3,1_4,0_3,6_4,0_2,1_1,5_4,2_1)$,
$(1_2,4_2,1_4,3_3,2_3,5_4,6_1,0_1,4_1,5_1)$, $(3_3,1_4,6_3,5_3,4_4,0_1,2_1,1_1,5_1,3_1)$,
$(5_1,5_4,5_3,1_4,3_4,2_3,4_1,5_2,0_3,4_4)$, $(2_1,4_3,1_4,3_2,1_2,0_2,4_4,6_2,2_4,2_2)$,
$(4_2,5_3,6_2,4_4,4_3,0_3,1_1,2_2,4_1,2_1)$, $(0_2,1_2,0_3,5_3,1_4,0_4,3_4,6_4,4_1,5_1)$,
$(0_1,2_1,5_1,0_2,3_2,1_2,4_2,1_1,0_3,3_1)$, $(0_1,0_2,1_1,1_2,1_4,0_3,4_3,3_1,6_3,5_1)$,
$(0_1,0_2,4_1,1_3,6_3,2_2,3_4,3_1,1_4,2_1)$, $(0_1,2_2,5_2,2_1,0_4,3_3,1_3,1_1,2_3,6_3)$.

$5K_{34}$ $V=\{i_j \mid 0\le i\le 16;\ j=1,2\}$. P-design follows, cycled modulo 17_-:

$(3_2,13_2,7_1,16_1,10_2,4_1,14_2,1_1,15_2,12_2)$, $(9_2,4_1,1_2,16_2,1_1,12_1,16_1,2_2,2_1,5_2)$,
$(8_2,13_1,15_2,7_2,9_2,0_2,15_1,14_2,5_1,13_2)$, $(0_2,11_1,7_1,5_1,6_2,15_1,9_2,10_2,3_1,2_2)$,
$(3_1,16_1,11_1,13_1,15_1,2_2,14_2,8_2,3_2,11_2)$, $(12_2,1_2,9_1,1_1,5_2,3_2,5_1,12_1,11_2,3_1)$,
$(10_2,3_2,2_1,9_1,14_2,4_1,13_2,0_2,3_1,16_2)$, $(0_1,1_1,2_1,3_1,4_1,5_1,8_1,7_1,10_1,15_1)$,
$(0_1,3_1,6_1,12_1,0_2,7_1,13_1,2_1,6_2,1_2)$, $(0_1,6_1,0_2,2_1,3_2,1_2,5_1,15_2,12_2,14_2)$,
$(0_1,4_2,16_1,6_2,7_2,8_2,11_2,15_1,3_2,5_2)$.

$5K_{37}$ $V=\mathbb{Z}_{37}$. P-design follows, cycled modulo 37:

$(34,11,8,31,4,27,24,9,16,17)$, $(16,14,25,7,6,22,9,17,32,20)$,
$(25,8,2,21,36,32,30,5,6,29)$, $(0,1,2,4,7,3,8,12,6,14)$,
$(0,4,8,12,16,9,17,1,13,25)$, $(0,8,17,31,20,19,34,10,27,7)$.

$5K_{43}$ $V=\mathbb{Z}_{43}$. P-design follows, cycled modulo 43:

$(21,40,10,4,7,39,22,36,1,6)$, $(30,8,38,12,25,32,29,41,36,24)$,
$(10,18,17,9,6,23,3,38,41,11)$, $(29,22,2,30,31,15,9,33,1,11)$,
$(33,37,10,20,38,17,27,42,18,23)$, $(0,1,2,3,9,6,13,20,10,21)$,
$(0,7,23,6,26,18,39,5,25,40)$.

$5K_{49}$ $V=\mathbb{Z}_{49}$. P-design follows, cycled modulo 49:

$(32,14,4,37,26,21,25,33,41,22)$, $(38,35,40,13,17,44,24,27,15,37)$,
$(17,28,21,20,1,10,46,30,6,7)$, $(48,32,13,22,3,47,38,9,18,42)$,
$(3,47,19,10,38,44,9,7,16,12)$, $(0,1,3,5,2,4,9,12,6,16)$,
$(0,7,15,1,12,8,29,46,20,38)$, $(0,12,29,3,26,31,14,48,35,18)$.

$5K_{52}$ $V=\{i_j \mid 0\le i\le 12;\ j=1,2,3,4\}$. P-design follows, cycled modulo 13_-:

$(3_4,9_4,12_4,7_4,8_4,10_2,11_2,5_3,1_4,4_3)$, $(11_3,4_4,9_4,5_4,10_4,6_4,8_1,8_2,7_1,4_2)$,
$(11_1,9_2,12_2,8_4,6_3,11_4,2_4,4_3,7_3,7_1)$, $(12_1,2_4,7_2,7_3,3_2,0_3,11_3,8_2,2_3,4_1)$,
$(7_3,12_4,11_3,3_4,6_4,9_2,11_1,7_2,1_1,5_4)$, $(10_3,4_4,2_4,5_4,8_4,11_4,7_2,5_1,5_2,0_2)$,
$(4_3,8_3,6_3,1_4,5_3,8_4,0_1,9_2,1_1,5_1)$, $(11_1,6_3,3_3,3_2,7_4,10_3,8_3,4_2,6_2,2_4)$,
$(10_2,8_4,0_3,7_4,7_3,11_1,12_4,10_4,3_2,8_2)$, $(0_3,11_4,5_4,4_4,8_3,0_1,3_1,6_4,6_2,7_3)$,

$(4_3, 0_4, 12_4, 5_3, 8_3, 4_2, 5_2, 0_3, 3_2, 11_3)$,
$(4_3, 3_4, 4_4, 2_4, 7_3, 9_3, 12_1, 9_4, 4_1, 1_1)$,
$(7_2, 1_3, 0_3, 3_4, 6_3, 11_1, 4_3, 4_1, 10_4, 12_2)$,
$(0_2, 0_4, 1_2, 10_3, 4_4, 11_4, 1_1, 8_1, 4_2, 7_1)$,
$(4_2, 12_3, 8_2, 4_4, 10_2, 2_3, 8_3, 2_2, 3_1, 9_1)$,
$(2_1, 7_4, 0_2, 10_4, 9_2, 3_2, 5_1, 12_2, 7_1, 4_2)$,
$(12_2, 4_3, 5_4, 3_3, 10_3, 9_3, 10_2, 12_3, 7_1, 5_1)$,
$(3_2, 4_4, 5_3, 6_3, 4_2, 5_4, 1_3, 0_3, 10_4, 10_1)$,
$(4_1, 8_4, 2_2, 5_2, 12_1, 8_2, 0_1, 9_1, 4_3, 1_2)$,
$(0_1, 1_1, 3_1, 5_1, 4_2, 6_1, 11_1, 0_2, 9_1, 2_3)$,
$(0_1, 3_2, 11_1, 0_3, 4_3, 5_2, 2_3, 10_1, 8_3, 7_2)$,
$(0_1, 5_3, 6_1, 11_3, 0_4, 2_4, 8_4, 2_1, 11_4, 4_1)$,
$(9_3, 1_4, 3_4, 11_3, 9_4, 4_3, 6_1, 8_3, 8_4, 1_1)$,
$(5_3, 0_4, 7_3, 1_4, 5_4, 12_3, 4_1, 4_3, 3_2, 0_1)$,
$(10_1, 12_2, 12_1, 6_2, 7_3, 3_3, 8_2, 10_4, 11_1, 11_2)$,
$(12_2, 7_3, 0_4, 12_4, 2_4, 10_3, 8_1, 9_2, 5_2, 7_2)$,
$(2_2, 8_2, 4_3, 7_3, 11_2, 4_1, 5_3, 12_2, 1_4, 3_1)$,
$(3_2, 8_4, 8_2, 11_4, 10_2, 4_1, 10_1, 9_3, 2_1, 3_1)$,
$(8_1, 8_4, 5_2, 7_2, 1_4, 2_4, 12_1, 2_1, 0_1, 11_3)$,
$(10_2, 12_3, 3_4, 6_3, 0_3, 7_1, 11_1, 8_2, 0_1, 3_1)$,
$(4_1, 5_3, 8_2, 1_3, 11_2, 11_1, 4_2, 5_1, 1_1, 0_1)$,
$(0_1, 5_1, 0_2, 6_1, 2_3, 2_2, 10_1, 1_2, 8_2, 7_3)$,
$(0_1, 5_2, 2_2, 12_2, 7_4, 10_4, 8_3, 3_1, 5_4, 10_2)$,
$(0_1, 2_4, 11_2, 0_3, 6_4, 12_4, 8_4, 12_1, 11_4, 1_2)$.

$5(K_{19} \setminus K_4)$ $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2,3\} \cup \{A, B, C, D\}$; hole on $\{A, \ldots, D\}$. P-design follows, cycled modulo 5_-:

$(2_1, 3_2, A, 2_3, 0_1, 4_2, 1_1, B, 3_3, C)$,
$(0_1, 3_3, A, 2_3, 2_1, 1_2, 3_2, B, 4_2, D)$,
$(1_1, 0_1, A, 3_1, 3_3, 4_1, 1_2, C, 0_2, D)$,
$(0_1, 1_1, 2_1, 0_3, 0_2, 1_3, 2_3, B, 3_3, D)$,
$(0_2, 1_2, 1_1, 0_3, 2_2, 1_3, 2_3, 3_3, 4_3, 0_1)$,
$(0_2, 2_2, 4_2, 2_3, 0_3, 4_3, 1_3, 0_1, 3_3, 1_2)$.
$(0_2, 1_1, A, 4_2, 3_1, 1_2, 2_3, B, 2_1, C)$,
$(0_2, 4_2, A, 3_1, 0_1, 0_3, 2_3, C, 1_1, D)$,
$(4_1, 4_2, B, 3_2, 1_1, 2_2, 2_3, C, 0_1, D)$,
$(0_1, 1_1, 0_2, 2_1, 1_2, 2_2, 4_1, 3_2, 0_3, 1_3)$,
$(0_2, 1_2, 2_2, 0_3, 2_3, 4_3, 1_3, 0_1, 3_3, 1_1)$,

$5(K_{19} \setminus K_7)$ $V = \mathbb{Z}_{19}$; hole on $\{0, 1, 2, 3, 4, 5, 6\}$.

Take five copies of the following ten non-cycled copies of P:

$(0, 7, 1, 8, 9, 10, 11, 2, 12, 3)$,
$(0, 13, 1, 9, 14, 15, 10, 5, 11, 3)$,
$(1, 15, 4, 7, 17, 18, 8, 5, 9, 10)$,
$(3, 7, 14, 6, 17, 16, 10, 8, 15, 11)$,
$(6, 8, 11, 9, 16, 10, 17, 18, 12, 13)$,
$(0, 8, 2, 7, 11, 12, 10, 4, 13, 5)$,
$(0, 16, 1, 12, 17, 18, 14, 2, 13, 6)$,
$(2, 15, 6, 7, 16, 18, 9, 4, 14, 13)$,
$(3, 13, 15, 5, 18, 14, 12, 4, 17, 16)$,
$(7, 8, 14, 11, 18, 9, 17, 15, 16, 12)$.

$5(K_{22} \setminus K_7)$ $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2,3\} \cup \{A, B, \ldots, G\}$; hole on $\{A, B, \ldots, G\}$. P-design follows, cycled modulo 5_-:

$(2_2, 1_3, A, 2_3, 4_2, 2_1, 4_3, B, 0_2, C)$,
$(2_3, 2_2, A, 3_2, 1_2, 4_3, 1_3, C, 3_3, E)$,
$(4_3, 3_1, A, 1_1, 3_2, 1_2, 4_1, D, 2_1, G)$,
$(0_1, 1_1, B, 4_1, 2_1, 3_1, 0_3, E, 0_2, F)$,
$(0_1, 0_2, C, 1_1, 1_2, 2_2, 3_1, F, 3_2, G)$,
$(0_1, 1_2, D, 0_3, 3_2, 1_3, 2_3, E, 0_2, G)$,
$(0_2, 1_2, 3_1, 1_3, 2_3, 4_3, 3_3, 2_1, 0_3, 0_1)$,
$(3_2, 4_2, A, 4_3, 1_2, 3_1, 0_2, B, 0_1, D)$,
$(4_3, 0_2, A, 0_1, 2_3, 1_3, 3_1, C, 1_1, F)$,
$(4_1, 1_1, B, 2_1, 3_1, 0_1, 1_2, D, 3_2, E)$,
$(0_1, 0_3, B, 1_3, 2_1, 0_2, 2_3, E, 1_1, G)$,
$(0_1, 1_2, C, 0_2, 3_2, 4_2, 0_3, D, 1_3, F)$,
$(0_2, 4_1, 2_2, 0_3, 2_3, 2_1, 1_3, F, 3_3, G)$,
$(0_2, 1_2, 2_2, 2_3, 3_3, 0_3, 4_3, 1_1, 1_3, 4_1)$.

$5K_{3,3,3,3}$ $V = \{i_j \mid 0 \le i \le 2;\ j = 1,2,3,4\}$. P-design follows, cycled mod 3_-:

$(2_2, 1_1, 1_3, 2_4, 0_3, 1_4, 2_1, 0_2, 2_3, 1_2)$,
$(0_1, 0_2, 1_1, 1_2, 0_3, 2_2, 1_3, 2_1, 2_3, 0_4)$,
$(0_1, 0_2, 1_1, 0_3, 2_4, 1_4, 1_3, 1_2, 0_4, 2_1)$,
$(2_1, 2_2, 2_4, 2_3, 0_4, 0_2, 1_3, 1_2, 0_3, 1_4)$,
$(0_1, 0_2, 1_1, 2_2, 0_4, 2_3, 1_4, 2_1, 0_3, 2_4)$,
$(0_1, 1_2, 2_1, 0_3, 0_4, 2_3, 2_4, 1_1, 1_4, 2_2)$.

$5K_{3,3,3,3,3}$ $V = \mathbb{Z}_{15}$. (Part $i+1$ contains the vertices $\{0+i, 5+i, 10+i\}, 0 \le i \le 4$). P-design follows, cycled modulo 15:

(0, 1, 2, 3, 4, 6, 5, 7, 9, 12), (0, 2, 4, 1, 9, 7, 11, 13, 6, 10).

$5K_{4,4,4,4}$ $V = \mathbb{Z}_{16}$. (Part $i+1$ contains the vertices $\{0+i, 4+i, 8+i, 12+i\}, 0 \le i \le 3$). P-design follows, cycled modulo 16:

(0, 1, 2, 3, 5, 6, 4, 7, 8, 9), (0, 2, 5, 8, 11, 9, 14, 4, 13, 3).

$5K_{22,22,22,22}$ $V = \mathbb{Z}_{88}$. (Part $i+1$ contains the vertices $\{0+i, 4+i, \ldots 84+i\}, 0 \le i \le 3$). P-design follows, cycled modulo 88:

(73, 63, 46, 87, 54, 12, 71, 25, 24, 45), (46, 85, 72, 78, 55, 23, 26, 80, 63, 61),
(14, 79, 49, 12, 13, 44, 15, 26, 24, 85), (56, 17, 76, 69, 63, 82, 57, 46, 79, 28),
(10, 81, 80, 29, 63, 23, 86, 85, 42, 56), (24, 27, 52, 74, 51, 29, 66, 85, 58, 80),
(87, 2, 40, 63, 20, 46, 25, 18, 75, 32), (0, 2, 4, 1, 6, 3, 13, 20, 7, 12),
(0, 5, 12, 1, 10, 9, 22, 32, 43, 58), (0, 14, 29, 3, 21, 18, 48, 66, 84, 49),
(0, 19, 40, 75, 46, 43, 78, 16, 54, 13).

P-designs with $\lambda = 15$

$15K_{12}$ $V = \mathbb{Z}_{11} \cup \{\infty\}$. P-design follows, cycled modulo 11:

(0, 1, 2, 3, 4, 5, 6, 7, 8, 9), (0, 1, 2, 3, 4, 5, 6, 7, 8, ∞),
(0, 2, 4, 1, 3, 5, 7, 9, 6, ∞), (0, 2, 4, 1, 3, 5, 7, 9, 6, ∞),
(0, 2, 4, 1, 3, 5, 8, 10, 6, ∞), (0, 2, 7, 1, 6, 4, 10, 3, 8, ∞).

$15K_{14}$ $V = \mathbb{Z}_{13} \cup \{\infty\}$. P-design follows, cycled modulo 13:

(0, 1, 2, 3, 4, 5, 6, 7, 8, 9), (0, 1, 2, 3, 4, 5, 6, 7, 8, 10),
(0, 2, 4, 1, 3, 5, 7, 9, 6, ∞), (0, 2, 4, 1, 3, 5, 7, 9, 6, ∞),
(0, 2, 4, 1, 3, 5, 7, 11, 6, ∞), (0, 3, 6, 1, 7, 5, 10, 2, 8, ∞),
(0, 4, 8, 1, 7, 5, 12, 3, 10, ∞).

$15K_{17}$ $V = \mathbb{Z}_{17}$. P-design follows, cycled modulo 17:

(2, 1, 7, 8, 14, 13, 4, 10, 15, 5), (5, 13, 2, 12, 9, 7, 16, 8, 15, 10),
(10, 11, 5, 13, 8, 12, 16, 3, 7, 9), (8, 14, 2, 11, 15, 5, 13, 16, 4, 7),
(0, 1, 2, 3, 4, 5, 6, 7, 8, 9), (0, 1, 2, 4, 6, 3, 5, 8, 10, 12),
(0, 2, 4, 1, 6, 5, 11, 16, 7, 12), (0, 4, 8, 15, 10, 9, 16, 5, 12, 2).

$15K_{18}$ $V = \mathbb{Z}_{17} \cup \{\infty\}$. P-design follows, cycled modulo 17:

(3, 11, 5, 6, 1, 2, 15, 9, 14, 12), (8, 10, 9, 14, 6, 1, 3, 2, 7, 15),
(10, 5, 6, 11, 16, 12, 14, 13, 3, 15), (8, 10, 14, 4, 16, 1, 7, 6, 2, 11),
(15, 7, 10, 13, 14, 11, 3, 8, 5, ∞), (0, 1, 2, 3, 4, 5, 6, 8, 10, ∞),
(0, 2, 4, 1, 3, 5, 8, 11, 6, ∞), (0, 3, 6, 1, 4, 5, 12, 8, 14, ∞),

(0, 4, 10, 1, 11, 7, 16, 5, 12, ∞).

$15K_{23}$ $V = \mathbb{Z}_{23}$. *P*-design follows, cycled modulo 23:

(18, 11, 14, 4, 12, 8, 9, 5, 13, 2), (5, 8, 15, 4, 1, 14, 6, 21, 9, 17), (16, 11, 4, 9, 8, 18, 21, 13, 6, 20), (8, 20, 10, 13, 5, 21, 22, 16, 19, 2), (5, 13, 11, 17, 10, 22, 21, 6, 8, 3), (18, 22, 5, 17, 14, 19, 9, 2, 13, 6), (8, 1, 15, 6, 2, 4, 19, 3, 9, 20), (0, 1, 2, 3, 4, 5, 6, 7, 8, 10), (0, 2, 4, 1, 3, 5, 7, 9, 6, 10), (0, 2, 4, 6, 11, 5, 12, 1, 10, 19), (0, 5, 11, 3, 13, 7, 21, 4, 14, 20).

$15K_{24}$ $V = \mathbb{Z}_{23} \cup \{\infty\}$. *P*-design follows, cycled modulo 23:

(7, 5, 13, 9, 11, 6, 10, 1, 21, 22), (19, 9, 14, 10, 15, 18, 1, 12, 16, 13), (20, 9, 17, 19, 3, 21, 1, 14, 18, 5), (10, 20, 19, 7, 1, 22, 2, 8, 5, 18), (1, 4, 6, 2, 17, 16, 9, 18, 12, 7), (5, 15, 17, 14, 13, 21, 10, 4, 20, 2), (17, 2, 15, 7, 19, 4, 22, 1, 14, 6), (0, 1, 2, 3, 4, 5, 6, 7, 8, ∞), (0, 2, 4, 1, 3, 5, 7, 9, 11, ∞), (0, 2, 6, 1, 5, 4, 11, 16, 8, ∞), (0, 6, 12, 1, 9, 10, 20, 3, 15, ∞), (0, 6, 12, 1, 15, 9, 21, 7, 16, ∞).

$15K_{27}$ $V = \mathbb{Z}_{27}$. *P*-design follows, cycled modulo 27:

(16, 7, 18, 8, 6, 26, 14, 5, 23, 1), (3, 18, 5, 12, 2, 10, 11, 23, 26, 6), (9, 21, 2, 16, 15, 11, 18, 14, 19, 4), (21, 25, 5, 10, 6, 4, 8, 20, 11, 23), (2, 19, 12, 13, 21, 18, 11, 14, 9, 3), (3, 2, 11, 4, 12, 18, 22, 9, 23, 8), (20, 3, 1, 14, 11, 26, 9, 25, 2, 4), (5, 2, 4, 7, 6, 17, 10, 18, 13, 21), (9, 21, 22, 23, 15, 17, 20, 14, 13, 19), (0, 2, 4, 1, 3, 5, 7, 9, 6, 8), (0, 3, 6, 1, 5, 4, 10, 15, 8, 14), (0, 5, 10, 1, 11, 7, 21, 3, 14, 23), (0, 5, 13, 2, 16, 9, 22, 3, 14, 20).

$15K_{29}$ $V = \mathbb{Z}_{29}$. *P*-design follows, cycled modulo 29:

(21, 1, 2, 23, 5, 13, 11, 26, 19, 22), (26, 18, 12, 5, 21, 25, 10, 16, 17, 7), (9, 27, 21, 5, 18, 26, 14, 3, 7, 8), (25, 5, 18, 9, 27, 20, 2, 4, 16, 10), (24, 5, 21, 13, 9, 15, 11, 14, 3, 28), (7, 16, 12, 23, 6, 4, 22, 10, 21, 26), (19, 16, 14, 20, 17, 3, 27, 5, 26, 25), (6, 7, 8, 17, 20, 10, 12, 27, 26, 16), (15, 9, 5, 25, 11, 13, 12, 28, 2, 22), (15, 27, 20, 8, 1, 17, 14, 24, 28, 6), (13, 20, 22, 5, 19, 3, 14, 9, 12, 25), (0, 2, 4, 1, 5, 3, 7, 11, 6, 13), (0, 5, 10, 1, 6, 8, 2, 14, 26, 16), (0, 5, 13, 26, 14, 15, 27, 4, 20, 7).

$15K_{38}$ $V = \mathbb{Z}_{37} \cup \{\infty\}$. *P*-design follows, cycled modulo 37:

(10, 2, 16, 19, 20, 14, 32, 9, 26, 4), (36, 18, 35, 21, 24, 13, 26, 28, 14, 4), (23, 29, 36, 32, 9, 16, 4, 26, 20, 27), (18, 27, 31, 10, 30, 25, 5, 3, 8, 28), (24, 29, 6, 26, 23, 2, 15, 1, 17, 32), (27, 36, 14, 19, 26, 15, 11, 10, 29, 8), (4, 30, 6, 16, 12, 32, 23, 34, 2, 7), (30, 20, 1, 8, 11, 5, 10, 14, 17, 22), (26, 27, 16, 25, 23, 28, 18, 3, 22, 12), (2, 28, 14, 10, 35, 23, 15, 8, 6, 33), (7, 30, 29, 19, 8, 5, 2, 22, 4, 16), (25, 34, 23, 31, 30, 22, 3, 28, 32, 35), (35, 36, 18, 33, 20, 7, 25, 22, 8, 6), (0, 1, 2, 3, 4, 5, 7, 9, 11, 16), (0, 2, 6, 10, 16, 7, 13, 1, 8, ∞), (0, 6, 12, 1, 7, 8, 20, 26, 14, ∞), (0, 6, 12, 1, 13, 8, 20, 5, 21, ∞), (0, 6, 12, 4, 17, 8, 25, 1, 14, ∞), (0, 6, 19, 35, 16, 24, 3, 32, 14, ∞).

15K_{39} $V = \mathbb{Z}_{39}$. P-design follows, cycled modulo 39:

(5, 18, 38, 22, 19, 32, 2, 30, 1, 36), (19, 11, 4, 29, 13, 33, 16, 8, 38, 27),
(31, 34, 36, 15, 13, 20, 16, 0, 19, 23), (27, 19, 15, 5, 32, 21, 4, 23, 9, 16),
(26, 8, 25, 37, 5, 4, 32, 19, 27, 3), (7, 18, 8, 12, 4, 22, 20, 28, 9, 19),
(15, 24, 13, 29, 17, 23, 16, 14, 0, 32), (31, 8, 20, 12, 2, 6, 32, 30, 3, 22),
(2, 11, 18, 25, 9, 34, 37, 30, 36, 15), (26, 23, 12, 22, 30, 24, 8, 27, 19, 16),
(6, 18, 17, 36, 15, 35, 24, 14, 9, 26), (11, 23, 34, 16, 12, 10, 31, 26, 13, 38),
(6, 17, 24, 34, 32, 4, 5, 18, 14, 9), (26, 11, 33, 24, 6, 3, 29, 21, 4, 38),
(28, 11, 10, 34, 16, 20, 2, 29, 30, 21), (0, 1, 2, 3, 4, 5, 6, 18, 16, 21),
(0, 1, 3, 5, 18, 2, 8, 13, 7, 20), (0, 2, 8, 14, 20, 6, 21, 4, 17, 23),
(0, 6, 15, 30, 17, 22, 38, 32, 23, 7).

15K_{53} $V = \mathbb{Z}_{53}$. P-design follows, cycled modulo 53:

(28, 46, 22, 19, 24, 27, 50, 18, 8, 7), (28, 3, 5, 41, 38, 51, 22, 42, 29, 27),
(47, 45, 8, 19, 43, 25, 48, 16, 3, 50), (5, 47, 27, 1, 20, 13, 22, 41, 48, 23),
(43, 4, 42, 11, 22, 9, 21, 34, 49, 51), (27, 45, 9, 7, 28, 19, 13, 24, 20, 18),
(39, 15, 21, 16, 7, 32, 38, 18, 40, 6), (9, 40, 19, 36, 31, 27, 42, 21, 33, 14),
(29, 33, 9, 3, 48, 18, 7, 36, 13, 44), (45, 2, 37, 36, 42, 7, 8, 18, 48, 28),
(7, 43, 20, 34, 25, 42, 27, 36, 6, 44), (25, 2, 35, 32, 37, 18, 44, 16, 45, 52),
(33, 22, 43, 39, 29, 2, 1, 34, 13, 10), (11, 39, 45, 16, 18, 5, 17, 52, 36, 6),
(15, 5, 3, 36, 17, 34, 10, 46, 29, 32), (38, 31, 32, 26, 52, 1, 11, 12, 29, 45),
(43, 17, 31, 22, 35, 24, 26, 27, 5, 25), (36, 3, 22, 42, 38, 13, 6, 34, 16, 50),
(10, 47, 7, 41, 31, 2, 43, 25, 39, 29), (3, 2, 20, 4, 16, 19, 5, 38, 51, 45),
(4, 11, 46, 35, 40, 20, 42, 1, 10, 28), (0, 3, 6, 1, 4, 5, 11, 16, 8, 14),
(0, 3, 6, 1, 7, 5, 13, 4, 12, 21), (0, 3, 12, 1, 14, 9, 25, 39, 17, 31),
(0, 14, 29, 1, 27, 30, 47, 12, 40, 15), (0, 14, 29, 2, 38, 17, 45, 22, 49, 32).

15K_{54} $V = \mathbb{Z}_{53} \cup \{\infty\}$. P-design follows, cycled modulo 53:

(12, 48, 40, 49, 52, 13, 39, 20, 2, 6), (8, 9, 10, 13, 33, 41, 3, 25, 20, 52),
(41, 29, 48, 12, 20, 39, 32, 16, 17, 27), (14, 5, 40, 10, 44, 12, 50, 20, 1, 33),
(29, 42, 36, 24, 31, 28, 9, 44, 48, 2), (19, 17, 44, 35, 34, 7, 49, 39, 47, 20),
(44, 8, 6, 34, 29, 2, 35, 31, 14, 22), (13, 41, 4, 33, 26, 7, 43, 51, 24, 10),
(52, 41, 6, 8, 21, 14, 15, 43, 17, 27), (42, 49, 41, 13, 7, 8, 32, 30, 20, 19),
(37, 7, 43, 50, 1, 38, 42, 47, 0, 10), (15, 19, 31, 0, 44, 7, 51, 4, 2, 34),
(13, 22, 50, 29, 32, 36, 30, 11, 8, 17), (8, 27, 0, 17, 7, 20, 48, 29, 15, 4),
(18, 19, 24, 20, 37, 42, 7, 11, 33, 45), (50, 1, 14, 31, 10, 15, 9, 27, 44, 24),
(32, 44, 18, 33, 29, 12, 3, 23, 50, 40), (22, 42, 12, 52, 2, 23, 13, 39, 19, 8),
(18, 3, 0, 30, 33, 48, 19, 7, 5, 35), (42, 47, 34, 46, 49, 36, 39, 6, 43, 14),
(12, 17, 33, 48, 39, 24, 38, 32, 2, 0), (21, 52, 14, 22, 11, 50, 34, 16, 7, 13),
(0, 2, 4, 1, 3, 8, 12, 23, 5, ∞), (0, 4, 12, 1, 11, 8, 22, 33, 15, ∞),
(0, 8, 19, 1, 22, 14, 35, 4, 26, ∞), (0, 14, 28, 3, 21, 16, 44, 5, 30, ∞),
(0, 14, 28, 4, 29, 21, 42, 5, 36, ∞).

15K_{57} $V = \mathbb{Z}_{57}$. P-design follows, cycled modulo 57:

(1, 25, 5, 9, 10, 27, 29, 38, 20, 23),
(25, 34, 44, 49, 6, 4, 11, 41, 35, 53),
(25, 32, 33, 30, 29, 36, 26, 20, 31, 1),
(29, 55, 28, 34, 16, 20, 53, 46, 15, 19),
(51, 22, 10, 26, 3, 25, 1, 29, 54, 19),
(28, 33, 3, 39, 5, 50, 20, 40, 36, 41),
(2, 3, 21, 17, 47, 12, 50, 39, 49, 52),
(27, 16, 17, 37, 50, 51, 53, 8, 5, 42),
(42, 8, 25, 21, 9, 51, 37, 23, 22, 10),
(42, 54, 8, 32, 40, 5, 12, 41, 33, 39),
(14, 39, 49, 13, 42, 4, 31, 8, 22, 5),
(46, 48, 13, 25, 51, 56, 28, 2, 19, 35),
(0, 1, 6, 12, 18, 7, 13, 3, 9, 19),
(0, 13, 26, 1, 32, 36, 51, 7, 43, 17),
(3, 51, 53, 2, 50, 14, 8, 45, 34, 5),
(41, 55, 27, 45, 26, 30, 6, 39, 37, 20),
(53, 13, 43, 39, 38, 35, 3, 40, 6, 26),
(45, 50, 19, 33, 11, 12, 37, 48, 46, 36),
(17, 32, 34, 36, 20, 15, 50, 38, 35, 56),
(50, 36, 52, 22, 37, 16, 11, 35, 34, 27),
(13, 53, 38, 32, 29, 9, 14, 6, 44, 1),
(2, 16, 51, 54, 43, 39, 31, 12, 34, 20),
(21, 9, 16, 3, 17, 18, 51, 29, 41, 52),
(41, 36, 13, 56, 38, 6, 8, 11, 51, 7),
(42, 17, 54, 16, 22, 7, 13, 5, 32, 11),
(17, 32, 53, 38, 21, 36, 24, 31, 2, 30),
(0, 7, 15, 2, 18, 10, 28, 3, 33, 46),
(0, 13, 29, 10, 31, 19, 50, 12, 32, 51).

$15K_{59}$ $V = \mathbb{Z}_{59}$. P-design follows, cycled modulo 59:

(2, 50, 23, 32, 1, 28, 30, 53, 7, 51),
(10, 57, 22, 19, 15, 23, 4, 37, 54, 5),
(9, 44, 7, 5, 29, 20, 24, 16, 28, 18),
(41, 57, 31, 42, 11, 50, 54, 37, 18, 13),
(26, 35, 24, 40, 7, 10, 16, 41, 57, 51),
(26, 14, 47, 55, 5, 13, 16, 40, 36, 56),
(39, 6, 20, 58, 1, 22, 56, 11, 47, 5),
(50, 10, 43, 49, 30, 7, 32, 25, 3, 23),
(14, 29, 18, 35, 15, 27, 56, 6, 1, 31),
(11, 16, 41, 28, 50, 55, 54, 46, 47, 24),
(38, 6, 18, 49, 21, 7, 2, 5, 48, 30),
(8, 26, 23, 45, 52, 35, 41, 58, 19, 57),
(19, 18, 6, 38, 40, 29, 23, 57, 47, 30),
(0, 2, 4, 6, 8, 5, 10, 18, 16, 24),
(0, 10, 24, 3, 38, 15, 53, 26, 48, 27).
(40, 30, 53, 24, 17, 50, 14, 20, 45, 4),
(20, 57, 58, 26, 21, 45, 2, 50, 44, 54),
(8, 49, 37, 22, 7, 40, 58, 44, 15, 51),
(27, 8, 40, 44, 16, 24, 37, 30, 56, 31),
(24, 25, 40, 14, 57, 52, 27, 19, 2, 11),
(46, 7, 45, 22, 20, 8, 23, 47, 18, 58),
(57, 37, 10, 6, 53, 49, 41, 44, 30, 22),
(30, 44, 55, 40, 26, 32, 4, 37, 50, 52),
(30, 4, 26, 36, 24, 5, 20, 27, 3, 13),
(50, 18, 11, 35, 41, 15, 39, 44, 4, 6),
(29, 15, 28, 8, 25, 48, 39, 57, 22, 18),
(25, 24, 54, 15, 22, 28, 53, 46, 19, 33),
(10, 1, 35, 33, 29, 45, 16, 28, 6, 51),
(0, 5, 10, 1, 15, 8, 20, 30, 45, 33),

$15(K_{12} \setminus K_2)$ $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2\} \cup \{A, B\}$; hole on $\{A, B\}$.

P-design follows, cycled modulo 5_-:

$(3_2, 4_2, 4_1, 1_1, 1_2, 0_2, 3_1, A, 0_1, B)$,
$(4_2, 3_2, 3_1, 1_1, 2_2, 0_2, 0_1, A, 4_1, B)$,
$(1_1, 2_2, 0_1, 2_1, 3_2, 1_2, 4_1, A, 0_2, B)$,
$(0_2, 1_2, 0_1, 1_1, 2_2, 3_2, 2_1, A, 3_1, B)$,
$(0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, A, 2_2, B)$,
$(0_1, 2_1, 4_1, 1_2, 3_2, 2_2, 4_2, 1_1, 0_2, 3_1)$,
$(0_1, 2_1, 4_1, 2_2, 0_2, 4_2, 3_2, 3_1, 1_2, 1_1)$.
$(3_2, 4_2, 0_2, 2_2, 1_1, 0_1, 1_2, A, 2_1, B)$,
$(1_1, 0_1, 4_2, 1_2, 2_2, 0_2, 3_2, A, 3_1, B)$,
$(0_2, 0_1, 1_2, 2_2, 3_2, 4_2, 1_1, A, 2_1, B)$,
$(0_1, 1_1, 2_1, 3_1, 4_1, 0_2, 1_2, A, 2_2, B)$,
$(0_1, 1_1, 0_2, 3_1, 1_2, 2_1, 4_1, A, 3_2, B)$,
$(0_1, 2_1, 4_1, 1_2, 3_2, 4_2, 2_2, 3_1, 0_2, 1_1)$,

$15(K_{13} \setminus K_3)$ $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2\} \cup \{A, B, C\}$; hole on $\{A, B, C\}$.

P-design follows, cycled modulo 5_-:

$(2_1, 1_2, 2_2, 4_1, 1_1, 4_2, 0_2, A, 3_1, B)$,
$(1_2, 3_1, 1_1, 4_2, 3_2, 4_1, 0_2, A, 0_1, B)$,
$(0_1, 2_1, 3_1, 1_1, 4_2, 4_1, 0_2, A, 2_2, B)$,
$(4_1, 1_2, 1_1, 4_2, 0_2, 2_2, 3_1, A, 3_2, B)$,

$(0_2,0_1,2_2,4_1,3_2,2_1,1_2,A,4_2,B)$, $(1_2,3_1,2_1,2_2,1_1,0_1,4_1,A,3_2,C)$,
$(3_2,1_2,2_2,1_1,2_1,3_1,4_1,A,0_2,C)$, $(4_2,3_1,2_2,0_1,1_2,2_1,0_2,A,4_1,C)$,
$(4_2,3_2,1_1,0_1,2_1,1_2,3_1,A,4_1,C)$, $(3_2,2_2,0_2,1_2,1_1,4_2,0_1,A,3_1,C)$,
$(0_1,1_1,0_2,1_2,2_2,2_1,3_1,B,3_2,C)$, $(0_1,1_1,2_1,4_1,3_1,0_2,1_2,B,2_2,C)$,
$(0_1,1_1,4_1,2_1,0_2,3_1,1_2,B,2_2,C)$, $(0_1,0_2,2_2,3_2,1_2,4_2,2_1,B,1_1,C)$,
$(0_2,2_2,4_2,1_2,0_1,3_2,3_1,B,1_1,C)$.

$15(K_{14} \setminus K_4)$ $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2\} \cup \{A,\ldots,D\}$;

hole on $\{A,B,C,D\}$. P-design follows, cycled modulo 5_-:

$(2_2,0_2,C,2_1,1_1,4_2,1_2,A,0_1,B)$, $(1_2,3_2,C,4_1,0_2,3_1,2_1,A,2_2,B)$,
$(4_1,0_2,C,2_2,3_1,3_2,4_2,A,2_1,B)$, $(2_1,1_1,D,3_2,3_1,2_2,1_2,A,0_1,B)$,
$(2_1,3_2,D,2_2,1_2,4_2,3_1,A,1_1,B)$, $(2_2,2_1,D,3_1,0_2,4_1,1_2,A,0_1,B)$,
$(3_1,4_1,D,0_1,4_2,1_1,2_1,A,2_2,C)$, $(1_1,3_1,D,1_2,0_2,2_2,4_2,A,2_1,C)$,
$(4_2,0_2,D,4_1,1_2,2_2,2_1,A,3_2,C)$, $(3_2,1_2,D,2_2,4_2,0_2,4_1,A,2_1,C)$,
$(0_2,0_1,1_2,1_1,2_1,2_2,3_1,B,3_2,C)$, $(0_1,1_1,2_1,3_1,4_1,0_2,1_2,B,2_2,D)$,
$(0_1,2_1,0_2,1_2,3_1,2_2,1_1,C,4_1,D)$, $(0_1,2_1,0_2,1_1,1_2,3_1,2_2,B,4_1,C)$,
$(0_1,2_1,0_2,1_1,3_2,2_2,1_2,B,4_1,D)$, $(0_1,2_1,4_1,1_2,0_2,2_2,3_2,3_1,4_2,1_1)$,
$(0_1,2_1,4_1,1_2,0_2,2_2,4_2,1_1,3_2,3_1)$.

$15(K_{17} \setminus K_2)$ $V = \mathbb{Z}_{15} \cup \{A,B\}$; hole on $\{A,B\}$.

P-design follows, cycled modulo 15:

$(0,1,2,3,4,5,6,7,8,9)$, $(0,1,2,3,4,5,6,7,8,10)$,
$(0,2,4,1,3,5,7,9,6,8)$, $(0,2,4,1,3,5,7,9,6,8)$,
$(6,3,10,7,12,8,4,A,5,B)$, $(10,2,7,12,5,14,11,A,9,B)$,
$(9,3,14,5,1,4,8,A,10,B)$, $(13,5,12,3,9,8,2,A,14,B)$,
$(0,4,9,1,6,8,3,A,10,B)$.

$15(K_{18} \setminus K_3)$ $V = \mathbb{Z}_{15} \cup \{A,B,C\}$; hole on $\{A,B,C\}$.

P-design follows, cycled modulo 15:

$(0,1,2,3,4,5,6,7,8,A)$, $(0,1,2,3,4,5,6,7,8,A)$,
$(0,1,3,5,2,4,6,8,10,A)$, $(0,2,4,1,3,5,7,9,6,A)$,
$(0,2,4,1,3,5,7,10,6,A)$, $(2,11,8,14,9,12,5,B,4,C)$,
$(1,9,6,12,8,11,2,B,13,C)$, $(9,4,10,5,14,13,3,B,11,C)$,
$(1,5,10,6,11,9,3,B,14,C)$, $(0,4,10,1,6,8,2,B,11,C)$.