A Note on Dickson-Stirling Numbers

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This note shows that Dickson-Stirling numbers of the second kind has some naive applications to the combinatorial enumeration of arithmetic mappings satisfying certain conditions.

Recall that Dickson polynomial $D_n(x,a)$ of degree $n \geq 1$ in x and with a real number parameter a may be defined by

$$D_n(x,a) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} (-a)^k x^{n-2k} \tag{1}$$

where $\lfloor\rfloor$ denotes the greatest integer function (cf. [3]).

Very recently we and Mullen [1] have introduced two kinds of generalized Stirling numbers, (so called Dickson-Stirling numbers), of which the second kind is defined by

$$D_n(x,a) = \sum_{k=0}^{n} S(n,k;a)(x-a)_k, \quad n = 1, 2, \cdots, \tag{2}$$

where $(x-a)_k = (x-a)(x-a-1)\cdots(x-a-k+1)$, with $(x-a)_0 = 1$. For $n = 0$ we define $S(0,0;a) = 1$. Note that $D_n(x,0) = x^n$, $(n \geq 1)$ so that the $a = 0$ case of (2) yields the ordinary Stirling numbers of the second kind, viz. $S(n,k;0) = S(n,k)$.

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Clearly, Newton's interpolation formula gives

$$S(n,k;a)=\frac{1}{k!}\,\Delta^k D_n(x,a)\big|_{x=a} \tag{3}$$

where Δ is the difference operator defined by $\Delta f(x)=f(x+1)-f(x)$.

In this note we are concerned with arithmetic functions defined on the set $[1,n]$, i.e., the set of positive integers from 1 to n. Some notations to be used may refer to, e.g., Stanley's [4]. We shall need a terminology "$\bar{a}$-property" as defined by the following:

For a given positive integer a, an arithmetic function f defined on $[1,n]$ is said to have $\bar{a}$-property if the cyclic sequence $f(1)$, $f(2),\cdots$, $f(n)$, $f(1)$ does not possess any pair of consecutive terms which take the same integer in $[1,a]$.

What we wish to prove are the following.

Proposition 1. *Let $m\geq 1, n\geq 1$ and a with $0\leq a<m$ be integers. Then $D_n(m,a)-D_n(a,a)$ counts the number of functions $f:[1,n]\to[1,m]$ that have the $\bar{a}$-property and take at least one integer of the set $[a+1,m]$.*

Proposition 2. *Let $n\geq k\geq 1$ and $a\geq 0$ be integers. Then $k!S(n,k;a)$ counts the number of functions $f:[1,n]\to[1,a+k]$ that have the $\bar{a}$-property and take all the k values of $[a+1,a+k]$.*

Our proof will make use of Kaplansky's cycle theorem which asserts that the number of ways of selecting k disjoint pairs of consecutive objects from n objects arranged in a cycle is given by $\frac{n}{n-k}\binom{n-k}{k}$, where $k\leq[n/2]$ and the pairs are counted in a clockwise fashion (cf. [2]).

Proof of Propositions 1 and 2. In accordance with (1) one may write

$$D_n(m,a)-D_n(a,a)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}a^k\left(m^{n-2k}-a^{n-2k}\right). \tag{4}$$

Using Kaplansky's cycle theorem, one may see that the general term without the factor $(-1)^k$ occurring on the RHS of (4) just counts the number of such functions $f:[1,n]\to[1,m]$ that there are k pairs of consecutive terms of the cyclic sequence $f(1)$, $f(2)$, $\cdots$, $f(n)$, $f(1)$, each of which takes the same integer of $[1,a]$, and that the remaining $(n-2k)$ values of f take at least one integer of the set $[a+1,m]$. Applying the inclusion-exclusion principle to the RHS of (4), it is clear that the alternating sum just counts the number of functions f satisfying the conditions of Proposition 1. Hence the combinatorial meaning of $D_n(m,a)-D_n(a,a)$ is verified. □

In proving Proposition 2, it suffices to note that (using (3))

$$k!S(n,k;a)=\Delta^k\left(D_n(x,a)-D_n(a,a)\right)\big|_{x=a} \tag{5}$$

Clearly one may rewrite (5) in the form

$$k!S(n,k;a) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{k-j} (D_n(a+k-j,a) - D_n(a,a)) \quad (6)$$

Here it may be observed that for every $(k-j)$ element subset of $[a+1, a+k]$ say $\{a_1, a_2, \cdots, a_{k-j}\}$ with $(a+1) \leq a_1 < \cdots < a_{k-j} \leq (a+k)$, the factor $(D_n(a+k-j,a) - D_n(a,a))$ always counts the number of functions $f : [1,n] \to [1,a] \cup \{a_1, \cdots, a_{k-j}\}$ that have the $\bar{a}$-property and take at least one value of $\{a_1, \cdots, a_{k-j}\}$. Consequently, the general term apart from the factor $(-1)^j$ on the RHS of (6) gives the total counting number for which there are exactly j values of $[a+1, a+k]$ not included in the range of f's. Thus, an application of the inclusion-exclusion principle to (6) just yields the conclusion of Proposition 2. □

Remark 1. Just like the case for the ordinary Stirling numbers $S(n,k)$, one can also interpretate $S(n,k;a)$ as a number of partitions of some set. To be more precise, let S be a set of n distinct elements, and denote it by $\bar{S}$ when the elements are cyclically ordered. Let $n \geq k \geq 1$ and $a \geq 0$ be integers with $a+k \leq n$. Then $S(n,k;a)$ counts the number of ways of partitioning the set S into k non-empty unordered subsets and, in addition, into at most a ordered subsets, of which no one contains two consecutive elements in $\bar{S}$.

Obviously, in the case $a = 0$ the above statement and Proposition 2 yield the classical combinatorial meanings of $S(n,k)$ and $S(n,k)\,k!$ respectively.

Remark 2. Proposition 2 can be slightly extended to the form: Let $n \geq k \geq 1$, $a \geq 0$ and $m > a+k$ be integers. Then $S(n,k;a)\,(m-a)_k$ counts the number of functions $f : [1,n] \to [1,m]$ that have the $\bar{a}$-property and take exactly k distinct values in $[a+1, m]$. Actually, this follows easily from the combinatorial meaning of $\binom{m-a}{k}$ and the fact that $(m-a)_k = k!\binom{m-a}{k}$, (cf. Proposition 2).

Finally, it may be worth mentioning that a shorter proof of Proposition 2 can even be obtained with the aid of Proposition 2.2.2 of Stantley's *Enumerative Combinatorics* [4]. Related details may be left to the interested reader.

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