A Theorem on 2-concurrence Designs

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ABSTRACT. In this paper we define the imbalance of equi-replicate incomplete block designs. We prove that the imbalance measure of an equi-replicate incomplete block design has a lower bound, and this bound is attained if and only if the design is a 2-concurrence design. This result allows one to formulate the construction of 2-concurrence designs as an optimization problem.

1 Introduction

An equi-replicate incomplete block (EIB) design is a collection of b blocks of size k on a v-set such that every element occurs in r blocks. Clearly we have $vr = bk$. We shall refer to the elements s and t as ith associates if they occur together in λ_i blocks. Then, we define a 2-concurrence design (see [3]) as an equi-replicate incomplete design satisfying the conditions below:

(1) Any two elements are either first or second associates.

(2) Each element has exactly n_i ith associates ($i = 1, 2$).

The numbers v, b, r, k, λ_1, λ_2, n_1 and n_2 are called the parameters of the design. It is not hard to see that these parameters satisfy:

$$vr = bk, \quad v - 1 = n_1 + n_2, \quad r(k-1) = n_1\lambda_1 + n_2\lambda_2. \tag{1}$$

The *concurrence* matrix of an EIB design with parameters v, b, r and k is a $v \times v$ matrix $B = (b_{ij})$, where b_{ij} is the number of blocks containing both the i and j elements. Note that an EIB design is a 2-concurrence design if and only if each row of its concurrence matrix consists of n_1 λ_1's, n_2 λ_2's and r on the diagonal.

A 2-concurrence design is called a regular graph design if $\lambda_2 = \lambda_1 + 1$. For the rest of this section, v, b, r, k, λ_1, λ_2, n_1 and n_2 will be integers satisfying (1), and $\lambda_2 = \lambda_1 + 1$.

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In [1] and [2] the *imbalance* of an EIB design with parameters v, b, r and k was defined as

$$\sigma = \sum_{i<j}(b_{ij} - \lambda_1)^2, \tag{2}$$

where (b_{ij}) is the concurrence matrix of the design. Let $\sigma_0 = b\binom{k}{2} - \lambda_1\binom{v}{2}$. It is not hard to check that for regular graph designs, $\sigma_0 = vn_2/2$. The author of [1] proved that for any EIB design with parameters v, b, r and k, $\sigma \geq \sigma_0$ and $\sigma = \sigma_0$ if and only if the design is a regular graph design with parameters v, b, r, k λ_1, λ_2, n_1 and n_2. This result was used in [2], [4], [5], [6], [7], [8] and [9] to formulate the construction of regular graph designs as a minimization problem with objective function σ. Then using a hill-climbing algorithm in [4] and [7], a tabu search procedure in [5], [6] and [9], and genetic algorithm in [8] some regular graph designs have been constructed. However, this result cannot be applied to construct non-regular designs.

In this work, we shall prove a similar result for general 2-concurrence designs. This also allows one to formulate the construction of these designs as an optimization problem.

2 The Theorem

Given two integers α and β we define the function

$$H_{\alpha,\beta}(x) = \begin{cases} (x-\alpha)^2, & \text{if } x \neq \beta, \\ 1, & \text{if } x = \beta. \end{cases}$$

Let v, b, r, k, λ_1, λ_2, n_1 and n_2 be integers satisfying (1), and $\lambda_1 < \lambda_2$. Let D be an EIB design with parameters v, b, r and k. Then, the (λ_1, λ_2)-*imbalance* of design D is defined as

$$\sigma_{\lambda_1,\lambda_2}(D) = \sum_{i<j} H_{\lambda_1,\lambda_2}(b_{ij}), \tag{3}$$

where (b_{ij}) is the concurrence matrix of D. It is not hard to see that the (λ_1, λ_2)-imbalance of a 2-concurrence design is $vn_2/2$. Note that the functions (2) and (3) are equal whenever $\lambda_2 = \lambda_1 + 1$.

Theorem 1 *Let v, b, r, k, λ_1, λ_2, n_1 and n_2 be integers satisfying* (1), *and $\lambda_1 < \lambda_2$. Let D be an EIB design with parameters v, b, r and k. Then, $\sigma_{\lambda_1,\lambda_2}(D) \geq vn_2/2$. Furthermore, $\sigma_{\lambda_1,\lambda_2}(D) = vn_2/2$ if and only if D is a 2-concurrence design with parameters v, b, r, k, λ_1, λ_2, n_1 and n_2.*

Proof: The entries of the concurrence matrix B of D are nonnegative integers. So, without loss of generality we can suppose that the ith row of B has

$$n_{i1} \text{ elements } \lambda_1, \; n_{i2} \text{ elements } (\lambda_1 + h_{i2}), \ldots, n_{im} \text{ elements } (\lambda_1 + h_{im})$$

and r on the diagonal, where n_{ij} is a nonnegative integer, $|h_{ij}| \geq 1$ and the h_{ij}'s are different for $j = 2, \ldots, m$. To simplify the notation we will omit the subindices λ_1 and λ_2 of H. It is easy to see that

$$\sigma_{\lambda_1,\lambda_2}(D) = \frac{1}{2}\sum_{i=1}^{v} n_{i2}H(\lambda_1 + h_{i2}) + \cdots + n_{im}H(\lambda_1 + h_{im}). \qquad (4)$$

We shall prove that this value is not less than $vn_2/2$. From the concurrence matrix definition, the sum of all entries excepting the diagonal element of each row of B is $r(k-1)$, and $(v-1) = n_{i1} + \cdots + n_{im}$. Thus we get

$$r(k-1) = (v-1)\lambda_1 + n_{i2}h_{i2} + \cdots + n_{im}h_{im}. \qquad (5)$$

Let $h = \lambda_2 - \lambda_1$. Since $v - 1 = n_1 + n_2$, then $r(k-1) = (v-1)\lambda_1 + n_2 h$. This and (5) imply that

$$n_2 h = n_{i2}h_{i2} + \cdots + n_{im}h_{im},$$

which gives

$$n_2 \leq n_2|h| \leq n_{i2}|h_{i2}| + \cdots + n_{im}|h_{im}|. \qquad (6)$$

If $h_{ij} \neq h$ for $j = 2, \ldots, m$, then by definition of function H, $H(\lambda_1 + h_{ij}) = h_{ij}^2 \geq |h_{ij}|$ for $j = 2, \ldots, m$. It follows from (6) that

$$n_2 \leq n_{i2}H(\lambda_1 + h_{i2}) + \cdots + n_{im}H(\lambda_1 + h_{im}).$$

Suppose now that some $h_{ij} = h$; to simplify the notation take $h_{i2} = h$. Note that $h_{ij} \neq h$ for $j = 3, \ldots, m$ and $H(\lambda_1 + h) = 1$. We consider two cases:

If $n_2 \leq n_{i2}$ then $n_2 \leq n_{i2}H(\lambda_1 + h)$. Since $n_{i3}H(\lambda_1 + h_{i3}) + \cdots + n_{im}H(\lambda_1 + h_{im}) \geq 0$, we have

$$n_2 \leq n_{i2}H(\lambda_1 + h) + n_{i3}H(\lambda_1 + h_{i3}) + \cdots + n_{im}H(\lambda_1 + h_{im}).$$

If $n_2 > n_{i2}$ then $-n_2(|h| - 1) \leq -n_{i2}(|h| - 1)$ because $|h| \geq 1$. This and (6) show that

$$n_2|h| - n_2(|h| - 1) \leq n_{i2}|h_{i2}| - n_{i2}(|h| - 1) + n_{i3}|h_{i3}| \cdots + n_{im}|h_{im}|. \quad (7)$$

Since $n_{i2}H(\lambda_1 + h) = n_{i2}|h| - n_{i2}(|h| - 1)$ and $H(\lambda_1 + h_{ij}) = h_{ij}^2 \geq |h_{ij}|$ for $j = 3, \ldots, m$, then from (7) we obtain

$$n_2 \leq n_{i2}H(\lambda_1 + h_{i2}) + n_{i3}H(\lambda_1 + h_{i3}) + \cdots + n_{im}H(\lambda_1 + h_{im}).$$

In any case, we have shown that the (λ_1, λ_2)-imbalance of each row of the concurrence matrix B is greater than or equal to n_2. Hence, from (4) we have $\sigma_{\lambda_1,\lambda_2}(D) \geq n_2 v/2$.

Now let us prove the second part of the theorem. Clearly, if the design D is a 2-concurrence design, then $\sigma_{\lambda_1,\lambda_2}(D) = vn_2/2$. We divide the proof of the converse implication into two cases:

Case (1) if $|h| = 1$. Since $\lambda_1 < \lambda_2$, we have $\lambda_2 = \lambda_1 + 1$. Therefore, the functions (2) and (3) are equal; and the rest of the theorem follows from Proposition 1 of [1].

Case (2) if $|h| > 1$. From the first part of the theorem, the (λ_1, λ_2)-imbalance of each row of the concurrence matrix B is greater than or equal to n_2. The hypothesis $\sigma_{\lambda_1,\lambda_2}(D) = vn_2/2$ and (4) imply that, for any $1 \leq i \leq v$,

$$n_2 = n_{i2}H(\lambda_1 + h_{i2}) + \cdots + n_{im}H(\lambda_1 + h_{im}), \tag{8}$$

so that

$$n_2|h| = n_{i2}|h|H(\lambda_1 + h_{i2}) + \cdots + n_{im}|h|H(\lambda_1 + h_{im}).$$

This and the second inequality of (6) show that

$$n_{i2}|h|H(\lambda_1 + h_{i2}) + \cdots + n_{im}|h|H(\lambda_1 + h_{im}) \leq n_{i2}|h_{i2}| + \cdots + n_{im}|h_{im}|. \tag{9}$$

Subtracting left-hand side from right-hand side of (9) we obtain

$$0 \leq n_{i2}(|h_{i2}| - |h|H(\lambda_1 + h_{i2})) + \cdots + n_{im}(|h_{im}| - |h|H(\lambda_1 + h_{im})). \tag{10}$$

From definition of the function H, $|h_{ij}| - |h|H(\lambda_1 + h_{ij}) = 0$ for $h_{ij} = h$, and for the other $|h_{ij}| - |h|H(\lambda_1 + h_{ij}) < 0$ because $|h| > 1$. Therefore, (8) and (10) imply that $n_{ij} = 0$ for all $h_{ij} \neq h$, and $n_{ij} = n_2$ for some $h_{ij} = h$; to simplify the notation take $h_{i2} = h$. Since $n_1 + n_2 = v - 1 = n_{i1} + n_{i2}$ we must have $n_{i1} = n_1$. In consequence each row of the concurrence matrix B of the designs D has n_1 elements λ_1, n_2 elements λ_2 and r on the diagonal. Hence, D is a 2-concurrence design. □

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