The spectrum problem for closed m-trails, $m \leqslant 10$

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ABSTRACT: For every connected, even-degree graph G with 10 or fewer edges, the problem of finding necessary and sufficient conditions for the existence of a decomposition of K_v into edge-disjoint copies of G is completely settled.

1 Introduction

A decomposition of a graph H into edge-disjoint copies of a given graph G is called a G-design of H. The problem of determining all values of v for which there is a G-design of K_v is called the spectrum problem for G.

The spectrum problem has been considered for many graphs. For example, if G is a complete graph on k vertices, then a G-design of K_v is a $(v, k, 1)$-BIBD. A great deal of work has been done on the spectrum problem in the case where G is an m-cycle (see [17]). Other graphs for which the spectrum problem has been considered include all graphs on 5 vertices or fewer (see [8] and [9]), cubes (see [4] and [10]), Platonic graphs (see [2]) and the Petersen graph (see [1, 5]).

A *closed m-trail* is a connected graph with exactly m distinct edges and with all vertices of even degree. When $m = 3, 4$ and 5, the only closed m-trails are cycles. Thus, the spectra for closed 3-trails, 4-trails and 5-trails are well-known (see [16], [6] and [14], respectively). There are two non-isomorphic closed 6-trails (Graphs 6.1 and 6.2 in Figure 1). The spectrum

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for 6.1 can be found in [6] and for 6.2 in [9]. Adams and Bryant in [3] study cyclically generated closed m-trail designs of K_{2m+1} for $m \leqslant 10$. In this paper we completely settle the spectrum problem for every closed m-trail, $7 \leqslant m \leqslant 10$.

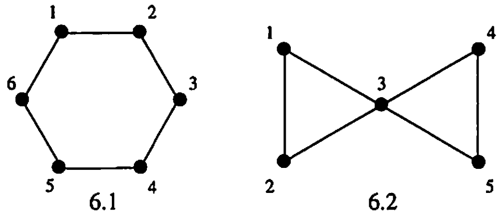

Figure 1

It is straightforward to check that the following conditions are necessary for the existence of a G-design of K_v when G is a closed m-trail.

(1) $|V(G)| \leq v$;

(2) v is odd; and

(3) $v(v-1) \equiv 0 \pmod{2m}$.

We make use of *group divisible designs*. A *group divisible design*, $(K, \lambda, M; v)$GDD, is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, where the v-set is partitioned into disjoint subsets (called groups) of size $g \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in λ blocks. If $M = \{g\}$ and $K = \{k\}$ we write $(k, \lambda, g; v)$GDD. Also, a GDD with exactly one group of size g_2 and the remaining groups of size g_1 is denoted by $(K, \lambda, \{g_1, g_2^*\}; v)$GDD. Similarly, a GDD with exactly one block of size k_2 and the remaining blocks of size k_1 is denoted by $(\{k_1, k_2^*\}, \lambda, M; v)$GDD.

We will need some notation. We denote by $K_{r(s)}$ the complete multipartite graph with r parts each of size s. The complete graph of order v with a hole of size u (that is, the graph with vertex set V and edge set $\{ab : a, b \in V \setminus U, a \neq b\} \cup \{ab : a \in V \setminus U \text{ and } b \in U\}$ where $|V| = v$, $|U| = u$ and $U \subseteq V$) is denoted by $K_v \setminus K_u$. The vertices in U are said to be *the vertices in the hole.*

In Figures 1 to 5 all of the non-isomorphic closed m-trails, $6 \leq m \leq 10$, are shown (see [3]), and the distinct closed m-trails are labelled $m.1$, $m.2$, and so on. For example, when $m = 7$, the closed m-trails are labelled 7.1, 7.2 and 7.3.

The Appendix gives decompositions of a number of graphs into closed m-trails. If an m-trail contains n distinct vertices, then we denote by

$[v_1, v_2, v_3, \ldots, v_n]$ the closed m-trail with edges shown in the corresponding figure. For example, Figure 1 shows a 6-trail labelled 6.2, sometimes called a *bowtie*. When considering this 6-trail, we denote by $[v_1, v_2, v_3, v_4, v_5]$ the bowtie with edges v_1v_2, v_2v_3, v_1v_3, v_3v_4, v_4v_5, v_3v_5.

2 Constructions

We will make use of the following well-known construction.

Lemma 1 *Suppose there exists a $(K, 1, M; v)$GDD and let $g' \in M$. If there exists*

(1) a closed m-trail design of $K_{r(s)}$ for each $r \in K$;

(2) a closed m-trail design of $(K_{sg+h} \setminus K_h)$ for each $g \in M \setminus \{g'\}$; and

(3) a closed m-trail design of $K_{sg'+h}$,

then there exists a closed m-trail design of K_{sv+h}.

2.1 Closed 7-trail designs

There are three non-isomorphic closed 7-trails (see Figure 2).

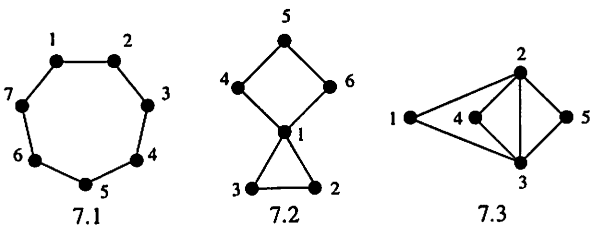

Figure 2

Theorem 2 *For each closed 7-trail G, there exists a G-design of K_v if and only if $v \equiv 1, 7 \pmod{14}$.*

Proof: From the necessary conditions we have $v \equiv 1, 7 \pmod{14}$. Now let G be a closed 7-trail. If G is Graph 7.1 or Graph 7.3 then the result follows by [14] or [9], respectively. For Graph 7.2 we apply Lemma 1 with GDDs (see [11]) and G-designs as shown in Table 1. The G-designs used in this table, except a G-design of K_{15} are given in the Appendix. For a G-design of K_{15} see [3]. □

v	GDDs used	G-designs of
$7(6w)+1,\ w \geqslant 1$	$(3,1,2;6w)$GDD	$K_{3(7)}$, K_{15}
$7(6w+2)+1,\ w \geqslant 1$	$(3,1,2;6w+2)$GDD	$K_{3(7)}$, K_{15}
$7(6w+4)+1,\ w \geqslant 1$	$(3,1,\{2,4^*\};6w+4)$GDD	$K_{3(7)}$, K_{15}, K_{29}
$7(6w+1),\ w \geqslant 0$	$(3,1,1;6w+1)$GDD	$K_{3(7)}$, K_7
$7(6w+3),\ w \geqslant 0$	$(3,1,1;6w+3)$GDD	$K_{3(7)}$, K_7
$7(6w+5),\ w \geqslant 0$	$(\{3,5^*\},1,1;6w+5)$GDD	$K_{3(7)}$, $K_{5(7)}$, K_7

Table 1

2.2 Closed 8-trail designs

There are five non-isomorphic closed 8-trails (see Figure 3).

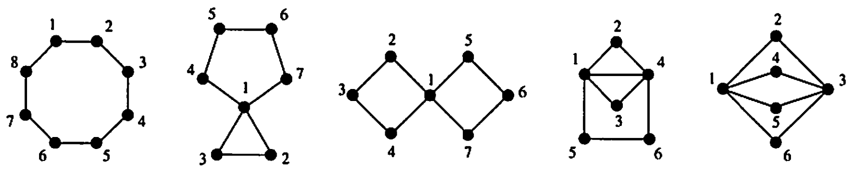

Figure 3

Theorem 3 *For each closed 8-trail G, there exists a G-design of K_v if and only if $v \equiv 1$ (mod 16).*

Proof: From the necessary conditions we have $v \equiv 1 \pmod{16}$. Now let G be a closed 8-trail. If G is Graph 8.1 then the result follows by [6]. For the other graphs we proceed as follows. Let $v = 16w+1$. If $w \equiv 0, 1 \pmod 3$, $w \geqslant 3$, then we apply Lemma 1 with a $(3,1,2;2w)$GDD, a G-design of K_{17} (see [3]) and a G-design of $K_{8,8,8}$ (see the Appendix). If $w \equiv 2 \pmod 3$, $w \geqslant 5$, then we apply Lemma 1 with a $(3,1,\{2,4^*\};2w)$GDD, a G-design of K_{17}, a G-design of K_{33} (see the Appendix) and a G-design of $K_{8,8,8}$. □

2.3 Closed 9-trail designs

There are nine non-isomorphic closed 9-trails (see Figure 4).

Theorem 4 *For each closed 9-trail G, there exists a G-design of K_v if and only if $v \equiv 1, 9$ (mod 18), and $v \neq 9$ for Graph 9.8.*

Proof: From the necessary conditions we have $v \equiv 1, 9 \pmod{18}$. Now let G be a closed 9-trail. If G is Graph 9.1, Graph 9.8 or Graph 9.9 then

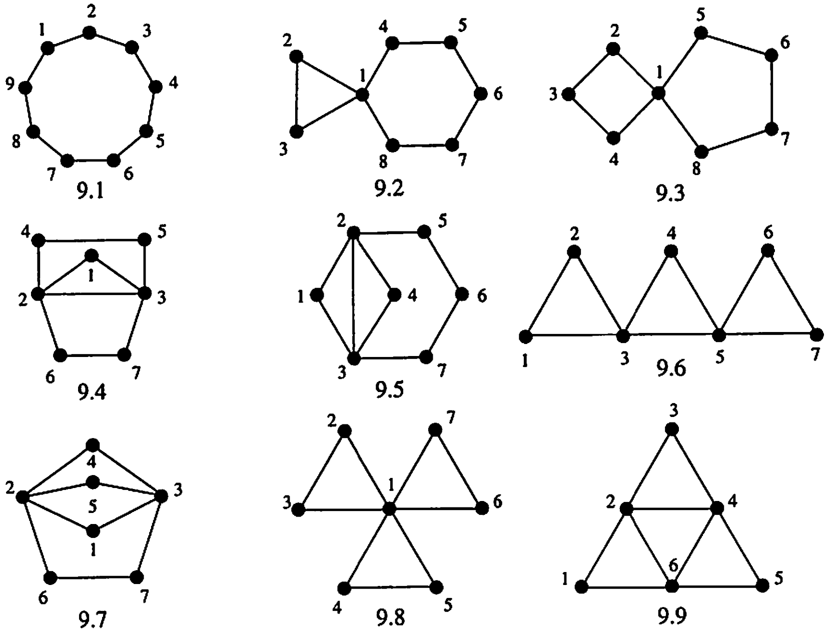

Figure 4

the result follows by [14], [15] or [18], respectively. For the other graphs we proceed as follows. If $v = 3(6w) + 1$, $w \geqslant 3$, then we apply Lemma 1 with a $(3, 1, 6; 6w)$GDD (see [11]), a G-design of K_{19} (see [3]) and a G-design of $K_{3,3,3}$ (see the Appendix). A G-design for the isolated case, $v = 37$, is given in the Appendix. If $v = 3(6w + 3)$, $w \geqslant 1$, then we apply Lemma 1 with a $(3, 1, 3; 6w+3)$GDD (see [11]), a G-design of K_9 (see the Appendix) and a G-design of $K_{3,3,3}$. □

2.4 Closed 10-trail designs

There are nineteen non-isomorphic closed 10-trails (see Figure 5).

Lemma 5 *Let G be a 10-trail. There exists a G-design of $K_{25} \setminus K_5$.*

Proof: If G is Graph 10.1 or Graph 10.19 then the result follows by [7] or [13], respectively. For the other graphs we proceed as follows. First we note that a G-design of $K_{25} \setminus K_5$ has 29 copies of G. Secondly we can decompose a $K_{10} \setminus F$, where F is a 1-factor of K_{10}, into four copies of G (see the Appendix). Finally, there exists a cyclic G-design of $(K_{25} \setminus K_5) \setminus (K_{10} \setminus F)$ (where the K_5 and the K_{10} are vertex disjoint) with five starters; see the Appendix. □

Lemma 6 *Let G be Graph 10.i, $i \neq 1, 17, 18, 19$. If $v \equiv 1, 5$ (mod 20) and $v \neq 5$ then there exists a G-design of K_v.*

Proof: If $v = 10(2w) + 1$, $w \geqslant 3$, then we apply Lemma 1 with a $(3, 1, \{2, 4\}; 2w)$GDD (see [11]), a G-design of $K_{10,10,10}$ (see the Appendix), a G-design of K_{21} (see [3]) and if $w \equiv 2$ (mod 3) a G-design of K_{41} (see the Appendix). If $v = 10(2w) + 5$, $w \geqslant 3$, then we apply Lemma 1 with a $(3, 1, \{2, 4\}; 2w)$GDD (see [11]), a G-design of $K_{10,10,10}$, a G-design of $K_{25} \setminus K_5$ (see Lemma 5) and if $w \equiv 0, 1$ (mod 3) a G-design of K_{25} (see the Appendix) otherwise a G-design of K_{45} (see the Appendix). □

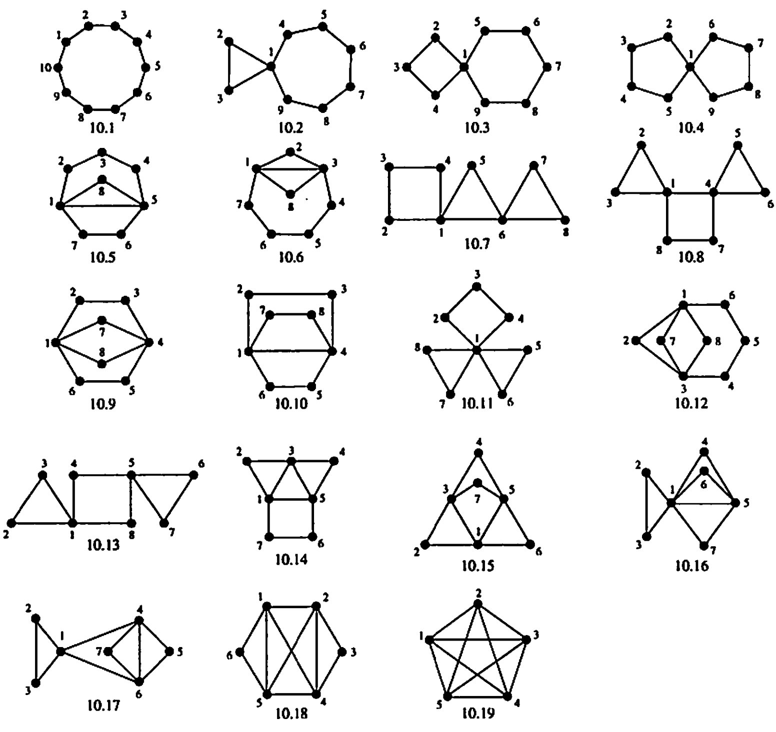

Figure 5

The above construction does not work for Graph 10.18 since this graph is not tripartite. Moreover a decomposition of $K_{10,10,10}$ into Graph 10.17 has not been found. For Graphs 10.17 and 10.18 we use a different construction and need the following lemma.

Lemma 7 *Let $w \equiv 0$ (mod 4) and $w \geqslant 20$.*

(1) *There exists a $(5, 1, 4; w)$GDD for all $w \equiv 0, 4$ (mod 20).*

(2) *There exists a* $(5,1,\{4,8^*\};w)$*GDD for all* $w \equiv 8, 16 \pmod{20}$*,* $w \neq 28, 48$.

(3) *There exists a* $(5,1,\{4,12^*\};w)$*GDD for all* $w \equiv 12 \pmod{20}$*,* $w \neq 32$.

Proof: For part (1) see [13] and for parts (2) and (3) see [12]. □

Lemma 8 *Let G be Graph* 10.17 *or* 10.18*. If* $v \equiv 1, 5 \pmod{20}$ *and* $v \neq 5$ *then there exists a G-design of* K_v*.*

Proof: Let $v \equiv 1, 5 \pmod{20}$. We apply Lemma 1 with GDDs (see Lemma 7) and G-designs as shown in Table 2. The G-designs used in this table, except a G-design of K_{21}, are given in the Appendix. For a G-design of K_{21} see [3]. For G-designs of K_v, $v \in \{81, 85, 141, 145, 161, 165, 241, 245\}$, the remaining cases, we apply Lemma 1 with GDDs and G-designs as shown in Table 3. The G-designs used in this table are given in the Appendix. □

v	GDDs used	G-designs of
$100w+1,\ w \geqslant 1$	$(5,1,4;20w)$GDD	$K_{5(5)}$,K_{21}
$100w+5,\ w \geqslant 1$	$(5,1,4;20w)$GDD	$K_{5(5)}$,K_{25},$K_{25} \setminus K_5$
$100w+21,\ w \geqslant 1$	$(5,1,4;20w+4)$GDD	$K_{5(5)}$,K_{21}
$100w+25,\ w \geqslant 1$	$(5,1,4;20w+4)$GDD	$K_{5(5)}$,K_{25},$K_{25} \setminus K_5$
$100w+41,\ w \geqslant 3$	$(5,1,\{4,8^*\};20w+8)$GDD	$K_{5(5)}$, K_{21},K_{41}
$100w+45,\ w \geqslant 3$	$(5,1,\{4,8^*\};20w+8)$GDD	$K_{5(5)}$,K_{45},$K_{25} \setminus K_5$
$100w+61,\ w \geqslant 2$	$(5,1,\{4,12^*\};20w+12)$GDD	$K_{5(5)}$, K_{21},K_{61}
$100w+65,\ w \geqslant 2$	$(5,1,\{4,12^*\};20w+12)$GDD	$K_{5(5)}$,K_{65},$K_{25} \setminus K_5$
$100w+81,\ w \geqslant 1$	$(5,1,\{4,8^*\};20w+16)$GDD	$K_{5(5)}$, K_{21},K_{41}
$100w+85,\ w \geqslant 1$	$(5,1,\{4,8^*\};20w+16)$GDD	$K_{5(5)}$,K_{45},$K_{25} \setminus K_5$

Table 2

v	GDDs used	G-designs of
81	$(4,1,1;4)$GDD	$K_{4(20)}$, K_{21}
85	$(4,1,1;4)$GDD	$K_{4(20)}$, K_{25}, $K_{25} \setminus K_5$
141	$(4,1,2;14)$GDD	$K_{4(10)}$, K_{21}
145	$(4,1,2;14)$GDD	$K_{4(10)}$, K_{25}, $K_{25} \setminus K_5$
161	$(8,1,1;8)$GDD	$K_{8(20)}$, K_{21}
165	$(8,1,1;8)$GDD	$K_{8(20)}$, K_{25}, $K_{25} \setminus K_5$
241	$(12,1,1;12)$GDD	$K_{12(20)}$, K_{21}
245	$(12,1,1;12)$GDD	$K_{12(20)}$, K_{25}, $K_{25} \setminus K_5$

Table 3

Theorem 9 *For each closed 10-trail G, there exists a G-design of K_v if and only if $v \equiv 1, 5 \pmod{20}$ and $v \neq 5$ except that there exists a G-design of K_5 when G is Graph 10.19.*

Proof: The result for Graph 10.1 and Graph 10.19 follows by [17] and [13], respectively. For the other graphs the result follows by Lemmas 6 and 8.

References

[1] P. Adams and D.E. Bryant, *The spectrum problem for the Petersen graph*, Journal of Graph Theory, **22** (1996), 1–6.

[2] P. Adams and D.E. Bryant, *Decomposing the complete graph into Platonic graphs*, Bulletin of the Institute of Combinatorics and its Applications, **17** (1996), 19–26.

[3] P. Adams and D.E. Bryant, *Cyclically generated closed m-trail systems of order* $(2m + 1)$, $m \leqslant 10$, Journal of Combinatorial Mathematics and Combinatorial Computing, **17** (1995), 3–19.

[4] P. Adams, D.E. Bryant and S. El-Zanati, *Lambda-fold cube decompositions*, Australasian Journal of Combinatorics, **11** (1995), 197–210.

[5] P. Adams, D.E. Bryant and A. Khodkar, *The spectrum problem for λ-fold Petersen graph designs*, Journal of Combinatorial Mathematics and Combinatorial Computing (to appear).

[6] B. Alspach and B.N. Varma, *Decomposing complete graphs into cycles of length $2p^e$*, Discrete Mathematics, **9** (1980), 155–169.

[7] D. E. Bryant, C. A. Rodger and E. R. Spicer, Embeddings of m-cycle systems and incomplete m-cycle systems: $m \leq 14$, Discrete Mathematics, **171** (1997), 55–75.

[8] J.-C. Bermond and J. Schönheim, *G-Decomposition of K_n where G has four vertices or less*, Discrete Mathematics, **19** (1977), 113–120.

[9] J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau, *Decomposition of Complete Graphs into Isomorphic Subgraphs with Five Vertices*, Ars Combinatoria, **10** (1980), 211–254.

[10] D.E. Bryant, S.I. El-Zanati and R.B. Gardner, *Decompositions of $K_{m,n}$ and K_n into cubes*, Australasian Journal of Combinatorics, **9** (1994), 285-290.

[11] C.J. Colbourn, D.G. Hoffman and R. Rees, *A new class of group divisible designs with block size three*, Journal of Combinatorial Theory, Ser. A **59** (1992), 73–89.

[12] A.M. Hamel, W.H. Mills, R.C. Mullin, R. Rees, D.R. Stinson and J. Yin, *The spectrum problem of PBD* $(\{5, k^*\}, v)$ *for* $k = 9, 13$, Ars Combinatoria, **36** (1993), 7–26.

[13] H. Hanani, *Balanced incomplete block designs*, Discrete Mathematics, 11 (1975), 255–369.

[14] D.G. Hoffman, C.C. Lindner and C.A. Rodger, *On the construction of odd cycle systems* Journal of Graph Theory, **13** (1989) 417–426.

[15] P. Horak and A. Rosa, *Decomposing Steiner triple systems into small configurations*, Ars Combinatoria, **26** (1988), 91–105.

[16] Rev. T.P. Kirkman, *On a problem in combinatorics*, Cambr. and Dublin Math., J. **2** (1984), 191–204.

[17] C.C. Lindner and C.A. Rodger, *Decomposition into cycles II: Cycle systems* in Contemporary design theory: a collection of surveys (J. H. Dinitz and D. R. Stinson, eds), John Wiley and Sons, New York, pp. 325-369 (1992).

[18] R.C. Mullin, A.L. Poplove and L. Zhu, *Decomposition of Steiner triple systems into triangles*, Journal of Combinatorial Mathematics and Combinatorial Computing, **1** (1987), 149–174.

Appendix

This appendix contains decompositions of various graphs H into copies of various m-trails G, $7 \leq m \leq 10$. In each case, the graph H is listed first, followed by a table containing the required G-designs of H. Each G-design is given as a pair (V, M), where V is the vertex set of H, and M is the collection of copies of G.

Often the decompositions are cyclic, so M is actually a starter set of m-trails. Vertices of the form (i, j) are written as i_j. Cycling a vertex i_j modulo a_- means i is cycled modulo a, and j remains unchanged. For brevity, in some cases the commas between vertices in a trail have been omitted; this is only done where there is no possible confusion.

Closed 7-trail designs

K_7 $V = \{0, 1, \ldots 6\}$. M as follows:

7.2	[0, 1, 2, 3, 4, 5]	[1, 3, 5, 4, 2, 6]	[6, 0, 4, 3, 2, 5]

$K_{7,7,7}$ $V = \{i_j \mid 0 \leq i \leq 6;\ j = 1, 2, 3\}$. M as follows, cycled modulo 7_-:

7.2	$[0_1, 0_2, 0_3, 1_2, 2_1, 4_2]$	$[0_1, 3_2, 1_3, 2_3, 4_1, 3_3]$	$[0_2, 6_3, 2_1, 1_3, 5_2, 2_3]$

$K_{7,7,7,7,7}$ $V = \mathbb{Z}_{35}$. M as follows, cycled modulo 35:
(Part $i + 1$ contains the vertices $\{0 + i, 7 + i, 14 + i, 21 + i, 28 + i\}$, $0 \leq i \leq 6$.)

7.2	[0, 1, 3, 4, 10, 17]	[0, 9, 21, 8, 24, 11]

K_{29} $V = \mathbb{Z}_{29}$. M as follows, cycled modulo 29:

7.2	[0, 1, 3, 4, 9, 16]	[0, 6, 14, 9, 21, 10]

Closed 8-trail designs

K_{33} $V = \mathbb{Z}_{33}$. M as follows, cycled modulo 33:

8.2	[0, 1, 3, 4, 9, 2, 10]	[0, 6, 15, 11, 23, 3, 17]
8.3	[0, 1, 3, 6, 4, 9, 16]	[0, 8, 19, 9, 12, 27, 13]
8.4	[0, 1, 2, 5, 6, 14]	[0, 7, 10, 21, 16, 1]
8.5	[0, 1, 3, 6, 7, 8]	[0, 9, 25, 10, 11, 12]

$K_{8,8,8}$ $V = \{i_j \mid 0 \leq i \leq 7;\ j = 1, 2, 3\}$. M as follows, cycled modulo 8_-:

8.2	$[0_1 0_2 0_3 1_2 2_1 4_2 1_3]$	$[0_1 3_2 2_3 4_2 0_3 1_1 4_3]$	$[0_1 5_2 6_3 6_2 0_3 2_2 5_3]$
8.3	$[0_1 0_2 1_1 2_2 3_2 5_1 0_3]$	$[0_1 4_2 7_1 1_3 4_3 6_1 5_3]$	$[0_2 0_3 1_2 2_3 3_3 6_2 4_3]$
8.4	$[0_1 0_2 1_2 0_3 2_2 3_1]$	$[0_1 3_2 4_2 1_3 2_3 6_1]$	$[0_1 5_2 6_2 7_3 6_3 3_2]$
8.5	$[0_1 0_2 1_1 2_2 4_2 6_2]$	$[0_1 0_3 1_1 2_3 4_3 6_3]$	$[0_2 0_3 1_2 2_3 4_3 6_3]$

Closed 9-trail designs

$\boxed{K_9}$ $V = \{0, 1, \ldots 8\}$. M as follows:

9.2	[0,1,2,3,4,5,6,7]	[0,4,6,5,1,3,2,8]	[7,1,4,2,6,3,5,8]
	[8,1,6,3,7,5,2,4]		
9.3	[0,1,2,3,4,5,6,7]	[0,2,4,6,5,1,3,8]	[7,1,4,3,5,2,6,8]
	[8,2,7,4,1,6,3,5]		
9.4	[0,1,2,3,4,5,6]	[0,3,5,2,7,6,8]	[0,4,8,5,2,6,7]
	[4,1,7,6,0,8,3]		
9.5	[0,1,2,3,4,5,6]	[0,3,4,6,5,7,2]	[0,5,8,2,1,7,6]
	[3,7,8,4,0,6,1]		
9.6	[0,1,2,3,4,5,6]	[0,3,5,2,7,1,4]	[0,4,8,2,6,1,3]
	[0,6,7,3,8,1,5]		
9.7	[0,1,2,3,4,5,6]	[0,3,4,5,6,7,8]	[0,5,6,7,8,2,1]
	[0,7,8,1,2,4,3]		

$\boxed{K_{37}}$ $V = \mathbb{Z}_{37}$. M as follows, cycled modulo 37:

9.2	[0,1,3,4,9,2,8,16]	[0,9,19,11,24,4,26,12]
9.3	[0,1,3,6,4,9,2,10]	[0,9,26,11,12,25,2,18]
9.4	[0,1,3,5,10,7,15]	[0,9,19,20,4,22,5]
9.5	[0,1,3,7,6,13,21]	[0,9,20,30,23,8,32]
9.6	[0,1,3,7,12,2,20]	[0,6,21,1,8,19,31]
9.7	[0,1,2,5,7,8,16]	[0,9,10,25,27,33,21]

$\boxed{K_{3,3,3}}$ $V = \{0, 1, 2\} \cup \{3, 4, 5\} \cup \{6, 7, 8\}$. M as follows:

9.2	[0,3,6,4,1,5,2,7]	[2,4,8,3,1,7,5,6]	[8,0,5,1,6,4,7,3]
9.3	[0,3,1,6,4,8,5,7]	[1,4,2,7,5,6,3,8]	[2,5,0,8,3,7,4,6]
9.4	[0,3,6,1,4,2,5]	[1,5,7,0,4,8,3]	[4,2,8,6,1,7,0]
9.5	[0,3,6,1,2,7,4]	[0,4,8,1,2,6,5]	[0,5,7,1,2,8,3]
9.6	[0,3,6,1,4,2,7]	[0,4,8,1,5,2,6]	[0,5,7,1,3,2,8]
9.7	[0,3,6,1,2,7,4]	[1,4,7,2,0,8,5]	[2,5,8,0,1,6,3]

Closed 10-trail designs

$\boxed{K_{10} \setminus F}$ $V = \{0, 1, \ldots, 9\}$. $F = \{(0,1), (2,3), (4,5), (6,7), (8,9)\}$. M as follows:

10.2	[0,1,3,2,4,5,7,6,8]	[0,4,6,5,1,8,2,3,7]	[9,1,4,6,3,8,5,2,7]
	[9,3,5,2,6,1,7,4,8]		
10.3	[0,1,3,2,4,5,7,6,8]	[0,3,8,5,6,1,4,9,7]	[2,5,1,7,4,8,9,3,6]
	[9,1,8,2,5,3,7,4,6]		
10.4	[0,1,3,2,4,5,7,6,8]	[0,2,5,1,6,3,8,4,7]	[9,1,4,5,8,3,7,2,6]
	[9,2,8,1,7,4,6,3,5]		
10.5	[0,1,3,2,4,5,7,6]	[0,2,6,1,5,9,3,8]	[1,9,6,3,7,2,8,4]
	[8,3,5,2,9,7,6,4]		
10.6	[0,1,3,2,4,5,7,6]	[0,2,5,1,7,9,4,8]	[4,1,6,2,9,3,8,7]
	[8,1,9,5,3,7,2,6]		
10.7	[0,1,3,2,4,5,7,9]	[0,3,9,8,6,7,1,4]	[2,5,3,8,4,9,1,6]
	[6,2,7,3,4,8,1,5]		
10.8	[0,1,3,2,4,5,6,7]	[0,4,6,5,1,7,9,8]	[2,3,7,8,1,4,6,9]
	[3,5,8,9,4,7,1,6]		
10.9	[0,1,3,2,4,5,6,7]	[0,2,5,1,6,3,4,8]	[7,3,5,9,8,4,1,6]
	[8,5,7,9,4,6,2,3]		
10.10	[0,1,3,2,4,5,6,7]	[0,3,5,8,6,4,7,9]	[1,4,7,5,2,8,6,9]
	[3,6,2,9,1,7,8,4]		
10.11	[0,1,3,2,4,5,6,7]	[7,1,6,2,3,5,4,9]	[8,0,3,6,1,4,2,5]
	[9,2,4,6,1,5,3,8]		
10.12	[0,1,3,2,4,5,6,7]	[0,2,5,7,6,4,3,8]	[1,4,9,7,2,6,5,8]
	[8,2,9,1,7,4,3,6]		
10.13	[0,1,3,2,4,5,7,6]	[1,4,9,5,2,3,6,8]	[5,8,9,0,7,1,6,3]
	[8,0,4,3,9,2,7,6]		
10.14	[0,1,3,2,5,4,6]	[0,2,4,1,7,3,8]	[1,5,8,2,9,3,6]
	[6,2,7,5,9,4,8]		
10.15	[0,1,3,2,4,5,6]	[0,2,7,1,8,6,9]	[2,5,8,3,9,6,4]
	[5,3,7,4,1,9,6]		
10.16	[0,1,3,2,4,5,7]	[2,6,9,3,5,7,8]	[6,1,7,0,8,3,4]
	[9,3,7,4,1,5,8]		
10.17	[0,1,3,2,4,5,7]	[1,5,8,6,0,7,3]	[6,2,8,4,1,9,7]
	[8,0,4,3,2,9,5]		
10.18	[0,1,3,2,5,7]	[0,3,8,1,9,6]	[1,4,2,9,7,6]
	[2,0,4,9,5,8]		

$(K_{25} \setminus K_5) \setminus (K_{10} \setminus F)$ $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2,3,4,5\}$. M as follows, cycled modulo 5_-:

10.2	$[0_1 0_2 0_3 1_2 2_1 4_2 1_1 2_3 0_4]$	$[0_3 1_4 0_5 2_4 0_4 3_5 1_3 2_5 4_5]$	$[0_1 2_3 1_4 3_3 4_1 2_4 3_1 0_4 0_5]$
	$[0_1 1_5 2_5 3_5 0_2 0_4 1_2 2_4 4_5]$	$[0_2 2_4 3_4 0_5 1_2 3_5 0_3 0_4 1_5]$	
10.3	$[0_1 0_2 1_1 2_2 3_2 3_3 2_1 0_3 0_4]$	$[0_1 0_3 1_1 2_4 2_3 0_4 2_1 1_4 0_5]$	$[0_1 1_5 0_2 2_5 3_5 3_2 0_4 1_2 4_5]$
	$[0_2 0_4 1_3 3_4 1_4 0_3 0_5 2_3 4_5]$	$[0_4 1_4 3_4 0_5 1_5 0_3 4_5 2_5 3_5]$	
10.4	$[0_1 0_2 1_1 2_2 2_3 3_2 0_4 2_1 0_3]$	$[0_1 2_2 0_4 1_1 1_4 1_3 2_1 4_4 0_5]$	$[0_3 3_4 4_3 4_5 2_5 1_5 2_4 0_4 3_5]$
	$[0_2 1_4 0_3 0_4 4_4 0_5 1_3 3_4 3_5]$	$[0_1 1_5 0_2 0_4 2_5 3_5 1_2 0_5 4_5]$	

$(K_{25} \setminus K_5) \setminus (K_{10} \setminus F)$ (continued)

10.5	$[0_1 0_2 1_1 2_2 2_3 2_1 1_3 0_4]$ $[0_2 1_4 2_2 4_4 2_5 0_3 4_5 3_5]$	$[0_1 2_2 4_1 2_3 1_4 2_1 0_5 2_4]$ $[0_3 0_4 2_4 1_5 1_4 4_3 0_5 3_5]$	$[0_1 3_4 0_2 0_4 1_5 1_2 2_5 4_5]$
10.6	$[0_1 0_2 0_3 1_1 2_2 3_1 1_3 0_4]$ $[0_2 1_4 3_4 1_2 1_5 0_3 2_5 3_5]$	$[0_1 2_2 1_4 2_1 0_2 0_4 2_3 2_4]$ $[0_3 1_4 0_5 1_5 0_4 1_3 4_5 2_4]$	$[0_1 3_4 0_5 1_1 4_5 0_2 1_5 2_5]$
10.7	$[0_1 0_2 1_1 2_2 3_2 3_3 2_1 0_4]$ $[0_2 1_4 2_2 1_5 2_4 3_4 0_3 0_4]$	$[0_1 0_3 1_1 0_4 2_3 1_4 4_1 0_5]$ $[0_3 0_5 2_3 1_5 1_4 2_5 2_4 4_5]$	$[0_1 0_5 0_2 2_5 3_5 4_5 1_2 1_4]$
10.8	$[0_1 0_2 0_3 1_2 2_1 0_4 3_1 1_3]$ $[0_2 0_4 2_4 1_4 0_3 1_5 3_3 3_5]$	$[0_1 2_2 0_4 2_3 3_1 2_4 1_4 0_5]$ $[0_3 2_4 4_5 2_5 4_4 0_5 3_2 3_5]$	$[0_1 1_4 2_4 1_5 2_1 0_5 0_2 2_5]$
10.9	$[0_1 0_2 1_1 2_2 4_1 0_3 2_3 0_4]$ $[0_2 2_4 2_3 2_5 0_3 1_5 0_5 3_5]$	$[0_1 2_2 1_4 2_1 0_3 2_4 3_4 0_5]$ $[0_3 3_4 0_4 1_4 0_5 4_4 3_5 4_5]$	$[0_1 4_3 0_4 0_2 1_4 1_5 2_5 4_5]$
10.10	$[0_1 0_2 1_1 2_2 4_1 0_3 2_3 0_4]$ $[0_2 0_3 0_4 1_4 2_3 4_4 2_4 0_5]$	$[0_1 3_3 4_1 0_4 1_1 3_4 0_5 0_2]$ $[0_3 1_4 3_4 0_5 0_4 1_5 3_5 2_5]$	$[0_1 1_5 0_2 2_5 4_2 3_5 4_5 0_3]$
10.11	$[0_1 0_2 1_1 2_2 3_2 3_3 0_3 0_4]$ $[0_3 0_5 1_1 4_5 2_4 4_4 1_5 3_5]$	$[0_1 1_3 2_1 1_4 2_3 3_4 2_4 0_5]$ $[0_4 1_5 0_1 2_5 2_2 0_5 2_3 4_5]$	$[0_2 0_4 1_2 0_5 1_4 2_4 1_5 2_5]$
10.12	$[0_1 0_2 1_1 2_2 4_1 0_3 3_3 0_4]$ $[0_2 1_4 1_3 0_4 0_5 2_4 4_4 2_5]$	$[0_1 2_2 2_3 3_1 0_4 1_4 3_4 0_5]$ $[0_3 2_4 1_5 0_4 3_4 0_5 2_5 4_5]$	$[0_1 1_5 0_2 0_4 2_2 2_5 3_5 4_5]$
10.13	$[0_1 0_2 0_3 1_2 2_1 4_2 0_4 1_3]$ $[0_2 3_4 0_5 1_5 0_3 3_5 4_5 2_5]$	$[0_1 3_2 0_4 2_3 4_1 1_4 3_4 0_5]$ $[0_4 0_5 3_5 0_3 3_4 2_3 2_5 1_3]$	$[0_1 1_4 2_5 3_5 0_2 0_4 4_4 4_5]$
10.14	$[0_1 0_2 0_3 1_1 0_4 2_1 1_2]$ $[0_2 0_4 1_4 4_2 4_5 0_3 3_4]$	$[0_1 2_2 1_4 4_1 2_3 1_1 0_5]$ $[0_3 1_4 1_5 4_4 2_4 3_5 0_5]$	$[0_1 3_2 1_5 0_2 2_5 0_3 3_5]$
10.15	$[0_1 0_2 0_3 1_1 2_2 0_4 3_1]$ $[0_2 1_4 3_5 0_3 1_5 4_5 0_4]$	$[0_1 3_2 2_4 1_1 4_4 1_3 2_2]$ $[0_3 0_4 0_5 1_3 2_4 4_4 1_4]$	$[0_1 3_3 0_5 1_1 2_5 3_5 0_2]$
10.16	$[0_1 0_2 0_3 1_2 0_4 2_2 3_2]$ $[0_4 2_4 2_5 4_2 1_5 3_3 4_3]$	$[0_1 4_2 4_4 1_3 1_4 2_3 3_3]$ $[0_5 2_5 4_2 1_1 4_5 0_2 0_3]$	$[0_1 4_3 0_5 2_4 3_4 1_5 2_5]$
10.17	$[0_1 0_2 0_3 1_2 2_1 0_4 3_1]$ $[0_2 0_4 1_4 3_4 0_3 2_5 1_3]$	$[0_1 2_2 4_4 1_3 2_1 0_5 3_1]$ $[0_3 1_4 3_5 0_4 2_4 0_5 1_5]$	$[0_1 2_3 1_4 1_5 0_2 4_5 1_2]$
10.18	$[0_1 0_2 1_1 2_2 0_4 0_3]$ $[0_2 0_5 2_1 2_4 4_3 1_5]$	$[0_1 2_2 4_1 0_3 1_4 0_5]$ $[0_3 0_5 1_2 3_4 2_4 3_5]$	$[0_1 3_3 0_4 4_1 2_4 2_5]$

K_{25} $V = \{i_j \mid 0 \le i \le 4;\ j = 1,2,3,4,5\}$. M as follows, cycled modulo 5_{-}:

10.2	$[0_1 1_1 0_2 2_1 3_2 1_2 3_1 0_3 2_2]$ $[0_2 0_3 2_3 1_3 2_2 0_4 1_2 1_4 0_5]$	$[0_1 3_4 0_5 1_5 2_1 4_5 0_2 1_2 3_5]$ $[0_2 1_4 1_5 2_4 0_3 0_4 1_3 0_5 3_5]$	$[0_1 0_3 1_3 3_3 4_1 0_4 1_1 1_4 2_4]$ $[0_3 1_4 3_4 0_5 2_3 3_5 0_4 1_5 2_5]$
10.3	$[0_1 1_1 3_1 0_2 1_2 2_1 0_3 4_1 4_3]$ $[0_2 0_3 2_2 0_4 1_3 2_3 4_3 1_4 4_4]$	$[0_1 2_4 0_2 2_5 3_5 2_2 1_2 0_3 4_5]$ $[0_2 1_4 0_3 0_5 3_5 1_3 0_4 4_4 4_5]$	$[0_1 3_2 0_2 2_3 0_4 1_1 2_4 4_1 0_5]$ $[0_3 0_4 2_5 3_4 1_5 4_5 0_5 2_4 3_5]$
10.4	$[0_1 1_1 3_1 0_2 1_2 3_2 4_1 4_2 0_3]$ $[0_2 0_3 1_2 3_3 0_4 1_4 2_2 1_5 3_4]$	$[0_1 2_4 0_2 2_2 0_5 1_5 2_1 4_5 3_5]$ $[0_2 3_3 2_3 0_4 0_5 1_5 0_3 1_4 2_5]$	$[0_1 1_3 2_1 0_3 2_3 0_4 1_1 2_4 3_4]$ $[0_3 0_4 1_3 1_5 4_5 2_5 3_4 1_4 3_5]$
10.5	$[0_1 1_1 3_1 0_2 1_2 2_1 0_3 3_2]$ $[0_2 2_3 3_2 1_3 1_4 1_2 3_4 1_5]$	$[0_1 2_4 3_1 1_4 0_5 1_1 2_5 3_5]$ $[0_2 0_5 0_3 1_3 2_5 1_4 4_5 3_5]$	$[0_1 0_2 0_3 1_1 2_3 1_2 0_4 1_4]$ $[0_3 2_3 0_4 3_4 2_4 1_3 3_5 4_5]$
10.6	$[0_1 1_1 0_2 2_1 4_1 1_2 0_3 1_3]$ $[0_2 1_2 2_4 3_2 1_3 0_3 0_5 1_5]$	$[0_1 2_4 3_4 4_1 0_5 2_1 2_5 4_5]$ $[0_2 2_2 4_5 0_3 2_3 1_4 3_4 3_5]$	$[0_1 1_2 3_3 1_1 0_3 0_2 0_4 1_4]$ $[0_3 0_4 2_5 0_5 2_3 3_4 1_5 2_4]$
10.7	$[0_1 1_1 3_1 0_2 1_2 3_2 4_1 0_3]$ $[0_2 1_2 1_3 3_3 0_4 2_5 3_2 4_3]$	$[0_1 3_4 0_2 0_5 1_5 2_5 1_2 0_3]$ $[0_2 1_4 0_3 2_4 4_4 3_5 0_4 0_5]$	$[0_1 0_3 1_1 3_3 0_4 1_4 2_1 4_4]$ $[0_3 0_5 1_1 4_5 4_4 1_3 1_4 2_5]$

K_{25} (continued)

10.8	$[0_1 1_1 0_2 2_1 3_2 4_2 0_3 1_3]$ $[0_1 1_5 2_5 3_5 0_2 0_3 2_2 4_5]$	$[0_1 0_4 3_4 4_4 0_2 2_2 1_2 0_5]$ $[0_2 3_3 0_5 4_3 1_3 0_4 3_5 1_4]$	$[0_1 3_2 0_3 2_3 3_1 0_4 1_2 1_4]$ $[0_4 1_4 1_5 3_5 2_3 4_4 0_5 0_3]$
10.9	$[0_1 1_1 3_1 0_2 2_1 1_2 0_3 1_3]$ $[0_2 3_3 4_2 0_4 1_2 0_5 1_5 2_5]$	$[0_1 3_3 4_1 1_4 2_1 0_5 3_4 1_5]$ $[0_3 1_3 0_4 2_3 3_5 0_5 2_4 4_5]$	$[0_1 0_2 1_2 3_2 0_3 2_3 0_4 1_4]$ $[0_4 4_4 4_2 2_5 0_1 4_5 4_3 3_5]$
10.10	$[0_1 1_1 3_1 0_2 2_1 1_2 0_3 2_2]$ $[0_2 1_2 0_3 1_3 4_2 3_4 1_5 3_3]$	$[0_1 2_4 0_2 0_5 1_1 2_5 3_5 1_2]$ $[0_3 0_4 1_3 2_4 4_3 0_5 0_2 3_5]$	$[0_1 1_3 2_1 0_4 1_1 3_3 1_4 0_2]$ $[0_4 1_4 3_4 0_5 1_3 3_5 4_5 1_5]$
10.11	$[0_1 1_1 3_1 0_2 1_2 3_2 4_2 0_3]$ $[0_2 1_2 0_3 2_3 3_3 0_4 1_4 3_4]$	$[0_1 4_4 0_2 0_5 1_5 4_5 2_5 3_5]$ $[0_2 1_5 1_3 2_5 0_3 3_5 2_4 4_5]$	$[0_1 1_3 2_1 0_4 2_3 3_3 1_4 2_4]$ $[0_3 0_4 0_5 1_4 3_4 4_5 4_4 2_5]$
10.12	$[0_1 1_1 3_1 0_2 2_1 1_2 0_3 1_3]$ $[0_2 0_3 1_2 2_3 4_2 2_4 2_5 4_5]$	$[0_1 2_4 3_1 1_4 0_2 0_5 1_5 2_5]$ $[0_3 2_3 0_4 1_3 1_4 2_4 0_5 1_5]$	$[0_1 0_2 1_2 3_2 0_3 4_3 0_4 1_4]$ $[0_3 1_4 0_5 2_4 0_4 2_5 3_5 4_5]$
10.13	$[0_1 1_1 0_2 2_1 3_2 1_2 0_3 2_2]$ $[0_1 1_5 2_5 3_5 0_2 1_3 2_4 4_5]$	$[0_1 1_4 2_4 3_4 0_2 0_3 0_4 0_5]$ $[0_2 1_4 4_4 1_5 0_3 2_3 0_5 2_5]$	$[0_1 3_2 1_3 0_3 1_1 3_3 4_3 0_4]$ $[0_3 2_4 4_5 3_4 3_5 0_4 1_5 4_4]$
10.14	$[0_1 1_1 0_2 2_1 0_3 3_1 4_3]$ $[0_2 1_2 2_3 3_2 0_4 0_3 3_3]$	$[0_1 2_1 0_4 1_1 2_2 0_2 1_4]$ $[0_2 4_4 0_5 0_3 1_5 2_3 4_5]$	$[0_1 2_4 0_5 1_1 2_5 0_2 3_5]$ $[0_3 1_3 2_4 1_5 4_4 3_4 3_5]$
10.15	$[0_1 1_1 0_2 2_1 0_3 3_1 4_1]$ $[0_2 1_2 1_3 0_3 1_4 0_5 3_3]$	$[0_1 2_2 4_3 0_2 0_4 1_4 1_2]$ $[0_2 2_2 3_5 0_1 2_5 2_4 0_3]$	$[0_1 2_4 4_4 1_1 0_5 1_5 1_2]$ $[0_3 2_4 0_5 1_3 0_4 1_5 2_5]$
10.16	$[0_3 1_3 1_4 3_2 3_5 4_2 2_5]$ $[0_1 1_3 1_5 4_3 2_4 0_4 1_4]$	$[2_5 3_1 1_2 2_1 4_2 1_4 4_4]$ $[0_2 3_3 2_5 1_2 4_4 3_2 0_3]$	$[0_1 1_1 0_2 2_1 0_3 1_2 2_3]$ $[0_4 0_5 2_5 1_1 4_5 2_1 3_3]$
10.17	$[0_1 1_1 0_2 2_1 3_2 0_3 4_2]$ $[0_1 1_5 4_5 2_5 0_3 3_5 2_3]$	$[0_1 3_2 1_3 2_3 3_1 0_4 2_2]$ $[0_3 0_4 4_5 1_3 2_2 2_4 3_3]$	$[0_1 1_4 3_4 4_4 0_2 0_5 2_2]$ $[0_4 1_4 3_5 4_2 1_2 0_5 3_2]$
10.18	$[0_1 1_1 3_1 0_2 0_3 1_2]$ $[0_1 0_5 1_1 2_5 2_4 3_5]$	$[0_1 3_2 0_2 1_3 0_4 2_3]$ $[0_2 0_4 1_2 2_5 4_2 4_5]$	$[0_1 3_3 1_2 4_4 3_4 1_4]$ $[0_3 2_3 3_3 0_5 1_5 2_4]$

K_{41} $V = \mathbb{Z}_{41}$. M as follows, cycled modulo 41:

10.2	[0, 1, 3, 4, 9, 2, 8, 16, 25]	[0, 10, 21, 12, 25, 6, 20, 2, 17]
10.3	[0, 1, 3, 6, 4, 9, 2, 10, 19]	[0, 10, 23, 11, 14, 30, 9, 32, 15]
10.4	[0, 1, 3, 6, 10, 5, 11, 2, 13]	[0, 7, 15, 1, 18, 12, 27, 2, 21]
10.5	[0, 1, 3, 6, 10, 2, 7, 16]	[0, 9, 20, 1, 14, 35, 17, 26]
10.6	[0, 1, 3, 7, 2, 8, 15, 11]	[0, 9, 22, 1, 15, 33, 16, 10]
10.7	[0, 1, 3, 6, 4, 9, 2, 17]	[0, 14, 34, 16, 10, 22, 5, 33]
10.8	[0, 1, 3, 4, 9, 15, 11, 19]	[0, 9, 21, 13, 3, 27, 31, 15]
10.9	[0, 1, 3, 6, 2, 7, 12, 14]	[0, 9, 19, 2, 13, 26, 18, 21]
10.10	[0, 1, 3, 6, 2, 7, 8, 17]	[0, 10, 25, 13, 32, 14, 16, 33]
10.11	[0, 1, 3, 6, 4, 9, 7, 15]	[0, 14, 34, 16, 10, 22, 11, 24]
10.12	[0, 1, 3, 6, 2, 7, 9, 11]	[0, 10, 30, 6, 28, 13, 12, 14]
10.13	[0, 1, 3, 4, 9, 2, 15, 17]	[0, 9, 19, 12, 33, 3, 17, 15]
10.14	[0, 1, 3, 7, 12, 2, 8]	[0, 7, 25, 1, 14, 34, 15]
10.15	[0, 1, 3, 7, 12, 18, 19]	[0, 9, 19, 5, 13, 24, 34]
10.16	[28, 22, 24, 10, 21, 11, 33]	[12, 9, 25, 20, 11, 26, 33]
10.17	[0, 1, 3, 4, 9, 15, 22]	[0, 10, 24, 8, 30, 21, 33]
10.18	[0, 1, 3, 6, 14, 18]	[0, 7, 27, 16, 31, 12]

K_{45} $V = \{i_j \mid 0 \le i \le 8;\ j = 1,2,3,4,5\}$. M as follows, cycled modulo 9_-:

10.2	$[0_37_16_43_23_37_53_16_37_2]$	$[1_14_22_15_22_44_18_35_54_4]$	$[1_43_54_50_27_28_53_34_36_3]$
	$[7_14_11_58_32_40_32_13_25_5]$	$[8_56_56_14_58_15_46_23_54_4]$	$[7_35_52_21_33_23_50_58_13_4]$
	$[5_44_13_51_22_36_16_32_21_5]$	$[0_18_30_42_40_23_27_21_45_4]$	$[0_12_10_24_11_25_14_23_26_3]$
	$[0_20_47_45_45_30_30_51_34_5]$	$[0_41_42_33_43_57_25_54_36_5]$	
10.3	$[3_10_55_24_45_41_38_55_30_2]$	$[2_46_38_37_17_41_38_12_31_2]$	$[0_24_26_45_58_26_36_12_57_5]$
	$[6_34_27_57_38_52_35_34_47_1]$	$[3_50_42_32_43_24_18_53_40_5]$	$[2_37_51_27_18_28_45_24_13_4]$
	$[5_11_40_44_55_57_40_22_25_2]$	$[0_17_22_31_56_38_12_44_12_5]$	$[0_11_13_15_24_17_12_21_35_3]$
	$[0_20_33_22_54_44_17_53_35_4]$	$[0_42_48_30_53_24_53_55_53_4]$	
10.4	$[7_54_23_13_34_37_30_35_37_1]$	$[8_26_20_21_23_36_10_42_11_1]$	$[5_15_25_41_34_20_58_51_28_3]$
	$[6_35_45_57_34_43_56_13_23_3]$	$[6_25_40_20_53_40_38_15_37_5]$	$[6_54_47_48_14_22_53_33_43_1]$
	$[4_25_31_52_46_40_25_56_18_4]$	$[0_15_30_28_34_45_48_10_52_5]$	$[0_12_15_11_14_22_33_17_31_4]$
	$[0_21_43_26_44_56_52_33_48_5]$	$[0_41_42_56_14_57_38_55_52_4]$	
10.5	$[1_44_31_36_30_40_38_13_1]$	$[5_15_52_54_36_52_40_48_1]$	$[4_25_57_54_16_52_20_23_3]$
	$[1_37_57_37_23_15_11_12_3]$	$[7_44_10_40_27_11_34_22_3]$	$[0_36_27_27_14_50_54_45_2]$
	$[2_46_40_23_10_31_40_58_5]$	$[0_12_31_24_24_53_38_55_5]$	$[0_11_10_22_13_27_18_45_3]$
	$[0_25_37_20_43_56_24_41_4]$	$[0_40_52_27_56_48_31_31_2]$	
10.6	$[3_12_21_52_56_47_44_40_2]$	$[7_34_48_45_26_48_51_48_1]$	$[8_16_45_50_25_37_14_32_3]$
	$[1_20_23_35_22_55_37_41_1]$	$[1_25_18_55_23_28_21_51_3]$	$[6_14_28_40_38_14_13_41_2]$
	$[8_22_23_32_53_15_15_33_5]$	$[0_13_45_40_28_30_32_50_5]$	$[0_11_12_28_12_16_31_54_4]$
	$[0_32_35_30_40_28_45_55_4]$	$[0_40_53_58_14_56_51_52_3]$	
10.7	$[4_53_40_23_35_26_55_58_2]$	$[6_15_51_27_46_57_33_45_2]$	$[4_23_31_46_15_25_47_10_2]$
	$[0_47_24_21_15_41_32_14_4]$	$[7_27_12_10_53_12_27_30_2]$	$[3_57_10_55_35_42_18_52_3]$
	$[8_54_10_43_34_55_47_40_5]$	$[0_11_21_34_35_20_43_16_5]$	$[0_11_13_15_36_13_35_12_2]$
	$[0_28_40_45_54_36_35_38_5]$	$[0_34_35_42_40_40_50_23_5]$	
10.8	$[6_35_36_58_55_42_23_43_3]$	$[6_18_52_54_25_57_50_35_1]$	$[0_25_10_48_32_52_24_25_4]$
	$[2_12_34_40_36_10_24_57_1]$	$[2_14_22_58_37_53_43_18_4]$	$[2_23_17_53_24_36_35_50_3]$
	$[2_53_21_41_55_53_42_45_3]$	$[0_12_11_53_13_28_25_38_4]$	$[0_45_28_14_35_13_40_37_4]$
	$[0_13_43_51_32_48_58_21_2]$	$[0_23_22_46_40_41_46_14_5]$	
10.9	$[3_10_55_24_45_41_38_55_3]$	$[0_40_22_46_38_37_17_42_3]$	$[8_52_35_36_53_56_48_27_1]$
	$[1_26_34_25_37_57_34_50_5]$	$[6_14_28_23_30_44_42_33_2]$	$[4_10_48_53_27_41_13_10_2]$
	$[4_31_27_18_20_27_44_10_5]$	$[0_12_16_15_23_13_43_31_4]$	$[2_20_26_46_54_11_33_48_5]$
	$[0_14_35_15_57_43_54_44_5]$	$[0_43_41_38_53_50_31_16_2]$	
10.10	$[2_51_32_31_23_58_36_48_1]$	$[2_38_48_22_12_51_54_55_1]$	$[5_20_37_31_16_48_38_56_5]$
	$[8_25_22_46_57_27_30_41_3]$	$[3_14_45_57_32_45_46_41_4]$	$[8_52_17_34_13_43_55_58_4]$
	$[4_33_15_35_40_28_20_46_2]$	$[0_11_13_13_21_25_10_41_4]$	$[4_57_34_36_10_51_42_47_2]$
	$[0_17_24_38_38_58_22_56_2]$	$[0_25_20_52_40_47_36_55_1]$	
10.11	$[3_35_56_58_51_37_34_36_4]$	$[4_42_53_17_53_28_21_55_5]$	$[7_17_41_45_53_18_46_32_4]$
	$[0_23_42_58_56_31_58_30_4]$	$[5_50_42_43_40_22_37_28_2]$	$[1_14_28_18_35_23_45_34_5]$
	$[5_33_53_15_50_22_42_25_2]$	$[0_11_14_11_36_23_47_21_5]$	$[5_24_10_57_23_15_16_34_4]$
	$[0_17_48_38_42_36_45_36_5]$	$[0_24_40_40_51_14_37_34_5]$	
10.12	$[7_45_41_13_40_43_51_38_1]$	$[3_42_40_35_22_37_31_55_1]$	$[7_11_52_42_30_27_24_58_1]$
	$[5_44_27_40_38_54_30_23_2]$	$[1_45_11_32_26_22_52_31_5]$	$[7_31_51_22_50_28_10_55_1]$
	$[2_31_10_55_15_50_35_48_5]$	$[0_13_25_12_31_20_43_36_4]$	$[4_12_56_18_57_16_21_11_2]$
	$[0_21_24_22_35_30_40_37_5]$	$[0_23_31_58_52_46_44_56_5]$	
10.13	$[8_37_30_54_38_53_26_43_4]$	$[5_11_36_50_37_41_17_34_3]$	$[2_35_24_30_24_44_18_58_2]$
	$[4_16_44_51_12_41_45_42_5]$	$[2_15_44_51_52_32_24_28_5]$	$[2_56_18_18_37_16_26_45_4]$
	$[7_52_58_54_22_13_24_36_2]$	$[0_14_10_21_14_21_23_45_4]$	$[8_15_23_42_37_40_41_56_3]$
	$[0_24_24_52_12_36_27_22_5]$	$[0_26_58_56_40_33_32_47_5]$	

10.14	$[3_3 1_1 6_4 7_4 5_3 5_1 4_2]$	$[2_2 4_4 7_5 1_1 8_3 0_3 5_1]$	$[3_4 5_5 5_1 0_4 3_1 1_5 7_3]$
	$[8_3 5_1 1_5 1_2 7_2 8_2 3_5]$	$[5_4 1_5 5_5 7_4 4_1 7_2 6_5]$	$[6_3 3_3 3_5 0_5 2_3 3_1 2_5]$
	$[3_5 6_2 2_2 0_2 6_4 0_4 4_4]$	$[0_1 1_1 1_2 8_1 3_5 1_5 2_5]$	$[4_3 2_4 6_2 2_1 8_1 1_4 1_2]$
	$[0_1 1_3 5_2 3_5 8_4 2_3 5_1]$	$[0_2 4_3 8_4 7_2 0_3 0_4 7_4]$	
10.15	$[6_2 3_3 4_2 5_2 5_1 1_2 2_4]$	$[3_5 4_3 8_1 0_5 3_3 3_1 2_2]$	$[5_5 3_5 2_4 6_3 7_4 6_4 2_2]$
	$[7_1 0_3 6_4 2_3 5_4 1_5 3_2]$	$[0_3 7_3 6_1 1_2 1_5 0_2 5_2]$	$[4_4 1_4 1_1 6_4 1_5 3_2 6_5]$
	$[3_3 0_3 5_2 8_3 7_3 1_5 3_1]$	$[0_1 1_1 6_1 7_4 8_3 6_2 3_5]$	$[1_1 3_5 8_2 7_5 5_4 3_1 5_5]$
	$[0_4 2_4 0_3 4_2 1_2 5_4 7_4]$	$[0_5 3_5 7_3 0_1 8_5 6_2 1_1]$	
10.16	$[9_1 5_2 1_4 3_1 7_1 2_3 7_3]$	$[2_1 3_3 10_3 10_2 8_2 5_3 8_4]$	$[10_2 1_4 8_4 4_2 1_1 3_3 0_4]$
	$[0_1 1_1 5_4 3_1 10_3 0_2 5_3]$	$[0_2 1_2 1_3 9_1 6_4 3_2 2_3]$	$[1_1 2_2 6_2 0_2 10_4 5_2 2_4]$
	$[6_3 9_3 5_4 10_2 5_3 6_4 8_4]$	$[8_4 10_3 7_4 2_4 6_1 6_4 \infty]$	$[8_2 6_3 1_4 2_3 0_3 5_4 \infty]$
10.17	$[2_3 6_2 3_4 3_1 5_4 0_3 6_5]$	$[2_3 7_3 0_5 6_1 4_5 2_2 5_5]$	$[8_1 6_1 1_3 4_4 0_3 3_5 3_4]$
	$[6_5 0_2 3_4 6_1 7_1 4_4 8_5]$	$[8_3 7_2 8_5 4_2 6_2 7_1 2_4]$	$[1_3 1_1 6_5 2_2 8_2 3_4 7_4]$
	$[2_4 3_2 5_4 8_1 2_3 0_4 6_3]$	$[0_1 0_2 0_4 3_2 1_3 4_2 6_3]$	$[0_1 3_1 1_2 4_1 6_2 1_5 3_4]$
	$[0_2 4_2 0_5 7_5 2_4 8_5 1_5]$	$[0_3 4_5 8_5 6_3 7_3 6_4 7_5]$	
10.18	$[4_3 1_4 9_3 3_4 3_2 5_1]$	$[3_1 4_1 0_2 7_4 3_2 0_4]$	$[8_1 6_3 3_3 10_4 10_1 2_2]$
	$[6_1 3_1 7_1 9_3 4_4 3_2]$	$[0_3 6_3 8_2 10_3 3_2 9_3]$	$[0_1 5_1 6_3 6_2 5_3 7_4]$
	$[2_1 1_4 2_4 8_4 6_2 10_3]$	$[2_2 7_2 1_3 1_4 9_2 10_2]$	$[9_1 3_4 0_4 2_3 \infty 0_2]$

$K_{10,10,10,10}$ $V = \{i_j \mid 0 \le i \le 9;\ j = 1,2,3,4\}$. M as follows, cycled modulo 10_-:

10.17	$[4_2 5_3 5_4 7_4 0_2 7_1 8_3]$	$[0_3 8_1 6_4 2_4 1_3 8_2 4_3]$	$[0_1 0_2 0_3 1_2 2_1 5_3 6_1]$
	$[0_1 2_2 2_4 4_2 8_1 3_4 9_3]$	$[0_1 8_2 6_4 4_3 6_1 7_4 8_1]$	$[0_2 7_3 2_4 8_3 1_1 5_4 9_2]$
10.18	$[1_1 5_3 5_1 0_2 8_4 0_3]$	$[1_1 3_3 2_1 3_4 2_2 8_3]$	$[5_4 0_3 6_4 2_2 2_1 9_2]$
	$[0_1 2_2 9_1 9_4 5_3 8_2]$	$[0_1 6_2 8_3 5_4 6_3 8_4]$	$[0_1 3_3 9_2 4_4 4_2 6_4]$

K_{61} $V = \mathbb{Z}_{61}$. M follows, cycled modulo 61:

10.17	$[0,1,3,4,9,15,22]$	$[0,8,17,10,39,23,43]$	$[0,14,35,19,53,31,55]$
10.18	$[0,1,3,6,10,17]$	$[0,8,22,45,20,35]$	$[0,18,51,29,48,21]$

K_{65} $V = \{i_j \mid 0 \le i \le 12;\ j = 1,2,3,4,5\}$. M as follows, cycled modulo 13_-:

10.17	$[8_2 1_1 10_5 6_3 0_2 4_4 6_5]$	$[6_2 2_5 10_5 5_5 5_1 7_5 1_5]$	$[3_3 9_3 1_5 12_4 5_3 4_5 11_4]$
	$[5_1 8_2 6_5 7_4 2_4 11_5 3_4]$	$[1_1 0_4 7_4 4_3 8_2 9_3 9_2]$	$[10_2 4_2 11_4 8_1 7_3 8_4 9_3]$
	$[2_1 7_1 12_2 1_1 2_4 9_5 6_4]$	$[0_1 2_1 6_1 3_1 1_2 4_2 3_2]$	$[1_5 1_2 6_3 9_2 11_2 4_5 6_4]$
	$[0_1 6_2 0_3 8_2 9_1 5_3 12_1]$	$[0_1 2_3 4_3 7_3 9_1 4_4 10_1]$	$[0_1 3_4 3_5 9_4 11_1 10_5 12_1]$
	$[0_2 4_2 3_3 8_2 0_4 3_4 2_5]$	$[0_2 4_3 12_4 2_3 11_3 2_4 4_4]$	$[0_3 1_3 4_5 5_4 5_2 2_5 12_2]$
	$[0_3 3_3 6_4 6_5 2_1 7_5 10_3]$		
10.18	$[0_3 5_4 4_1 7_4 0_5 11_3]$	$[7_2 5_5 11_5 6_5 10_2 7_3]$	$[7_4 11_4 1_1 1_5 12_4 8_1]$
	$[11_2 9_3 10_5 10_4 1_3 1_4]$	$[11_2 8_4 12_1 0_5 2_4 3_3]$	$[0_2 4_2 11_5 12_3 6_2 10_5]$
	$[7_1 9_2 5_4 10_2 2_5 11_1]$	$[4_1 10_1 6_5 12_1 2_5 2_2]$	$[0_1 1_1 4_1 0_2 5_2 1_3]$
	$[0_1 1_2 4_1 2_3 3_3 0_3]$	$[0_1 6_2 0_4 10_3 7_4 7_2]$	$[0_1 4_3 12_1 8_3 6_4 8_2]$
	$[0_1 6_3 9_5 12_3 10_5 12_5]$	$[0_1 11_4 9_1 2_5 7_3 0_4]$	$[0_2 2_4 10_1 5_4 1_5 7_3]$
	$[0_3 8_4 2_2 5_5 9_5 6_5]$		

$K_{10,10,10}$ $V = \{i_j \mid 0 \le i \le 4;\ 1 \le j \le 6\}$. M as follows, cycled modulo 5_-:
(Part $p+1$ contains vertices $\{i_j \mid 0 \le i \le 4;\ j \in \{2p, 2p+1\}\}$, $0 \le p \le 2$.)

10.2	$[4_1 0_3 1_6 3_5 1_4 4_2 2_5 4_3 2_6]$ $[0_1 3_4 3_5 1_6 0_2 2_3 4_5 2_2 4_6]$	$[0_1 0_3 0_5 2_3 3_1 1_3 0_2 3_3 2_5]$ $[0_2 4_3 0_5 0_4 3_5 2_2 1_5 2_4 0_6]$	$[0_1 0_4 1_5 1_4 2_1 4_4 0_2 0_3 0_6]$ $[0_2 1_4 3_6 3_4 2_6 3_3 0_6 4_4 4_6]$
10.3	$[2_3 1_2 1_5 1_1 0_6 3_3 2_5 2_4 2_1]$ $[0_2 2_3 0_5 2_4 0_4 2_2 1_5 3_1 1_6]$	$[0_1 2_3 3_1 1_4 3_3 0_2 0_3 1_2 2_4]$ $[0_2 2_5 4_2 2_6 0_6 0_3 0_5 3_3 4_6]$	$[0_1 4_4 0_2 1_5 2_5 3_1 0_6 1_1 1_6]$ $[0_4 1_5 0_3 4_6 2_5 3_4 1_6 1_4 2_6]$
10.4	$[4_6 2_4 0_6 0_4 0_2 0_3 2_6 3_4 1_1]$ $[0_1 3_4 0_5 1_1 0_6 1_6 0_2 4_3 2_6]$	$[0_1 0_3 1_1 2_3 0_5 3_3 0_2 1_3 1_5]$ $[0_2 1_4 0_5 1_2 1_5 2_4 3_6 2_3 3_5]$	$[0_1 2_3 0_2 0_3 2_5 0_4 1_1 2_4 3_5]$ $[0_2 3_4 1_5 4_2 2_6 4_4 4_5 0_3 0_6]$
10.5	$[2_4 0_6 4_2 0_3 4_6 3_3 1_5 2_1]$ $[0_2 0_3 1_2 3_3 3_5 1_1 0_6 1_4]$	$[0_1 0_3 1_1 2_3 1_5 2_1 1_4 3_4]$ $[0_2 3_3 4_5 2_2 1_5 4_3 4_6 0_4]$	$[0_1 2_3 4_1 2_5 2_4 0_2 0_5 1_6]$ $[0_2 4_4 0_6 0_1 3_6 0_3 2_6 3_4]$
10.6	$[3_1 3_6 3_4 3_5 2_3 2_1 1_4 2_6]$ $[0_2 0_3 0_5 1_2 2_3 3_2 0_4 3_3]$	$[0_1 1_3 0_5 1_1 0_3 2_1 3_5 2_3]$ $[0_2 2_3 2_6 0_3 3_6 4_3 0_6 4_4]$	$[0_1 1_4 3_6 1_1 3_5 0_2 1_6 2_4]$ $[0_2 1_5 3_4 4_5 2_4 1_2 4_6 2_5]$
10.7	$[3_3 4_5 1_2 4_6 3_5 3_2 4_3 3_6]$ $[0_1 0_6 1_1 2_6 2_4 3_6 1_2 0_3]$	$[0_1 0_3 1_1 2_3 3_3 0_5 2_1 0_4]$ $[0_2 2_3 0_5 1_4 1_5 0_4 1_2 0_6]$	$[0_1 0_4 1_1 2_5 1_4 4_5 0_2 2_4]$ $[0_3 0_6 3_4 2_6 4_5 2_2 0_4 3_6]$
10.8	$[1_5 3_2 1_4 0_3 4_1 0_5 0_6 2_2]$ $[0_2 0_3 2_6 1_3 2_2 4_5 0_6 2_3]$	$[0_1 0_3 2_5 2_3 3_1 1_5 4_1 0_4]$ $[0_4 1_5 0_2 2_5 2_2 3_4 3_1 3_6]$	$[0_1 2_4 0_5 3_4 4_1 0_6 1_2 2_6]$ $[0_4 1_6 1_2 4_6 0_2 3_3 1_1 0_6]$
10.9	$[4_6 2_1 2_6 2_4 2_5 1_3 1_1 3_2]$ $[0_1 1_3 0_2 1_5 4_2 2_5 3_3 4_4]$	$[1_1 2_5 3_3 4_1 1_3 2_6 4_4 4_5]$ $[0_2 0_3 1_2 4_3 1_5 0_4 3_6 4_6]$	$[4_1 4_3 4_5 4_2 1_3 3_6 1_4 3_5]$ $[0_2 1_4 3_6 4_4 2_5 3_4 0_6 2_6]$
10.10	$[4_2 1_4 2_1 2_4 2_2 4_5 4_6 1_1]$ $[0_2 0_3 1_2 3_3 0_5 1_4 4_4 1_5]$	$[0_1 0_3 1_1 2_3 4_1 2_4 0_5 1_2]$ $[0_2 1_5 0_3 3_6 2_3 4_6 3_5 3_3]$	$[0_1 1_5 2_1 1_6 1_1 3_5 2_6 0_2]$ $[0_4 0_5 1_3 0_6 1_4 2_6 3_5 2_4]$
10.11	$[1_4 4_5 3_2 3_6 1_1 2_6 1_2 0_6]$ $[0_1 1_3 2_2 4_6 2_3 2_5 4_4 4_5]$	$[3_3 0_2 2_5 3_2 3_1 1_6 2_2 0_5]$ $[0_2 1_4 0_1 2_4 3_4 3_6 4_4 0_5]$	$[1_1 4_5 0_4 3_6 4_3 2_5 4_4 1_5]$ $[0_3 1_5 2_3 2_6 1_1 1_6 3_2 4_6]$
10.12	$[1_5 3_3 0_6 0_3 1_2 4_4 0_2 2_2]$ $[0_1 3_4 3_2 1_3 0_2 2_5 1_5 0_6]$	$[2_2 2_5 0_3 1_5 2_1 2_3 3_6 1_6]$ $[0_1 1_6 0_4 1_2 2_4 2_6 3_6 4_6]$	$[0_1 1_3 2_1 0_3 3_1 0_4 1_4 0_5]$ $[0_3 0_5 0_4 3_5 2_4 4_6 3_2 4_5]$
10.13	$[0_5 1_2 4_4 4_2 4_5 2_1 4_3 2_4]$ $[0_1 1_3 3_5 4_3 1_2 0_3 4_6 1_6]$	$[4_5 0_1 3_3 3_1 4_4 3_2 4_6 4_1]$ $[0_1 4_4 2_6 2_4 4_6 2_2 2_3 3_4]$	$[1_6 3_1 3_3 1_1 0_6 1_2 1_4 0_3]$ $[0_2 2_4 2_5 1_3 0_5 2_2 1_4 2_3]$
10.14	$[4_1 0_3 2_6 0_1 4_4 0_2 4_6]$ $[0_1 0_3 0_5 2_2 4_3 1_2 3_5]$	$[3_6 1_4 0_2 1_5 4_3 2_1 3_4]$ $[0_3 0_6 0_2 0_4 1_6 2_1 3_6]$	$[3_1 1_3 4_5 2_1 0_4 3_2 2_5]$ $[0_4 4_6 2_2 3_3 2_5 2_4 3_5]$
10.15	$[1_2 2_5 2_3 3_2 0_5 3_4 0_1]$ $[0_1 0_3 2_5 2_4 3_5 4_3 4_4]$	$[1_3 3_2 1_6 3_3 4_5 1_2 0_1]$ $[0_3 2_6 2_1 3_4 3_2 4_6 3_5]$	$[2_6 3_4 0_1 4_4 0_2 1_4 4_6]$ $[0_4 3_6 0_1 1_3 2_6 2_2 2_4]$
10.16	$[4_5 3_2 2_3 2_1 4_3 3_1 4_1]$ $[0_1 0_4 3_5 4_5 3_4 1_6 4_6]$	$[0_6 1_2 3_4 0_1 1_4 2_1 3_2]$ $[0_2 1_3 4_5 2_3 3_5 1_4 4_4]$	$[2_6 1_2 1_3 3_3 0_1 4_3 2_4]$ $[0_2 0_4 0_5 2_5 3_3 0_6 3_6]$

$K_{5,5,5,5,5}$ $V = \mathbb{Z}_{25}$. M as follows, cycled modulo 25:
(Part $i+1$ contains the vertices $\{0+i, 5+i, 10+i, 15+i, 20+i\}$, $0 \le i \le 4$.)

10.17	$[0, 1, 4, 6, 15, 8, 19]$
10.18	$[0, 1, 3, 12, 18, 21]$

$K_{20,20,20,20}$ $V = \mathbb{Z}_{80}$. M as follows, cycled modulo 80:
(Part $i+1$ contains the vertices $\{0+i, 20+i, 40+i, 60+i\}$, $0 \le i \le 19$.)

10.17	[3,34,56,1,43,38,52]	[0,1,7,3,12,33,16]	[0,18,41,10,36,25,44]
10.18	[42,79,16,21,12,57]	[0,1,3,26,19,29]	[0,11,5,38,77,31]

$K_{8(20)}$ $V = \mathbb{Z}_{160}$. M as follows, cycled modulo 160:
(Part $i+1$ contains the vertices $\{0+i, 20+i, 40+i, 60+i, \ldots, 140+i\}$, $0 \leq i \leq 19$.)

10.17	[18,108,126,85,73,150,74]	[47,83,120,26,48,57,56]
	[57,8,135,96,89,116,139]	[0,14,69,17,64,114,143]
	[0,19,44,26,61,118,64]	[0,2,15,4,9,62,103]
	[105,31,102,5,86,11,120]	
10.18	[31,22,36,80,145,46]	[54,55,102,122,59,157]
	[89,14,42,55,139,53]	[0,7,34,89,122,53]
	[0,19,13,79,109,134]	[0,2,5,23,54,11]
	[140,2,61,19,150,67]	

$K_{12(20)}$ $V = \mathbb{Z}_{240}$. M as follows, cycled modulo 240:
(Part $i+1$ contains the vertices $\{0+i, 20+i, 40+i, 60+i, \ldots, 220+i\}$, $0 \leq i \leq 19$.)

10.17	[222,67,178,215,87,233,195]	[165,128,231,196,117,206,129]
	[105,128,193,62,60,220,204]	[173,124,181,11,180,66,183]
	[195,42,59,54,15,163,224]	[189,159,180,13,72,231,96]
	[0,1,122,3,7,53,8]	[0,15,182,19,46,93,169]
	[0,6,130,13,27,113,38]	[0,26,212,35,87,184,224]
	[0,29,206,62,164,95,196]	
10.18	[164,205,106,218,165,25]	[12,135,62,145,138,171]
	[193,197,121,239,223,74]	[19,169,23,225,42,108]
	[45,64,78,125,56,133]	[7,75,69,184,102,100]
	[0,5,14,49,20,37]	[0,18,39,70,92,25]
	[0,32,137,75,161,55]	[0,28,79,181,78,142]
	[0,39,175,110,155,58]	