NON-ISOMORPHIC SOLUTIONS FOR SOME TRIPLE SYSTEMS WITH BIPOINTS

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Abstract. We determine the number of non-isomorphic triple systems with bipoints in those cases for which the total number of triples does not exceed 20.

1. Introduction.

The concept of triple systems with singular bipoints was introduced in [3] and [2]. We have v points of which i are bipoints and v-i are regular points. Every pair of regular points occurs once in the system, whereas every pair that involves a bipoint occurs twice. We normally assume that there are at least two regular points, since a single regular point would behave just like a bipoint.

Now any regular point must occur with v-i-1 other regular points and with 2i bipoints; this requires that v+i-1 should be even. Hence the parity of v and that of i must differ. This result is important enough to be stated as a separate lemma.

Lemma 1. If i is even, then v must be odd, whereas, if i is odd, then v must be even.

The number of blocks in one of these singular triple systems with bipoints (we call such a system a TS(2,i,v)) is readily determined. The number of pairs appearing in the blocks is

$$v(v-1) - (v-i)(v-i-1)/2 = [v^2 + (2i-1)v - (i^2 + i)]/2.$$

These pairs must occur in b triples, and so we have

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Lemma 2. $b = [v^2 + (2i-1)v - (i^2 + i)]/6$, and a necessary condition for the existence of a TS(2,i,v) is

$$v^2 + (2i-1)v - (i^2 + i) \equiv 0, \text{ modulo } 6.$$

We now let i assume its possible values, modulo 6, and obtain (recalling the fact that i and v have different parities):

Lemma 3. If $i \equiv 0$, then $v \equiv 1$ or 3, modulo 6.
If $i \equiv 1$, then $v \equiv 4$, modulo 6.
If $i \equiv 2$, then $v \equiv 3$, modulo 6.
If $i \equiv 3$, then $v \equiv 0$ or 4, modulo 6.
If $i \equiv 4$, then $v \equiv 1$, modulo 6.
If $i \equiv 5$, then $v \equiv 0$, modulo 6.

It is clear from Lemma 3 that there are only a few systems TS(2,i,v) with a small number of blocks. We list the cases in which there are at most 15 points (for each v, the number of blocks is given in parentheses).

For i = 0, v = 3(1) or 7(7) or 9(12) or 13(26) or 15(35).
For i = 1, v = 4 (3) or 10(18).
For i = 2, v = 3(2) or 9(17) or 15(39).
For i = 3, v = 4(4) or 6(9) or 10(23) or 12 (32).
For i = 4, v = 7(13) or 13(40).
For i = 5, v = 6(10) or 12 (37).
For i = 6, v = 7(14) or 9(23) or 13(45) or 15(58).
For i = 7, v = 10(29).
For i = 8, v = 9(24) or 15(63).
For i = 9, v = 10(30) or 12(43).
For i = 10, v = 13(51).
For i = 11, v = 12(44).
For i = 12, v = 13(52) or 15(69).
For i = 14, v =15(70).

In this paper, we discuss the number of non-isomorphic solutions for those TS(2,i,v) where the number of blocks does not exceed 20.

We omit the case i = 0, since that corresponds to Steiner triple systems, and it is known that the systems are unique for v = 3 or 7 or 9, that there are 2 solutions for v = 13, and that there are 80 solutions for v = 15; cf. Chapter 5 of [1].

We also omit the cases when i = v-1. In those cases, the regular point can not be distinguished from a bipoint and so the solution is just a twofold triple system. It is known that these twofold triple systems are unique for v = 3, 4, or 6; that there are 4 solutions for v = 7, 36 solutions for v = 9, and 960 solutions for v = 10; cf. Chapter 5 of [1].

Consequently, the only "small" cases (that is, cases in which the number of blocks does not exceed 20) are those where (i,v,b) equals (1,4,3), (1,10,18), (2,9,17), (3,6,9), and (4,7,13).

2. The Cases (i,v,b) = (1,4,3) and (1,10,18).

For i > 0, we use numbers to denote regular points and letters a, b, c, ... to denote bipoints. In the case v = 4, the design is unique and is simply the set of three triples a12, a13, a23.

The case when v = 10, b = 18, is more interesting. The bipoint a must appear twice with each of the 9 regular points. So it appears in 9 blocks. There are also 9 triples that contain only regular points.

Since each symbol appears twice with a, the blocks that contain the symbol a must contain a cycle union on the nine symbols 1, 2, 3, 4, 5, 6, 7, 8, 9.

Case 1. There are three 3-cycles in the cycle union. These can be taken as 123, 456, 789. Then the blocks containing a are a12, a13, a23, a45, a46, a56, a78, a79, a89. However, the three triples 123, 456, 789, together with the 9 other triples that must appear as blocks in the TS(2,1,10), form the BIBD(9,12,4,3,1), and this design is unique. Consequently, Case 1 leads to a unique solution (this solution is just the one that was used to establish the existence of a TS(2,1,10) in [2]).

Case 2. The cycle union consists of a 5-cycle and a 4-cycle. We take these cycles to be 12345 and 6789. In this case, we have triples a12, a23, a34, a45, a51, a67, a78, a89, a97. The other 9 blocks must contain blocks that include the pairs 13, 14, 24, 25, and 35. However, if j is any symbol from {1,2,3,4,5}, it has already appeared with two numbers in the triples containing a and it must appear in 3 numerical triples (it appears with a total of 8 regular points). So there have to be 5 more blocks that contain j (j= 1, 2, 3, 4, 5). This requires a total of 10 numerical triples (a contradiction). So Case 2 can not occur.

Case 3. The cycle union consists of a 6-cycle and a 3-cycle. We may take these as 123456 and 789.

Then we have 9 blocks a12, a23, a34, a45, a56, a61, a78, a89, a97. Note that symbol a occurs with two 1-factors of the complete graph K_6 on the set {1,2,3,4,5,6}, namely, F_1 = {12,34,56} and F_2 = {23,45,61}. The other 9 blocks have the form $7F_3$, $8F_4$, and $9F_5$., where F_3, F_4, and F_5 are three further 1-factors. Since the 1-factorization of {1,2,3,4,5,6} is unique, we have a unique solution in Case 3.

Case 4. The cycle union is a Hamiltonian cycle 123456789. Then we have blocks a12, a23, a34, a45, a56, a67, a78, a89, a91. We form a skeleton for the nine remaining blocks as: 13-, 1--, 1--, 29-, 2--, 2--, 39-, 3--, 9--. The symbol 8 appears 3 times; it appears with 1, 2, 3, but not with 9. So we fill in the skeleton as: 13-, 18-, 1--, 29-, 28-, 2--, 39-, 38-, and 9--.The pairs 47, 46, and 57 must appear in the blocks 1--, 2--, 9--.

Case 4a. 157 is a block. Then 136 and 184 are also blocks. Now 7 must appear with elements 2, 3, 4, 9. Since it must appear with 39, we get blocks 397, 247. Then blocks 286, 385, 946, 295, are forced; this completes the design. So there is a unique solution in Case 4a; it is generated by cycling 157, modulo 9.

Case 4b. 146 is a block. In this case, 137 and 158 must be blocks. The remaining 6 blocks are 29-, 28-, 27-, 39-, 38-, 97-. Now symbol 4 must appear with 2, 7, 8, 9, and this forces triples 974 and 284. The remaining blocks are uniquely completable as 275, 296, 386, 395. This solution is also cyclic and is found by cycling 146, modulo 9. Since the starters 146 and 157 are equivalent under reflection in the basic 9-cycle, the solutions in Cases 4a and 4b are isomorphic.

Case 4c. 147 is a block; then 135 and 186 may also be blocks. The remaining 6 blocks are 29-, 28-, 2--, 39-, 38-. 9--. Now symbol 7 must appear with 2, 3, 5, 9, and so we must take blocks 397 and 257. Then symbol 5 must appear with 8 and 9; so completion is impossible.

On the other hand, if 147 is a block and 136, 185, are blocks, then the other blocks are 29-, 28-, 2--, 39-, 38-, 9--. Again, symbol 7 must appear with 2, 3, 5, 9, and so we must take blocks 397 and 257. occurred. Then symbol 5 must appear with 3 and 9; so this is also impossible, and thus Case 4c can not occur.

Our conclusions can be embodied in

Theorem 1. There exist three non-isomorphic systems TS(2,1,10). The first is found by taking the Kirkman Triple System on 9 points and expanding one resolution class as in [2]. The second is found by taking the five one-factors of a complete graph on the set {1,2,3,4,5,6}, adjoining elements 7, 8, 9, a, a, to these 1-factors, and completing the system with the blocks a78, a89, a97.. The third is found by cycling the starter blocks 157 and a12 (a being invariant), modulo 9.

3. The Case (i,v,b) = (2,9,17)

In the case that v = 9 and i = 2, we have 7 regular points, 2 bipoints, and a total of 17 blocks.

Case 1. The blocks containing ab are ab1 and ab2. Then there are 6 more blocks with a that collectively contain symbols 1 and 2, and two each of the symbols 3, 4, 5, 6, 7. A similar statement holds for b. This leaves 3 triples made up of only regular points. Since any regular point must occur with 6 other regular points, we see that 1 and 2 each occur with 4 regular points in these triples, whereas 3, 4, 5, 6, 7, only occur with 2 regular points. Hence we may take the triples as 125, 134, 267. The remaining pairs are 16, 17, 23, 24, 35, 36, 37, 45, 46, 47, 56, 57, and these occur in the 12 blocks that contain a or b. Hence we may assign the triples 16a, 17b, 23a, 24b.

Case 1a. If 36a is a block, we get 35b, 37b, 46b, 56b; this forces blocks 45a, 47a, 57a (Solution 1a).

Case 1b. If 36b is a block and we try 35a, 37b, we are forced to take 47a,57a, and then 56b, 45b, and 46a (Solution 1b).

Case 1c. If 36b is a block and we try 35b, 37a, then we obtain either 46b, 57b, with 45a, 47a, 56a (Solution 1c), or we obtain 47b, 56b, with 45a, 46a, 57a (Solution 1d).

This construction has produced four solutions. However, these solutions are not all non-isomorphic. Solutions 1b and 1c are isomorphic under the permutation (12)(36)(47). And Solutions 1a and 1d are isomorphic under the permutation (12)(37)(46)(ab).

By considering the cycle structure of the numbers in the blocks containing a and b, we see that Solutions 1a and 1b are not isomorphic. Each design has a group of order 2; that for the first solution is generated by (12)(36)(47), and that for the second solution is generated by (34)(67) (ab).

We pass on to:

Case 2. The blocks that contain ab are ab1 and ab1. Then there are 6 more blocks with a that contain 2 symbols each from the set G = {2,3,4,5,6,7}; the same statement is true for b. By considering frequencies, we see that the three regular triples must all contain 1 and they must contain 2,3,4,5,6,7, once each.

We thus see that 1 occurs with a 1-factor from G, and a and b each occur with cycle unions from G.

Case 2a. The cycle unions with a and b are each 6-cycles. This gives two one-factors with a and 2 one-factors with b. Since G has a unique one-factorization, the solution is unique as 1F1, aF2, aF3, bF4, bF5.

Case 2b. The cycle union with a is a 6-cycle and that with b is made up of two 3-cycles. In this case, we have three one-factors (one with symbol 1 and two with symbol a) as well as two 3-cycles. This is just the unique decomposition of G into three 1-factors and one triangle factor, namely, into F1 = (72,34,56), F2 = (23,45,67), F3 = (74,25,36), and T1 = 357, T2 = 246. The unique solution is thus aF1, aF2, 1F3, bT1 , bT2. We understand this notation to mean that b occurs with the pairs from T1 and T2.

Case 2c. The cycle unions with a and b consist of two 3-cycles each. We take the three numerical triples as 123, 145, 167. Then it is not possible to get two cycle unions on 2,3,4,5,6,7, to go with a and b. So this case is not possible.

Our result is

Theorem 2. There are four non-isomorphic systems TS(2,2,9).

4. The Cases (i,v,b) = (3,6,9) and (4,7,13).

The case i = 3, v = 6, b = 9 is straightforward. There are 3 regular points and 3 bipoints. Let p, q, r, and s be the numbers of blocks of types xxx, xxn, xnn, and nnn, respectively, where x is a bipoint (letter) and n is a regular point (number) Then 2q + 2r = 2(3)(3), that is, q + r = 9; this accounts for all the blocks. So the triples must be abn, abn, acn, acn, bcn, bcn, a23, b13, c12 (each letter must occur with 6 numbers).Then it is easy to complete the design as ab1, ab2, ac1, ac3, bc2, bc3.

In the case where l = 4, v = 7, b = 13, there are 4 bipoints and 3 regular points. We find that q+r = 12.

Case 1. There is a triple 123 and 2 triples with each of ab, ac, ad, bc, bd, cd.

One obvious solution is to place 1 with ab, ab, cd, cd; 2 with ac, ac, bd, bd; 3 with ad, ad, bc, bc.

If 1 does not occur with a repeated letter pair, then we may take 1ab, 1ac, and this forces 1db, 1dc. Then we may complete the design with 2ad, 2ad, 2bc, 2bc, which forces 3ab, 3ac, 3db, 3dc. Or we may complete the design with 2ad, 2ab, 2cd, 2cb, which forces 3ad, 3ac, 3bd, 3bc.

So we have three possible designs in Case 1.

Case 2. There is a triple abc. Then there is one triple with each of ab, ac, bc, and 2 triples with each of ad, bd, cd, as well as 3 triples that contain 12, 13, and 23. So symbol a occurs with 4 numbers in the pairs xxn; it thus can appear with only two numbers in the pairs xnn. The same statement is true for b and c. This proves that we must take the triples 12c, 13b, and 23a.

If we have triples ad1, ad1, bd2, bd2, then we are forced to take cd3, cd3, and then ab3, ac2, bc1.

If we have triples ad1, ad1, bd2, bd3, then we are forced to take cd2, cd3, and then bc1, ab2, ac3.

If we have triples ad1, ad2, bd1, bd3, cd2, cd3, then we reach a contradiction.

If we have triples ad1, ad2, bd2, bd3, cd1, cd3, then we are forced to take ab1, bc2, ac3.

So there are also three solutions in Case 2.

We thus have:

Theorem 3. There are six non-isomorphic systems TS(2,4,7).

6. Conclusion.

The next two cases are when (i,v,b) = (3,10,23) and (6,9,23). For these cases, the number of solutions increases markedly, and we have not completed the determination of the total number of possibilities.

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