Weakly Completable Sets in Latin Squares

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ABSTRACT. Until quite recently, very few weakly completable critical sets were known. The purpose of this note is to prove the existence of at least one latin square of each order greater than four in which a weakly completable set exists. This is done by actual construction of such a square. Non-existence of weakly completable sets in latin squares of orders 2, 3 and 4 is already known.

1 Introduction

A long-standing problem concerns the sizes of critical sets in latin squares: that is, subsets of the cell entries which are completable to only one latin square and such that, if one member of the subset is removed, the unique completion property ceases to hold. A problem of particular difficulty has been that of finding so-called weakly completable critical sets in latin squares (despite their anticipated abundance): in particular, to find a latin square of smallest order which possesses a weakly completable set. It is known that this smallest order must exceed 4 (see Keedwell [3] or Burgess [1]). In the present paper, we solve this problem by first constructing a latin square of order 5 which possesses a weakly completable set and, further, constructing a latin square of each order from 6 to 9 with that property, from which latin squares of higher orders can be easily produced.

2 Definitions

A *latin square* of order n is a square matrix with n^2 cells (i, j), each filled with a number k in the range [1, n] such that each of these numbers occurs exactly once in each row and once in each column. A *uniquely completable set* C for a latin square L of order n is a subset of the triples (i, j, k) of L, where the entry of the i th row and j th column of L is k, such that (i) L is the only latin square of order n which has these particular triples. If also (ii) no proper subset of C satisfies (i), then C is said to be a *critical set* for L. That is, C is an irreducible set of triples from which L can be reconstructed uniquely. A *minimal critical set* for L is a critical set for L of smallest cardinal.

Given any subset C of triples of a latin square L of order n, the following process may be followed.

Find, if such a one exists, a triple $(i, j, k) \in (L \setminus C)$ which satisfies at least one of the following three conditions:

(i) $\forall\, h \neq i, \exists\, z$ such that $(h, j, z) \in C$ or $(h, z, k) \in C$;
(ii) $\forall\, h \neq j, \exists\, z$ such that $(z, h, k) \in C$ or $(i, h, z) \in C$;
(iii) $\forall\, h \neq k, \exists\, z$ such that $(i, z, h) \in C$ or $(z, j, h) \in C$.

JCMCC 34 (2000), pp. 65-69

We say that (i, j, k) is *forced* in C.

Add (i, j ,k) to C. Repeat the process (*forcing*) until no such further triples can be found. This final set C is called the *strong partial completion* of the original set C in L. If the strong partial completion of C in L is L itself, C is called a *strongly completable set.* A uniquely completable set which is not a strongly completable set is a *weakly completable set.*

(The latter concepts are given different names by some authors.)

3 Existence Result

We shall require the following lemma from [2]:

Lemma. *Let L_k be an arbitrary latin square of order k. Then, if $k \leq n/2$, there exists at least one latin square L_n of order n such that L_k is a latin subsquare in L_n.*

Theorem. *There is at least one latin square of each order greater than or equal to 5 for which a weakly completable set exists.*

Proof: Let W_5={(1, 1, 1), (1, 2, 2), (1, 3, 3), (2, 1, 2), (2, 2, 1), (2, 5, 3), (4, 1, 4), (4, 2, 3), (4, 5, 2), (5, 5, 1)} and L_5 be as below (ignoring subscripts).

$$W_5 = \begin{bmatrix} 1 & 2 & 3 & . & . \\ 2 & 1 & . & . & 3 \\ . & . & . & . & . \\ 4 & 3 & . & . & 2 \\ . & . & . & . & 1 \end{bmatrix} \qquad L_5 = \begin{bmatrix} 1 & 2 & 3 & 4_{10} & 5_8 \\ 2 & 1 & 4_{14} & 5_{12} & 3 \\ 3_0 & 5_3 & 1_6 & 2_{13} & 4_4 \\ 4 & 3 & 5_9 & 1_{11} & 2 \\ 5_1 & 4_2 & 2_5 & 3_7 & 1 \end{bmatrix}$$

If $n \geq 10$, there exists a latin square L_n of order n such that L_5 is a latin subsquare of L_n by the Lemma. Let $W_n=(L_n \setminus L_5) \cup W_5$.

Now consider the case where n=5 or $n \geq 10$.

First we confirm that no element is forced in W_n as follows. Suppose that (i, j, k) is forced in W_n by condition (i), (ii) or (iii). Note that there are no entries in row 3 or in column 4 and that the integer 5 is not used. If $i \neq 3$, we may contradict the supposition that (i, j, k) can be forced by means of condition (i) by choosing h=3 in that condition. (It is clearly false that $\exists$ z such that (3, j, z)$\in W_n$ or (3, z, k)$\in W_n$.) Similarly if $j \neq 4$, we may contradict the supposition that (i, j, k) can be forced using condition (ii) by choosing h=4 in that condition and if $k \neq 5$, we may contradict the supposition that (i, j, k) can be forced using condition (iii) by choosing h=5 in that condition. Suppose now that i=3 in condition (i) or j=4/k=5 in conditions (ii)/(iii) respectively. In the first case, put h=5 in condition (i) unless j=5 or k=1 to get a contradiction to the supposition that (i, j, k) can be forced using that condition. In the latter sub-cases, put h=1, 4 respectively. In the second case, put h=3 in condition (ii) unless i=1 or k=3: in the latter sub-cases, put h=5, 1 respectively. Finally, in the third case, put h=4 unless i=4 or j=1: in the latter sub-cases, put h=1, 3 respectively.

We now show that W_n is a uniquely completable (and, hence, a weakly completable) set in L_n as follows. Let L'_n be any latin square of order n containing W_n. L'_n must contain (3, 1, 3) or (3, 1, 5). If (3, 1, 5)$\in L'_n$, then (3, 2, 4) and (3, 5, 4) are both forced (by condition (iii)). But then the third row contains two fours, contradicting the fact that L'_n is a latin square. Therefore (3, 1, 3)$\in L'_n$. All of the remaining triples are now forced and the subscripts of the representation above show one order of forcing by condition (iii). That is, $L'_n = L_n$.

Next we consider in turn the cases n=6, 7, 8, 9.

Let W_6 and L_6 be as below (ignoring subscripts). This example is taken from [4].

$$\underline{W_6} \qquad \begin{bmatrix} 1 & . & . & . & . & . \\ . & . & . & 6 & 4 & . \\ . & . & 2 & . & 6 & . \\ . & 5 & 6 & . & . & 3 \\ . & . & . & 3 & . & 2 \\ . & 4 & 5 & 2 & . & . \end{bmatrix}$$

$$\underline{L_6} \qquad \begin{bmatrix} 1 & 2_{15} & 3_3 & 4_{19} & 5_{21} & 6_9 \\ 2_{11} & 3_{12} & 1_0 & 6 & 4 & 5_2 \\ 3_{10} & 1_{13} & 2 & 5_{14} & 6 & 4_8 \\ 4_{17} & 5 & 6 & 1_{18} & 2_{20} & 3 \\ 5_6 & 6_{16} & 4_1 & 3 & 1_{22} & 2 \\ 6_5 & 4 & 5 & 2 & 3_4 & 1_7 \end{bmatrix}$$

First we confirm that no element is forced in W_6 as follows. Suppose that (i, j, k) is forced in W_6 by condition (i), (ii) or (iii). Note that each of {2, 3, 4, 5, 6} occurs at most three times in W_6. So, it is immediate to see that (i, j, k) cannot be forced by condition (i) if j=1 or by condition (ii) if i=1. Note also that each row and column of W_6 contains at most three entries. Thus, since the entry 1 occurs only once in W_6, (i, j, k) cannot be forced by condition (i) or by condition (ii) if k=1 and, since the first row and column each contain just one entry, (i, j, k) cannot be forced by condition (iii) if i=1 or j=1. Suppose next that $2 \le i,j,k \le 6$. Putting h=1 in conditions (i), (ii) and (iii) shows that, no triple (i, j, k) whose entries satisfy this inequality can be forced by any one of the three conditions. The only remaining cases are i=1 and $2 \le j,k \le 6$ for condition (i), j=1 and $2 \le i,k \le 6$ for condition (ii), and k=1 and $2 \le i,j \le 6$ for condition (iii). We may contradict the fact that (i, j, k) can be forced in W_n in these cases by putting h=2 if j=3 or h=j otherwise in condition (i); h=2 if i=3 or h=i otherwise in condition (ii); h=3 if i=2 or h=i otherwise in condition (iii).

We now show that W_6 is a uniquely completable (and, hence, a weakly completable) set in L_6 as follows. Let L'_6 be any latin square containing W_6. L'_6 must contain (2, 3, 1) or (2, 3, 3). If (2, 3, 3)$\in L'_6$, then the following triples are successively forced: (1, 3, 4), (5, 3, 1), (5, 2, 6), (3, 6, 4), (6, 1, 6), (3, 1, 3). Then (3, 4, 5) and (1, 4, 5) are both forced (by condition (ii) and condition (iii) respectively). This means that the fourth column contains two fives, contradicting the fact that L'_6 is a latin square. Therefore (2, 3, 1)$\in L'_6$. All of the remaining triples are now forced and the subscripts of the representation above show one order of forcing with one application of condition (i) (to force (6, 5, 3)) and the remainder by condition (iii). That is, $L'_6 = L_6$.

Let W_7 and L_7 be as below (ignoring subscripts).

$$W_7 = \begin{bmatrix} 1 & . & . & . & . & . & 7 \\ . & . & . & 7 & 4 & . & 6 \\ . & . & 7 & . & 6 & . & 2 \\ . & 5 & 6 & . & . & 7 & 3 \\ . & . & . & 3 & 7 & 2 & . \\ . & 7 & 5 & 2 & . & . & 4 \\ 7 & 4 & 2 & 6 & . & 3 & . \end{bmatrix} \qquad L_7 = \begin{bmatrix} 1 & 2_{15} & 3_3 & 4_{19} & 5_{21} & 6_9 & 7 \\ 2_{11} & 3_{12} & 1_0 & 7 & 4 & 5_2 & 6 \\ 3_{10} & 1_{13} & 7 & 5_{14} & 6 & 4_8 & 2 \\ 4_{17} & 5 & 6 & 1_{18} & 2_{20} & 7 & 3 \\ 5_6 & 6_{16} & 4_1 & 3 & 7 & 2 & 1_{24} \\ 6_5 & 7 & 5 & 2 & 3_4 & 1_7 & 4 \\ 7 & 4 & 2 & 6 & 1_{22} & 3 & 5_{23} \end{bmatrix}$$

Note that the construction of L_7 from L_6 is almost a prolongation as defined in [2]. [In L_6 observe that the entries (2, 4), (3, 3), (4, 6), (5, 5), (6, 2) form a partial transversal. Replace these entries by the symbol 7 and form an additional row and column with the entries so replaced and an entry 7 in cells (1, 7) and (7, 1) and an entry 5 in cell (7, 7).] We see that the entries which occur in a given row or column of W_7 (other than the last) are the same as those which occur in the corresponding row or column of W_6 together with an additional entry 7. Also, there are no cells of L_7 which are filled in W_6 but are not filled in W_7 and none of the other cells which are empty in W_7 (that is, none of (5, 7), (7, 5), (7, 7)) are forced in W_7. Finally, the only cell which is filled in W_7 but empty in W_6 is the cell (5, 5). In W_6, the empty cell (5, 5) can only be filled with the entries 1 or 5. In W_7, these two entries can go in the empty cells (5, 7) and (7, 5) instead and these cells are respectively in the same row and column as (5, 5).

Therefore, since no element is forced in W_6, neither is one in W_7. We now show that W_7 is a uniquely completable and, hence, a weakly completable set in L_7 as follows. Let L'_7 be any latin square containing W_7. We deduce that $(2, 3, 1) \in L'_7$ in the same manner as we deduced that $(2, 3, 1) \in L'_6$ above. All of the remaining triples are now forced and the subscripts of the representation above show one order of forcing. (For the first 21 triples this is the same order as for L_6.) That is, $L'_7 = L_7$.

Finally, for n=8 or n=9, we let W_n be as below. W_8 is obtained from W_7 by a prolongation using the transversal whose cells are (1, 1), (2, 4), (3, 5), (4, 7), (5, 6), (6, 3), (7, 2) and then interchanging the elements 1, 8 in the first and last rows. Similarly, W_9 is obtained from W_8 by a prolongation using the transversal whose cells are (1, 1), (2, 7), (3, 5), (4, 6), (5, 4), (6, 8), (7, 3), (8, 2) and then interchanging the elements 1, 9 in the first and last rows.

$\underline{W_8}$

$$\begin{bmatrix} 1 & . & . & . & . & . & 7 & 8 \\ . & . & . & 8 & 4 & . & 6 & 7 \\ . & . & 7 & . & 8 & . & 2 & 6 \\ . & 5 & 6 & . & . & 7 & 8 & 3 \\ . & . & . & 3 & 7 & 8 & . & 2 \\ . & 7 & 8 & 2 & . & . & 4 & 5 \\ 7 & 8 & 2 & 6 & . & 3 & . & 4 \\ 8 & 4 & 5 & 7 & 6 & 2 & 3 & 1 \end{bmatrix}$$

$\underline{W_9}$

$$\begin{bmatrix} 1 & . & . & . & . & . & 7 & 8 & 9 \\ . & . & . & 8 & 4 & . & 9 & 7 & 6 \\ . & . & 7 & . & 9 & . & 2 & 6 & 8 \\ . & 5 & 6 & . & . & 9 & 8 & 3 & 7 \\ . & . & . & 9 & 7 & 8 & . & 2 & 3 \\ . & 7 & 8 & 2 & . & . & 4 & 9 & 5 \\ 7 & 8 & 9 & 6 & . & 3 & . & 4 & 2 \\ 8 & 9 & 5 & 7 & 6 & 2 & 3 & 1 & 4 \\ 9 & 4 & 2 & 3 & 8 & 7 & 6 & 5 & 1 \end{bmatrix}$$

Since the empty cells in W_8 and W_9 are the same as those in W_7 and, since, in any particular row or column, the symbols available to fill these empty cells are the same in W_8 or W_9 as in W_7 (by the nature of a prolongation), W_8 and W_9 are weakly completable (uniquely) to latin squares L_8 and L_9 by virtue of the fact that W_7 is weakly completable to the latin square L_7.

This completes the proof.

Acknowledgement

The author thanks his PhD supervisor A D Keedwell for introducing him to this problem and for helpful comments and advice.

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