Recent bounds on domination parameters

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ABSTRACT. In this paper, we survey some recent bounds on domination parameters. A characterisation of connected graphs with minimum degree at least 2 and domination number exceeding a third their size is obtained. Upper bounds on the total domination number, $\gamma_t(G)$, of a graph G in terms of its order and size are established. If G is a connected graph of order n with minimum degree at least 2, then either $\gamma_t(G) \leq 4n/7$ or $G \in \{C_3, C_5, C_6, C_{10}\}$. A characterisation of those graphs of order n which are edge-minimal with respect to satisfying G connected, $\delta(G) \geq 2$, and $\gamma_t(G) \geq 4n/7$ is obtained. We establish that if G is a connected graph of size q with minimum degree at least 2, then $\gamma_t(G) \leq (q+2)/2$. Connected graphs G of size q with minimum degree at least 2 satisfying $\gamma_t(G) > q/2$ are characterised. Upper bounds on other domination parameters, including the strong domination number and the restrained domination number are presented. We provide a constructive characterisation of those trees with equal domination and restrained domination numbers. A constructive characterisation of those trees with equal domination and weak domination numbers is also obtained.

1 Introduction

In this paper, we follow the notation of [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a

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vertex in V. The open neighbourhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the open neighborhood of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. The subgraph of G induced by the vertices in S is denoted by $\langle S \rangle$. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$). A cycle of length n is an *n-cycle*. A graph of order n that is a path or a cycle is denoted by P_n or C_n, respectively. We refer to a vertex that is adjacent to an end-vertex as a *remote* vertex.

A set $S \subseteq V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. (That is, $N[S] = V$.) The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set S is called an *independent dominating set* of G if S is also independent. The *independent domination number* of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [2] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes, Hedetniemi and Slater [12, 13].

A set $S \subseteq V$ is a *total dominating set* if every vertex in V is adjacent to a vertex in S. (That is, $N(S) = V$.) Every graph without isolated vertices has a total dominating set, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3] and is now well studied in graph theory (see [12, 13]).

A set $S \subseteq V$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. Clearly, $\gamma_r(G) \geq \gamma(G)$. The concept of restrained domination was introduced by Telle and Proskurowski [24], albeit indirectly, as a vertex partitioning problem and further studied in [4, 5, 6, 7, 16].

A set $S \subseteq V$ is a *weak dominating set* of G if for every u in $V - S$, there exists a $v \in S$ such that $uv \in E$ and $\deg u \geq \deg v$. The weak domination number of G, denoted by $\gamma_w(G)$, is the minimum cardinality of a weak dominating set of G. A set $S \subseteq V$ is a *strong dominating set* of G if for every u in $V - S$, there exists a $v \in S$ such that $uv \in E$ and $\deg u \leq \deg v$. The strong domination number of G, denoted by $\gamma_{st}(G)$, is the minimum cardinality of a strong dominating set of G. The concept of weak and strong domination was introduced by Sampathkumar and Pushpa Latha in [22] and further studied in [10, 11, 20].

An *end-dominating set* of G is a dominating set of G that contains all end-vertices of G. The *end-domination number* of G, denoted by $\gamma_e(G)$, is the minimum cardinality of an end-dominating set of G. We call an end-dominating set of G of minimum cardinality a γ_e-set of G. The concept of an end-dominating set was introduced in [11] and further studied in [8].

A *2-packing* in a graph G is a set of vertices that are pairwise at distance at least 3 apart, i.e., if S is a 2-packing of G, then $d(u, v) \geq 3$ for all $u, v \in S$.

In this paper, we survey recent bounds on domination parameters, including the domination number, the total domination number, the restrained domination number, the weak domination number, and the strong domination numbers.

2 Bounds on the domination number

The decision problem to determine the domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the domination number of a graph. Various authors have investigated upper bounds on the domination number of a connected graph in terms of the minimum degree and order of the graph. The earliest such result is due to Ore [18].

Theorem 1 (Ore) *If G is a graph of order n with $\delta(G) \geq 1$, then $\gamma(G) \leq n/2$.*

A large family of graphs attaining the bound in Theorem 1 can be established using the following transformation of a graph. The *corona* of a graph G, denoted by G^+, is the graph obtained from G by adding an adjacent end-vertex to each vertex of G. Payan and Xuong [19] characterised those graphs with no isolated vertex and with domination number exactly half their order.

Theorem 2 (Payan, Xuong) *If G is a connected graph of order n, then $\gamma(G) = n/2$ if and only if $G \cong C_4$ or $G \cong H^+$ for some connected graph H.*

McCraig and Shepherd [17] investigated upper bounds on the domination number of a connected graph with minimum degree at least 2.

Theorem 3 (McCraig, Shepherd) *If G is a connected graph of order n with $\delta(G) \geq 2$, and if G is not one of seven exceptional graphs (one of order 4 and six of order 7), then $\gamma(G) \leq 2n/5$.*

McCraig and Shepherd [17] also characterised those graphs G of order n which are edge-minimal with respect to satisfying G connected, $\delta(G) \geq 2$, and $\gamma(G) \geq 2n/5$. This characterisation may be found in [12]. Reed [21] investigated upper bounds on the domination number of a connected graph with minimum degree at least 3.

Theorem 4 (Reed) *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.*

Sanchis [23] investigated upper bounds on the domination number of a connected graph in terms of the minimum degree and size of the graph.

Theorem 5 (Sanchis) *If G is a connected graph of size q with $\delta(G) \geq 2$, then $\gamma(G) \leq (q+2)/3$ with equality if and only if G is a cycle of length n where $n \equiv 1 \pmod 3$.*

We refer to a graph G as a **$\frac{q}{3}$-graph** if G is a connected graph of size q with minimum degree at least 2 satisfying $\gamma(G) > q/3$. In [14], $\frac{q}{3}$-graphs are characterised. To describe this characterisation, we introduce a family $\mathcal{G}$ of $\frac{q}{3}$-graphs and a collection $\mathcal{H}$ of five $\frac{q}{3}$-graphs. Let $\mathcal{H}$ be the collection of five $\frac{q}{3}$-graphs shown in Figure 1.

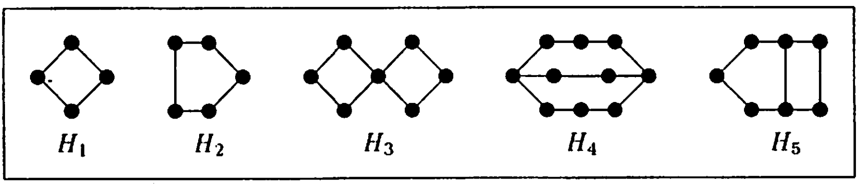


Figure 1. Graphs in the collection $\mathcal{H}$

We define a *unit* to be either a 4-cycle with a path of length 1 attached to a vertex of the 4-cycle, which we call a *type-1 unit*, or a 5-cycle, which we call a *type-2 unit*. If v is a vertex of a graph, then by *attaching a type-1 unit* to v we mean adding a 4-cycle and joining v with an edge to one vertex of the cycle (see Figure 2.(a)). By *attaching a type-2 unit* to v we mean adding a (disjoint) 5-cycle to the graph and identifying one of its vertices with v (see Figure 2.(b)). We now introduce a family $\mathcal{G}$ of $\frac{q}{3}$-graphs.

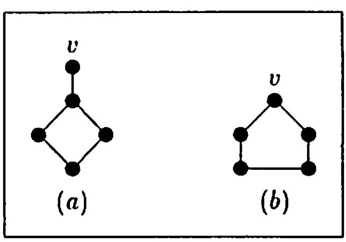


Figure 2. (a) type-1 unit and (b) type-2 unit

Let F be a forest that consists of $k \geq 1$ nontrivial components $F_1, \ldots, F_k$. For $i \in \{1, \ldots, k\}$, we let S_i be a distinguished set of vertices of F_i that satisfies the following two conditions: (i) every end-vertex of F_i belongs to S_i (but not every vertex of S_i is necessarily an end-vertex of F_i); (ii) if $V(F_i) \neq S_i$, then $F_i - S_i$ is a forest whose vertex set can be partitioned into $\ell \geq 1$ sets each of which induce a path P_3, the central vertex of which has degree 2 in F_i. We refer to the partition in (ii) as the *path-partition* of $V(F_i) - S_i$. Let $S_F = \cup_{i=1}^{k} S_i$.

If $k \geq 2$, then we construct a tree T from the forest F by adding $k-1$ edges $e_1, \ldots, e_{k-1}$ to F where both ends of e_i belong to S_F for $i = 1, \ldots, k-1$. Let $E^* = \{e_1, \ldots, e_{k-1}\}$ and let S_F^* denote the vertices incident with some edge of E^*. (Thus, $S_F^* \subseteq S_F$.) Let $S'_F = S_F$ if $k = 1$ and let $S'_F = S_F - S_F^*$ if $k \geq 2$. If $k = 1$, then we let $T = F$.

We now construct a graph G from T as follows. Notice that each component of the subgraph $\langle E^* \rangle$ induced by E^* is a nontrivial tree. Each component of $\langle E^* \rangle$ of order ℓ we replace with a $(3\ell - 1)$-cycle in which the ℓ vertices in the component are the ℓ vertices on the $(3\ell - 1)$-cycle in positions $1, 3, 6, \ldots, 3(\ell - 1)$. (In particular, each component of $\langle E^* \rangle$ that is a path P_2 is replaced with a 5-cycle in which the two vertices of the path are non-adjacent vertices on the cycle.) For example, if $\langle E^* \rangle \cong P_2 \cup K_{1,3}$, then $\langle E^* \rangle$ is replaced by the graph shown in Figure 3.

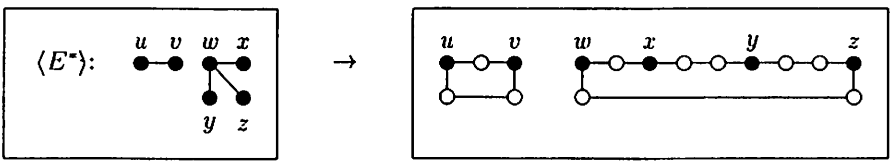


Figure 3. A graph replacing the subgraph $\langle E^* \rangle$

Finally, we attach a type-1 unit or a type-2 unit to each vertex of S'_F. Let G denote the resulting graph. We refer to the forest F as the **underlying forest** of G and the tree T as the **underlying tree** of G. The collection of all such graphs G we denote by $\mathcal{G}$.

If $F \cong K_2$, for example, then $T = F$ and G is one of the three graphs shown in Figure 4 (where u and v denote the two vertices of F).

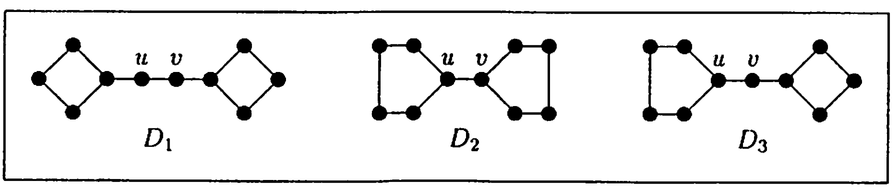


Figure 4. Three graphs in the family $\mathcal{G}$ constructed from $F = K_2$

As a further example of our construction, consider the graph G in the family $\mathcal{G}$ that is shown in Figure 5.

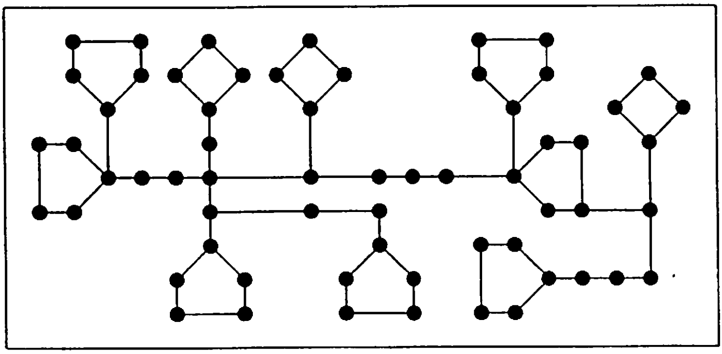


Figure 5. A graph G in the family $\mathcal{G}$

The underlying forest F of the graph G of Figure 5 is shown in Figure 6 with a set of distinguished (darkened) vertices S_F. In this example, the forest F consists of two components, namely a component F_1 containing the vertex named u and a component F_2 containing the vertex named v. The underlying tree T of G is constructed from F by adding the edge uv. The graph G is constructed from T by replacing the edge uv with a 5-cycle in which u and v are non-adjacent vertices on the 5-cycle, and by attaching a type-1 unit or a type-2 unit to each vertex of $S'_F = S_F - \{u, v\}$.

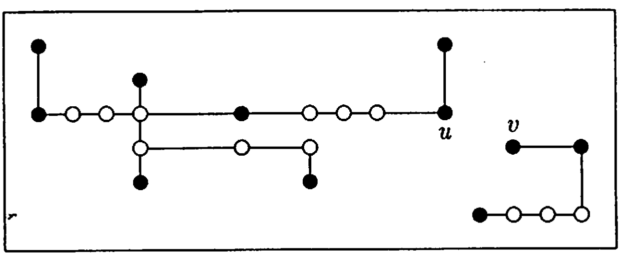


Figure 6. The underlying forest F of the graph G of Figure 5

The final two examples of our construction are shown in Figure 7 and Figure 8. These examples serve to illustrate two graphs G in the family $\mathcal{G}$ with different underlying trees T but with the same underlying forest F.

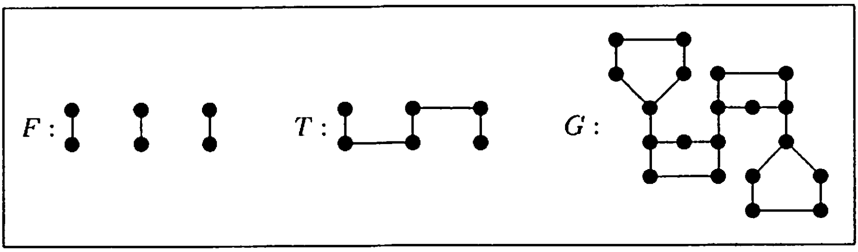


Figure 7.
A graph G in $\mathcal{G}$ with underlying tree T and underlying forest F

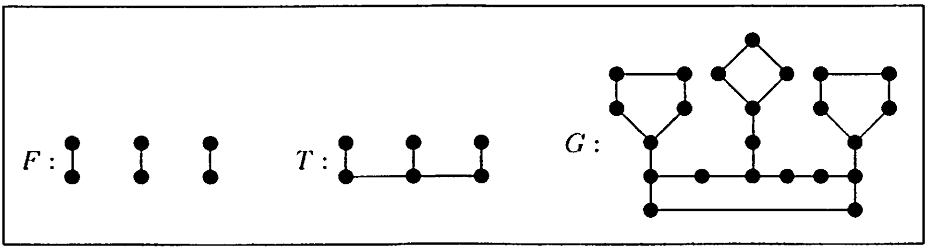


Figure 8.
A graph G in $\mathcal{G}$ with underlying tree T and underlying forest F

Let G be a nonempty graph. We define an *elementary* 3-*subdivision* of G as a graph obtained from G by subdividing some edge three times. A 3-*subdivision* of G is a graph obtained from G by a succession of elementary 3-subdivisions (including the possibility of none). We denote the family of all 3-subdivisions of G by G^*; that is, $G^* = \{H \mid H \text{ is a 3-subdivision of } G\}$. Let

$$\mathcal{G}^* = \bigcup_{G \in \mathcal{G}} G^* \text{ and } \mathcal{H}^* = \bigcup_{H \in \mathcal{H}} H^*.$$

For $i = 0, 1, 2$, let $\mathcal{C}_i = \{C_n \mid n \equiv i \,(mod 3)\,\}$. Notice that $H_1^* = \mathcal{C}_1$ and $H_2^* = \mathcal{C}_2$. In [14], the following characterisation of $\frac{q}{3}$-graphs is obtained.

Theorem 6 (Henning) *If G is a $\frac{q}{3}$-graph, then $G \in \mathcal{G}^* \cup \mathcal{H}^*$.*

As a consequence of Theorem 6, we have the following result.

Theorem 7 (Henning) *If G is a connected graph of size q with minimum degree at least 2, then $\gamma(G) \le q/3$ unless either $G \in \mathcal{C}_1$, in which case $\gamma(G) = (q+2)/3$, or $G \in \mathcal{G}^* \cup (\mathcal{H}^* - \mathcal{C}_1)$, in which case $\gamma(G) = (q+1)/3$.*

3 Bounds on the total domination number

The decision problem associated with the total domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the total domination number of a graph. Cockayne, Dawes, and Hedetniemi [3] obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 8 (Cockayne, Dawes, Hedetniemi) *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*

A large family of graphs attaining the bound in Theorem 8 can be established using the following transformation of a graph. The 2-*corona* of a graph H is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex disjoint. The 2-corona of a connected graph has total domination number 2/3 its order. The following characterisation of connected graphs of order at least 3 with total domination number exactly 2/3 their order is obtained in [1].

Theorem 9 (Brigham, Carrington, Vitray) *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6 or the 2-corona of some connected graph.*

The following property of minimal total dominating sets is established in [3].

Proposition 10 (Cockayne, Dawes, Hedetniemi) *If S is a minimal total dominating set of a connected graph $G = (V, E)$, then each $v \in S$ has at least one of the following two properties:*

P_1 : There exists a vertex $w \in V - S$ such that $N(w) \cap S = \{v\}$;
P_2 : $\langle S - \{v\} \rangle$ contains an isolated vertex.

The following result in [15] guarantees the existence of a minimum total dominating set satisfying certain desirable properties.

Theorem 11 (Henning) *If G is a connected graph of order $n \geq 3$, and $G \ncong K_n$, then G has a minimum total dominating set S in which every vertex has property P_1 or is adjacent to a vertex of degree 1 in $\langle S \rangle$ that has property P_1.*

Using Theorem 11, the result of Theorem 9 follows readily. Next we examine the total domination number of connected graphs G of order $n \geq 3$

with minimum degree $\delta(G) \geq 2$. We will refer to a graph G as an $\frac{4}{7}$-**minimal graph** if G is edge-minimal with respect to satisfying the following three conditions:

(*i*) $\delta(G) \geq 2$,
(*ii*) G is connected, and
(*iii*) $\gamma_t(G) \geq 4n/7$,

where n is the order of G. We shall characterise $\frac{4}{7}$-minimal graphs. For this purpose, we introduce a family $\mathcal{H}$ of $\frac{4}{7}$-minimal graphs. Let $\mathcal{H}$ be the collection of graphs that can be obtained from a nontrivial tree T as follows. For each vertex v of T, add a 6-cycle C_v and join v to one vertex of C_v. The subgraph induced by v and C_v we call a **unit** of $\mathcal{H}$. We refer to the tree T as the **underlying tree** of the resulting graph. A graph in the family $\mathcal{H}$ with four units and with underlying tree $T \cong P_4$ is shown in Figure 9.

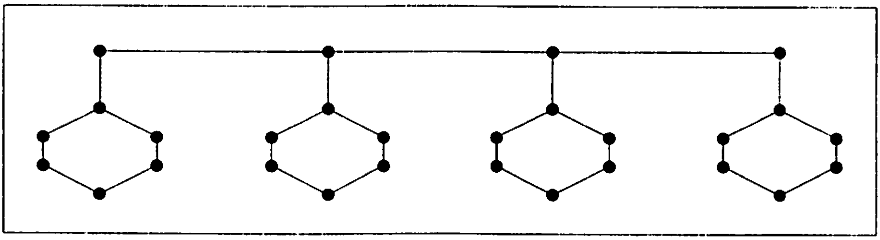


Figure 9. A graph in the family $\mathcal{H}$ of $\frac{4}{7}$-minimal graphs

Let $\mathcal{C} = \{C_3, C_5, C_6, C_7, C_{10}, C_{14}\}$. Let H_1 be the graph obtained from a 6-cycle by adding a new vertex and joining this vertex to two vertices at distance 2 apart on the cycle. The graph H_1 is shown in Figure 10.

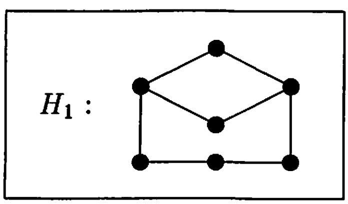


Figure 10. The graph H_1

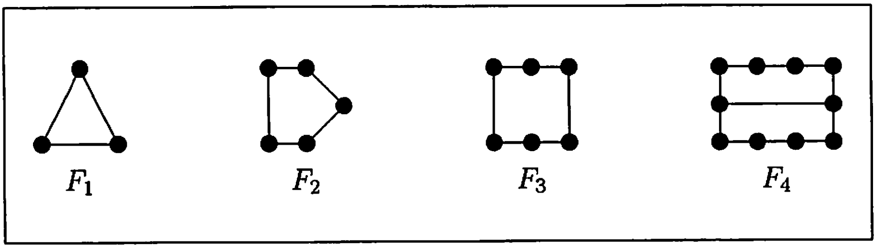
The following result in [15] characterises $\frac{4}{7}$-minimal graphs.

Theorem 12 (Henning) *If G is a $\frac{4}{7}$-minimal graph, then $G \in \mathcal{C} \cup \mathcal{H} \cup \{H_1\}$.*

An immediate consequence of Theorem 12 now follows.

Corollary 13 (Henning) *If G is a connected graph of order n with minimum degree at least 2 and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$.*

In [15], an upper bound on the total domination number in terms of the size of the graph is established. We will refer to a graph G as an $\frac{q}{2}$**-graph** if G is a connected graph of size q satisfying $\delta(G) \geq 2$ and $\gamma_t(G) > q/2$. In [15], $\frac{q}{2}$-graphs are characterised. To describe this characterisation, we introduce some families of $\frac{q}{2}$-graphs. For $i = 0, 1, 2, 3$, let $\mathcal{C}_i = \{C_n \mid n \equiv i \,(mod 4)\,\}$. Let $\mathcal{F}$ be the collection of four $\frac{q}{2}$-graphs shown in Figure 11.



Figue 11. The collection $\mathcal{F}$ of four $\frac{q}{2}$-graphs

Next we construct a family $\mathcal{G}$ of graphs G as follows. Let T be a nontrivial tree with a distinguished set S_T of vertices that satisfies the following two conditions: (i) every end-vertex of T belongs to S_T (but not every vertex of S_T is necessarily an end-vertex of T); (ii) if $V(T) \neq S_T$, then $T - S_T$ is a forest whose vertex set can be partitioned into $\ell \geq 1$ sets each of which induce a path on four vertices with the two central vertices having degree 2 in T. We refer to the partition in (ii) as the *path-partition* of $V(T) - S_T$. For each vertex v in S_T, add a 6-cycle C_v and join v to one vertex of C_v. Let G denote the resulting graph. We refer to the tree T as the **underlying tree** of G. The family of all such graphs G we denote by $\mathcal{G}$. A graph G in the family $\mathcal{G}$ with underlying tree T is shown in Figure 12 (here S_T consists of the four darkened vertices in T).

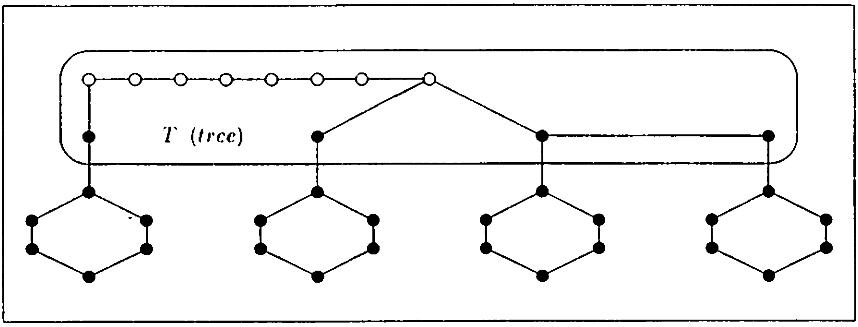


Figure 12. A graph G in the family $\mathcal{G}$ with underlying tree T

We define an *elementary 4-subdivision* of a nonempty graph G as a graph obtained from G by subdividing some edge four times. A *4-subdivision* of G

is a graph obtained from G by a succession of elementary 4-subdivisions (including the possibility of none). We denote the family of all 4-subdivisions of a graph G by G^*. Let

$$\mathcal{G}^* = \bigcup_{G \in \mathcal{G}} G^* \text{ and } \mathcal{F}^* = \bigcup_{F \in \mathcal{H}} F^*.$$

For $i = 0, 1, 2, 3$, let $\mathcal{C}_i = \{C_n \mid n \equiv i\,(mod 4)\}$. Notice that $F_1^* = \mathcal{C}_3$, $F_2^* = \mathcal{C}_1$, and $F_3^* = \mathcal{C}_2$. In [15], the following characterisation of $\frac{q}{2}$-graphs is obtained.

Theorem 14 (Henning) *If G is a $\frac{q}{2}$-graph, then $G \in \mathcal{F}^* \cup \mathcal{G}^*$.*

As a consequence of Theorem 14, we have the following result.

Theorem 15 (Henning) *If G is a connected graph of size q with minimum degree at least 2, then either $G \in \mathcal{C}_2$, in which case $\gamma_t(G) = (q+2)/2$, or $G \in (\mathcal{F}^* - \mathcal{C}_2) \cup \mathcal{G}^*$, in which case $\gamma_t(G) = (q+1)/2$, or $\gamma_t(G) \leq q/2$.*

4 Bounds on the restrained domination number

If $G = (V, E)$ is a connected graph of order n, then V is a restrained dominating set, so $\gamma_r(G) \leq n$. The family of stars $K_{1,n-1}$ shows that this bound can be attained. Domke, Hattingh, Henning, and Markus [6] investigated upper bounds on the restrained domination number of a connected graph with minimum degree at least two. Let $\mathcal{B}$ be the collection of graphs shown in Figure 13.

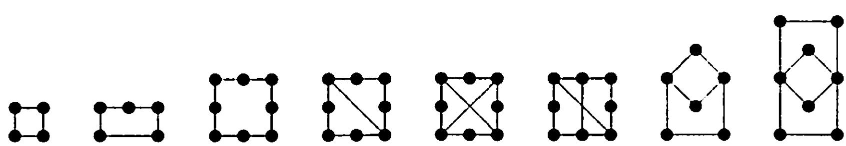


Figure 13. The collection $\mathcal{B}$ of graphs

Theorem 16 (Domke, Hattingh, Henning, Markus) *Let G be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$. If $G \notin \mathcal{B}$, then $\gamma_r(G) \leq (n-1)/2$.*

We will refer to a graph G of order n as an **$(\frac{n-1}{2})$-minimal graph** if G is edge-minimal with respect to satisfying the following three conditions:

(*i*) $\delta(G) \geq 2$,
(*ii*) G is connected, and
(*iii*) $\gamma_r(G) \geq (n-1)/2$.

A characterisation of $(\frac{n-1}{2})$-minimal graphs was obtained in [16]. To describe this characterisation, we let $\mathcal{B}^* = \{B_1, B_2, \ldots, B_5\}$ and we let $\mathcal{F} = \{F_1, F_2, \ldots, F_{22}\}$ be the collection of graphs shown in Figures 14 and 15.

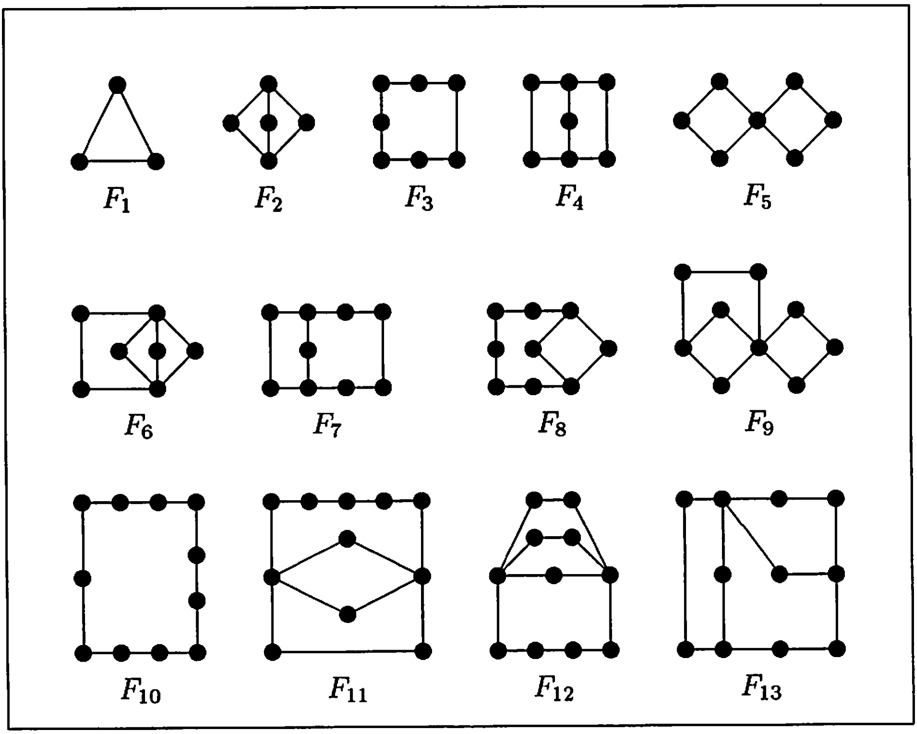


Figure 14. The collection $\{F_1, F_2, \ldots, F_{13}\}$ of graphs

We now construct a collection $\mathcal{H}$ of graphs as follows. Let $H_{1,m}$ be constructed from m disjoint 5-cycles by identifying a set of m vertices, one from each cycle, into one vertex. Let $\mathcal{H}_1 = \{H_{1,m} \,|\, m \geq 2\}$. For $i = 2, 3, \ldots, 7$, let $\mathcal{H}_i = \{H_{i,m} \,|\, m \geq 1\}$ where $H_{i,m}$ is the graph shown in Figure 16. For $i = 8, 9, 10$, let $\mathcal{H}_i = \{H_{i,m,\ell} \,|\, m \geq \ell \geq 1\}$ where $H_{i,m,\ell}$ is the graph shown in Figure 16. Let $\mathcal{H} = \{\mathcal{H}_i \,|\, 1 \leq i \leq 10\}$.

We are now in a position to state the characterisation obtained in [16] of the collection of all $(\frac{n-1}{2})$-minimal graphs.

Theorem 17 (Henning) *A graph G is an $(\frac{n-1}{2})$-minimal graph of order n if and only if $G \in \mathcal{B}^* \cup \mathcal{F} \cup \mathcal{H}$.*

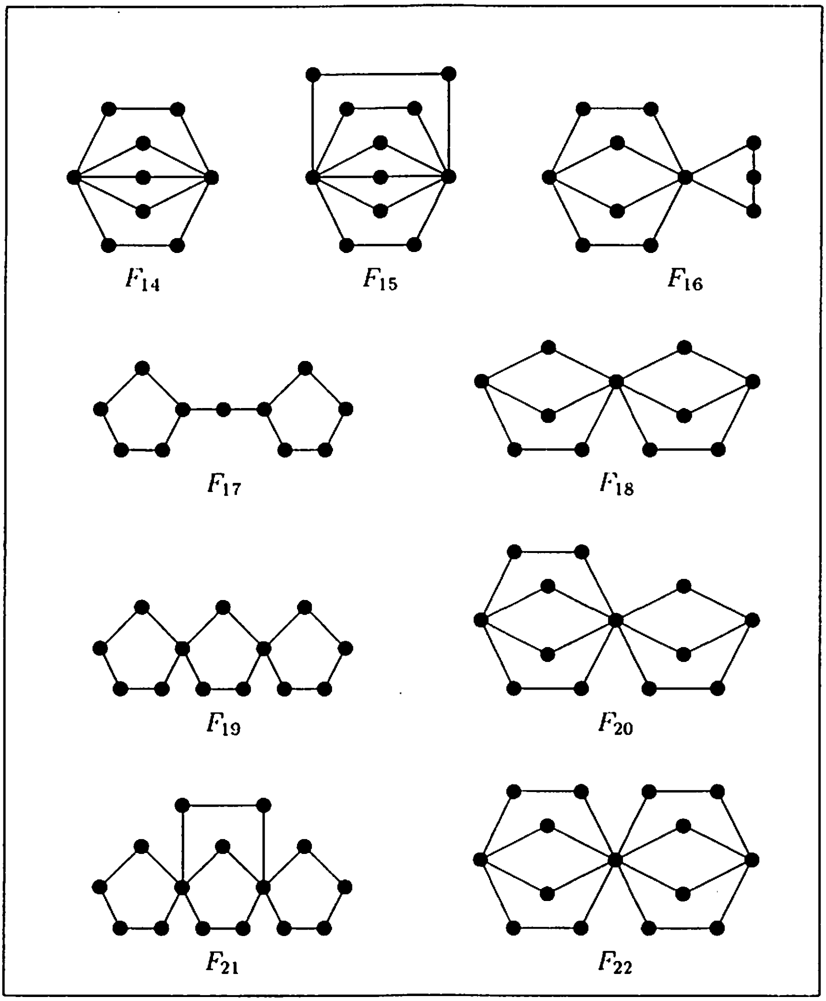


Figure 15. The collection $\{F_{14}, F_{15}, \ldots, F_{22}\}$ of graphs

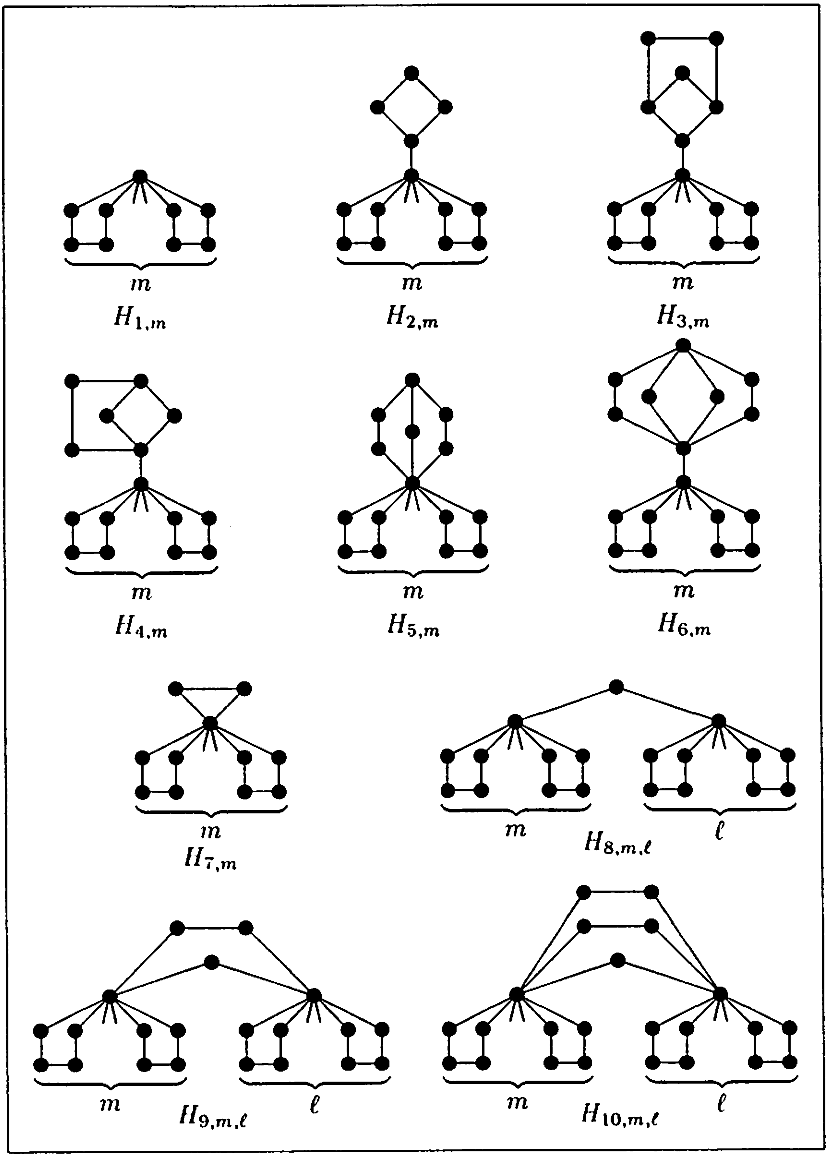


Figure 16. The collection $\mathcal{H}$ of graphs

5 Trees with equal domination and restrained domination numbers

Since every restrained dominating set is a dominating set, $\gamma(G) \leq \gamma_r(G)$ for every graph G. Hattingh and Rautenbach [11] have shown that $i(T) \leq$

$\gamma_e(T)$ for every tree T. Since every independent dominating set is a dominating set, $\gamma(T) \leq i(T)$, and since every restrained dominating set is an end-dominating set, $\gamma_e(T) \leq \gamma_r(T)$. Hence we have the following inequality chain.

Theorem 18 *For any tree T, $\gamma(T) \leq i(T) \leq \gamma_e(T) \leq \gamma_r(T)$.*

A constructive characterisation of trees with equal domination and restrained domination numbers was obtained by Domke, Hattingh, Laskar, and Markus in [5]. To state this characterization, we need to define two types of operations on a tree T.

Type-A operation: Attach a P_2 to a vertex v of T where v is a vertex such that $\gamma(T - v) = \gamma(T)$ and does not belong to some γ_r-set of T;

Type-B operation: Attach a P_3 to a vertex v of T which is in some γ_r-set of T.

We now define the family $\mathcal{F}_r'$ as $\mathcal{F}_r' = \{T \mid T$ is obtained from P_1 by a finite sequence of operations of type-A or type-B $\}$.

Theorem 19 (Domke, Hattingh, Laskar, Markus) *For any tree T, $\gamma(T) = \gamma_r(T)$ if and only if $T \in \mathcal{F}_r'$.*

Hattingh and Henning [8] characterized trees with equal independent domination and restrained domination numbers. To state this characterization, we introduce two types of operations that we use to construct trees with equal independent domination and restrained domination numbers.

Type-1 operation: Attach a path P_2 to a vertex of a tree T which is in no γ_e-set of T (a type-1 operation is illustrated in Figure 17);

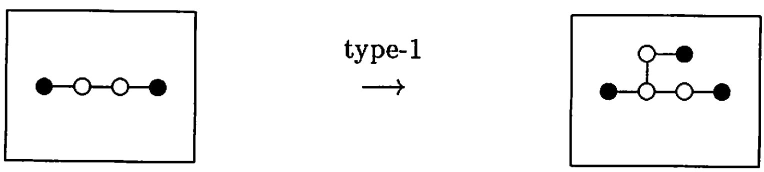


Figure 17. A type-1 operation

Type-2 operation: Attach a path P_3 to a vertex of a tree T which is in a γ_e-set of T (a type-2 operation is illustrated in Figure 18).

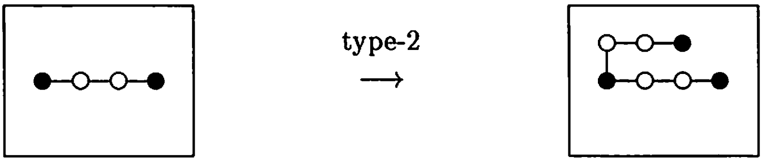


Figure 18. A type-2 operation

We now define the family $\mathcal{F}_r$ as $\mathcal{F}_r = \{T \mid T$ is obtained from P_1 by a finite sequence of operations of type-1 or type-2 $\}$. In [8], a constructive characterisation of those trees with equal independent domination and restrained domination numbers is obtained.

Theorem 20 (Hattingh, Henning) *For a tree T, the following statements are equivalent:*

(a) $\gamma(T) = \gamma_r(T)$.
(b) *Every* γ_e*-set of* T *is a* 2*-packing.*
(c) $i(T) = \gamma_e(T)$.
(d) $i(T) = \gamma_r(T)$.
(e) $T \in \mathcal{F}_r$.
(f) $T \in \mathcal{F}'_r$.

6 Trees with equal domination and weak domination numbers

Since every weak dominating set is a dominating set, $\gamma(G) \leq \gamma_w(G)$ for every graph G. In this section, we present a constructive characterization, due to Hatting and Henning [8], of the family of trees T satisfying $\gamma(T) = \gamma_w(T)$.

If $T \not\cong K_2$ is a tree, then every weak dominating set of T is an end-dominating set of T, and so $\gamma_e(T) \leq \gamma_w(T)$. Hence as a consequence of Theorem 18, we have the following inequality chain.

Corollary 21 *If* $T \not\cong K_2$ *is a tree, then* $\gamma(T) \leq i(T) \leq \gamma_e(T) \leq \gamma_w(T)$.

We now introduce the following type of operation that we use to construct trees with equal domination and weak domination number:

Type-3 operation: Let v be a vertex of a tree T which is in a γ_e-set of T and with $\deg y > \deg v$ for all neighbors y of v in T. Attach a path v, u, w, x to v, and then attach at least $\deg_T v - 1$ (disjoint) P_2's to u.

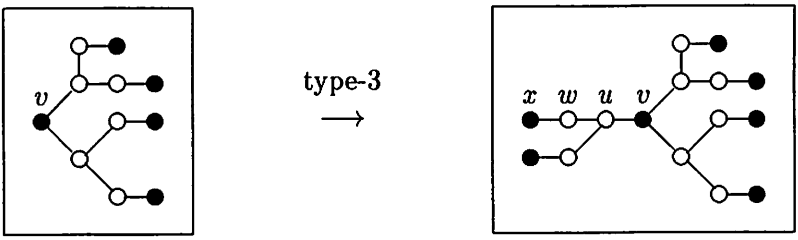


Figure 19. A type-3 operation

Next we define the family $\mathcal{F}_w$ as $\mathcal{F}_w = \{T \mid T$ is obtained from P_1 by a finite sequence of operations of type-1 or type-3 $\}$. The following result is established in [8].

Theorem 22 (Hattingh, Henning) *For a tree T, the following statements are equivalent:*

(a) $\gamma(T) = \gamma_w(T)$.
(b) $T \cong K_2$ or $T \in \mathcal{F}_w$.
(c) $i(T) = \gamma_w(T)$.

7 Bounds on the strong domination number

In this section, we investigate upper bounds on the strong domination number of a connected graph. Let $G = (V, E)$ be a graph, and let $u, v \in V$. We say that v is a *strong neighbour* of u if $uv \in E$ and $\deg u \leq \deg v$. Recall (see Theorem 1) that the domination number of a connected graph is at most half its order. However, the strong domination number of a connected graph of order p may exceed half its order as the following result in [9] shows.

Theorem 23 (Hattingh, Henning) *Let G be a connected graph of order n, and let W be the set of all vertices of G that have no strong neighbours; that is, $W = \{v \in V \mid \deg v > \deg u$ for all vertices u adjacent to $v\}$. Then, $\gamma_{st}(G) \leq (n + |W|)/2$.*

An immediate consequence of Theorem 23 now follows.

Corollary 24 *Let G is a connected graph of order n. If every vertex of G has a strong neighbour, then $\gamma_{st}(G) \leq n/2$.*

The following result in [9] establishes a sharp upper bound on the strong domination number of a connected graph.

Theorem 25 (Hattingh, Henning) *For any connected graph G of order $n \geq 3$, $\gamma_{st}(G) \leq 2(n-1)/3$, and this bound is sharp.*

That the upper bound in Theorem 25 is sharp, may be seen by taking a complete bipartite graph $K(k, k+2)$ and adding an adjacent end-vertex to each vertex of the partite set of cardinality $k+2$, i.e., for each vertex v in the partite set of cardinality $k+2$ we add a new vertex v' and the edge vv'. Let G denote the resulting graph. Then G is a connected graph of order $n = 3k+4$ with $\gamma_{st}(G) = 2(k+1) = 2(n-1)/3$.

Next, we establish a sharp upper bound on the strong domination number of a tree. Let T^* be the tree obtained from a star $K_{1,3}$ by subdividing each edge once. (The tree T^* is shown in Figure 20. The darkened vertices form a minimum strong dominating set of T^*.) Then T^* is a tree of order $n = 7$ with $\gamma_{st}(T^*) = 4 = 4n/7$.

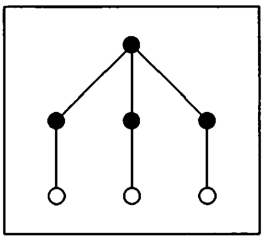


Figure 20. The tree T^* of order $n = 7$ with $\gamma_{st}(T^*) = 4 = 4n/7$

We close with the following result in [9].

Theorem 26 (Hattingh, Henning) *For any tree T of order $n \geq 2$ that is different from the tree T^* of Figure 20, $\gamma_{st}(T) \leq (4n-1)/7$, and this bound is sharp.*

That the upper bound in Theorem 26 is sharp, may be seen as follows. Let F_1 be the tree obtained from a star $K_{1,4}$ by subdividing each edge once, and, for $k \geq 2$, let $F_2, \ldots, F_k$, be $k-1$ disjoint copies of the tree T^* shown in Figure 20. For $i = 1, 2, \ldots, k$, let v_i denote the central vertex of F_i, and let w_i be a vertex adjacent to v_i in F_i. Let $W = \{v_1, v_2, \ldots, v_k\}$. For $k \geq 2$, let T_k be the tree obtained from the disjoint union $\cup_{i=1}^{k} F_i$ of $F_1, F_2, \ldots, F_k$ by the addition of the edges $w_i v_{i+1}$ for $i = 1, \ldots, k-1$. (The tree T_4 is shown in Figure 21. The darkened vertices form a minimum strong dominating set of T_4.) Then T_k is a tree of order $n = 7k+2$ with $\gamma_{st}(T) = |W| + |N(W)| = 4k+1 = (4n-1)/7$.

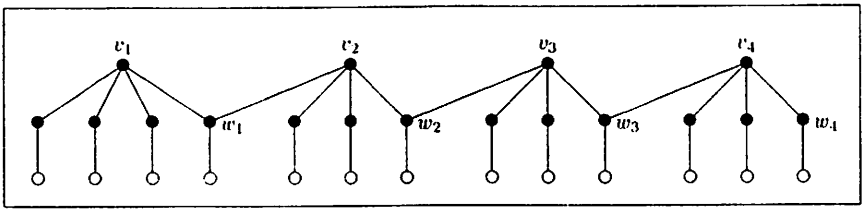


Figure 21. The tree T_4 of order n with $\gamma_{st}(T) = (4n-1)/7$

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