A sufficient condition of the Four Color Theorem*

Lianying Miao
Institute of Mathematics
Shandong University
Jinan, 250100
P.R. China

ABSTRACT. Grotzsch conjectured that if G is planar, bridgeless with $\Delta = 3$ and $n_2 \geq 2$, then G is of Class one. We prove that when $n_2 = 2$ the conjecture is equivalent to the statement: G is 3-critical if G is planar, bridgeless with $\Delta = 3$ and $n_2 = 1$ Then we prove that the conjecture implies the Four Color Theorem.

1 Introduction

Let G be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote $uv \in E(G)$ if $u, v \in V(G)$ are adjacent in G.The degree of u in G denoted by $d(u)$ is the number of vertices adjacent to u of G. $N_G(u)$ is the set of vertices adjacent to u of G and n_i is the number of vertices of degree i in G. A (proper) edge-coloring φ of G is a mapping from $E(G)$ to $\{1, 2, \cdots\}$ such that no two adjacent edges of G have the same image. The chromatic index $\chi'(G)$ of G is the minimum cardinality of the image set of all possible edge-colorings of G. G is said to be k-colorable if there is an edge-coloring of G with image set $\{1, 2, \cdots, k\}$. We denote by C and $C(u)$ the set of colors and the set of colors missing from the vertex u in G, respectively. The well-known theorem of Vizing says that if G is a graph with maximum degree Δ, then either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$. If $\chi'(G) = \Delta$, then G is said to be of Class one, otherwise G is said to be of Class two. A graph G is said to be (chromatic index) critical if G is connected of Class two and $\chi'(G - e) < \chi'(G)$ for any edge e of G. If G is critical and G has maximum degree Δ, we say that G is Δ-critical. The Four Color Thorem is equivalent to the statement: If G is a cubic, planar,

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bridgeless graph, then G is of Class one. Grotzsch posed the following conjecture [5]: If G is planar, bridgeless with $\Delta = 3$ and $n_2 \geq 2$, then G is of Class one. We prove that when $n_2 = 2$ the conjecture is equivalent to the statement: G is 3-critical if G is planar, bridgeless with $\Delta = 3$ and $n_2 = 1$. Then we prove that the statement implies the Four Color Theorem, so the conjecture implies the Four Color Theorem.

2 Main Results

Theorem 1. *Let G be a simple, bridgless, planar graph with $\Delta = 3$, then the following two statements are equivalent:*

(1) *if $n_2 = 2$, then G is of Class one, and*

(2) *if $n_2 = 1$, then G is 3-critical.*

Proof: (1) implies (2). Let $e = xy$ be an arbitrary edge of G. It is sufficient to prove that $G - e$ is 3-edge-colorable. Let G' be the graph obtained by inserting one vertex v in e. G' is 3-edge-colorable by (1), so $G - e = G' - vx - vy$ is also 3-edge-colorable.

(2) implies (1). Let $d(x_1) = d(x_2) = 2$, $x_1, x_2 \in V(G)$. We consider two cases.

Case 1. $x_1x_2 \in E(G)$. Let $N_G(x_1) = \{x_2, u\}$, $N_G(x_2) = \{x_1, v\}$. Let $H = G - x_2 + x_1v$. Then H is 3-critical. Let φ be a 3-edge-coloring of $H - x_1v$. This 3-edge-coloring φ of $H - x_1v$ can be modified to a 3-edge-coloring ψ of G as follows. For any edge e of G

$$\varphi(e) = \begin{cases} \varphi(x_1u), & \text{if } e = x_2v, \\ c \in C(x_1), & \text{if } e = x_1x_2, \\ \varphi(e), & \text{otherwise.} \end{cases}$$

Case 2. $x_1x_2 \notin E(G)$. Let $N_G(x_1) = \{u, v\}$, $N_G(x_2) = \{y, z\}$.

Subcase 2. $uv \notin E(G)$ or $yz \notin E(G)$. Without loss of generality we may assume that $uv \notin E(G)$.

Let $H = G - x_1 + uv$. H is 3-critical. Let φ be a 3-edge-coloring of $H - uv$. We choose a 3-edge-coloring ψ of G as follows. For any edge e of G

$$\psi(e) = \begin{cases} c_1 \in C(u), & \text{if } e = x_1u, \\ c_2 \in C(v), & \text{if } e = x_1v, \\ \varphi(e), & \text{otherwise.} \end{cases}$$

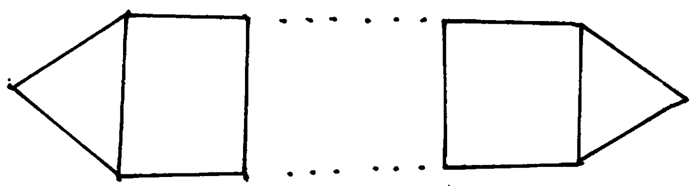
Subcase 2.2. $uv \in E(G)$, $yz \in E(G)$.

A_1. Starting with the vertex x_1, we can find i such that $u_iv_i \notin E(G)$, but $u_ku_{k+1} \in E(G)$, $v_kv_{k+1} \in E(G)$, $u_kv_k \in E(G)$ for $1 \le k \le i-1$, and $vv_1 \in E(G)$, $uu_1 \in E(G)$. Let

$$H = G - x_1 - u - v - u_1 - v_1 - \cdots - u_{i-2} - v_{i-2} - v_{i-1} + u_{i-1}v_i.$$

H has a 3-edge-colorable φ by Subcase 2.1, then we can get a 3-edge-coloring φ of G by setting $\psi(v_iv_{i-1}), \psi(v_{i-1}v_{i-2}), \ldots, \psi(v_1v), \psi(vx)$ and $\psi(u_{i-1}u_{i-2}), \psi(u_{i-2}u_{i-3}), \ldots, \psi(u_1u), \psi(ux_1)$ to be $\psi(u_{u-1}v_i)$ and $\psi(u_iu_{i-1})$ alternatively, and setting $\psi(u_iv_i)$ to be $\{1,2,3\} - \{\varphi(u_{i-1}v_i), \varphi(u_iu_{i-1})\}$ for all $1 \le k \le i-1$, $\psi(e) = \varphi(e)$ for any other edge e of G. So G is 3-edge-colorable.

A_2. G is a ladder-like graph as indicated in the following figure. It is easy to verify that G is 3-edge-colorable.



This completes the proof of the theorem.

Theorem 2. *The statement (1) of Theorem 1 implies the Four Color Theorem.*

Proof: Let G be a cubic, planar, bridgeless graph and $e = xy$ be an arbitrary edge of G. Let G' be the graph obtained by inserting one vertex v in e. $G' - vy$ is 3-edge-colorable as G is 3-critical and in any 3-edge-coloring φ of $G' - vy$, $\varphi(vx) \in C(y)$. Then φ can be modified to a 3-edge-coloring of G. This completes the proof of the theorem.

From the above theorems, we know that it is not easy to prove the conjecture of Grotzsch without using the Four Color Theorem.

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