SOME PROBLEMS ABOUT R-FACTORIZATIONS OF COMPLETE GRAPHS

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Abstract

We propose a number of problems about r-factorizations of complete graphs. By a completely novel method, we show that K_{2n+1} has a 2-factorization in which all 2-factors are non-isomorphic. We also consider r-factorizations of K_{rn+1} where $r \geq 3$; we show that K_{rn+1} has an r-factorization in which the r-factors are all r-connected and the number of isomorphism classes in which the r-factors lie is either 2 or 3.

1 Introduction

Let $r, n \geq 1$ and consider decompositions of K_{rn+1} into r-factors (spanning r-regular subgraphs). It is well-known that K_{rn+1} has an r-factorization (a decomposition into n edge-disjoint r-factors) if and only if either r is even or r and n are both odd. It is natural to inquire whether the set of r-factors in an r-factorization can satisfy some additional requirements.

We would like to propose the following problems, which are of decreasing difficulty.

Problem 1. Let $H_1, H_2, \ldots, H_n$ be r-regular graphs of order $rn+1$. Under what conditions is it true that K_{rn+1} has an r-factorization with r-factors $F_1, F_2, \ldots, F_n$, where F_i is isomorphic to H_i for $1 \leq i \leq n$?

Clearly a necessary condition for the existence of an r-factorization is that either r is even or r and n are both odd. But, if these necessary conditions are satisfied, it may not be the case that such an r-factorization exists. For example, there is no such factorization if $r = 2$, $n = 3$ and exactly two of F_1, F_2, F_3 consist of a 3-cycle and a 4-cycle. The non-existence in such a case seems to be a result of the fact that n is relatively small compared

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with r. Our intuition is that if n is large enough compared with r then such a factorization will exist.

If $r = 2$ and each of $H_1, \dots, H_n$ is a Hamilton cycle, then there is such an r-factorization. Buchanan [3] showed that if $r = 2$ and H_1 is any 2-regular graph of order $2n+1$ and $H_2, \dots, H_n$ are Hamilton cycles, then there is such an r-factorization. The well-known Oberwohlfach problem is the special case when $r = 2$ and $H_1, \dots, H_n$ are all the same 2-regular graph of order $2n + 1$; this problem has not been completely solved, although solutions in some interesting cases have been found [1]. If $r \geq 2$ and $H_1, \dots, H_n$ are all the same specific r-connected r-regular graph of order $rn + 1$ described in [4], then such an r-factorization exists.

We can relax Problem 1 slightly to obtain Problem 2.

Problem 2. Let $1 \leq s \leq n$ and let $H_1, \dots, H_s$ be non-isomorphic r-regular graphs of order $rn+1$. Under what conditions is it true that K_{rn+1} has an r-factorization with r-factors $F_1, \dots, F_n$ such that, for $1 \leq i \leq s$, at least one of $F_1, \dots, F_n$ is isomorphic to H_i and each of $F_1, \dots, F_n$ is isomorphic to one of $H_1, \dots, H_s$?

Of course, in any case where there is a solution to Problem 1, there is a solution to many corresponding cases in Problem 2.

Instead of specifying $H_1, \dots, H_n$ initially, we can weaken Problem 1 considerably in the following way.

Problem 3. Let $p_1 + \dots + p_s = n$ and $p_i \geq 1$ $(1 \leq i \leq s)$. Under what conditions is it true that K_{rn+1} has an r-factorization with r-factors $F_1, \dots, F_n$ such that there exist non-isomorphic r-regular graphs $H_1, \dots, H_s$ of order $rn + 1$ with exactly p_i of $F_1, \dots, F_n$ isomorphic to H_i $(1 \leq i \leq s)$?

Problem 3 lends itself to further modification, since we could require that the r-regular graphs $H_1, \dots, H_s$ have some further property. In [4] when $s = 1$ with Fleischner, we required that H_s be r-connected. Even in the case $s = 1$, one could vary this by requiring for example that H_s has connectivity κ for some given κ, where $0 \leq \kappa \leq r$. Another possibility is that $H_1, \dots, H_s$ all be cyclically r-connected. No doubt there are other modifications that would be of interest. Clearly any solution to Problem 1 is a solution to Problem 3.

We can similarly weaken Problems 2 and 3.

Problem 4. Let $1 \leq s \leq n$. Under what conditions is it true that K_{rn+1} has an r-factorization with r-factors $F_1, \dots, F_n$ such that there exist non-isomorphic r-regular graphs $H_1, \dots, H_s$ of order $rn+1$ with at least one of $F_1, \dots, F_n$ isomorphic to H_i, for each i, $(1 \leq i \leq s)$ and each of $F_1, \dots, F_n$ isomorphic to one of $H_1, \dots, H_s$?

The possibilities for modifying Problem 3 further by requiring that $H_1, \dots, H_s$ have some further property apply in this case also. A solution to any of Problems 1, 2 and 3 is also a solution to Problem 4.

In this paper we deal with two special cases of these problems.

Firstly we consider the case when $r = 2$. Let $J(2n+1; c_1, \dots, c_s)$ denote a regular graph of order $2n+1$ and degree 2 consisting of disjoint cycles $C_1, C_2, \dots, C_s$ with $|V(C_i)| = c_i$, where $c_i \geq 3$ $(1 \leq i \leq s)$ and $c_1 + \cdots + c_s = 2n+1$. For $2n+1 \geq 9$, let $H_1 = J(2n+1; 3, 3, 2n-5)$, $H_2 = J(2n+1; 3, 2n-2)$, $H_3 = J(2n+1; 4, 2n-3), \dots, H_{n-1} = J(2n+1; n, n+1)$, $H_n = J(2n+1; 2n+1)$.

Theorem 1. *Let $2n+1 \geq 9$. Then K_{2n+1} has a 2-factorization into 2-factors $F_1, \dots, F_n$, where F_i is isomorphic to H_i $(1 \leq i \leq n)$.*

In this form our first result solves a special case of Problem 1 and a special case of Problem 2. However, it is probably true to say that its corollary, solving a special case of Problem 3 and a special case of Problem 4, is more striking.

Corollary. *Let $2n+1 \geq 3$. Then K_{2n+1} has a 2-factorization into non-isomorphic 2-factors if and only if $2n+1 \geq 9$.*

Secondly we consider the case $r \geq 3$ with the extra condition that the r-factors are r-connected. Let $m = rn+1$. Throughout we assume that rm is even (or equivalently that either r is even, or r and n are both odd). We use methods developed from those found in [4] and [6].

In particular as mentioned above, in [4] it was shown that K_m has an r-factorization of the type described in Problem 3 with all the r-factors isomorphic to the same specific r-regular r-connected graph of order m. These r-factors were defined with the aid of arithmetic modulo $m-1$ as follows. One of the vertices of K_m is labeled ∞ and the others by the integers modulo $m-1$. The set of vertices labeled by the integers modulo $m-1$ will be denoted by V. If p is any integer, then G_p denotes the subgraph of K_m with edge-set given by:

$$E_p = \{\, \{x, y\} \mid x \neq y \text{ and } x + y \equiv p \,\} \cup \{\, \{x, \infty\} \mid 2x \equiv p \,\}.$$

Then:

$$\begin{aligned} K_m &= G_0 \cup G_1 \cup G_2 \cup \cdots \cup G_{m-2} \\ &= F_1 \cup F_2 \cup \cdots \cup F_n, \end{aligned}$$

where for each integer i $(1 \leq i \leq n)$ F_i is defined by:

$$F_i = G_{(i-1)r} \cup G_{(i-1)r+1} \cup \cdots \cup G_{(i-1)r+r-1}.$$

In this paper we generalize the construction of these r-factors, F_i, as follows. Let $P = \{X_i \mid 1 \leq i \leq n\}$ be a partition of the index set $I = \{j \in \mathbb{Z} \mid 0 \leq j \leq m-2\}$ into n (disjoint) subsets, each of cardinality r. Then P determines a factorization of K_m:

$$K_m = F_1 \cup F_2 \cup \cdots \cup F_n,$$

where for each i $(1 \leq i \leq n)$:

$$F_i = \bigcup_{j \in X_i} G_j.$$

If r and n are both odd, then m is even and hence it is clear that each factor F_i is r-regular. However if r is even, then m is odd and the factors will only be r-regular if the partitions, P, are chosen so that each X_i contains the same numbers of odd and even numbers. It is clear that in this way we obtain many r-factorizations of K_m. Indeed the above factorization of K_m is determined by a partition of this type; namely, the trivial partition:

$$[0, r-1], [r, 2r-1], \ldots, [(n-1)r, nr-1],$$

where, for any integers p, q with $p \leq q$, $[p, q]$ denotes the interval in $\mathbb{Z}$ consisting of all integers k such that $p \leq k \leq q$. In this paper we analyze two other cases and thereby show that, in particular, whenever $r \geq 3$ and $n \geq 2$, K_{rn+1} has an r-factorization in which the factors are all r-connected and in which at least two of the factors are not isomorphic. These factorizations are determined by the following partitions of I:

P_1: $[0, r+1] \setminus \{r-1, r\}$, $[r-1, 2r-1] \setminus \{r+1\}$, $[2r, 3r-1]$, $[3r, 4r-1]$, $\ldots$, $[(n-1)r, nr-1]$;

P_2: $[0, r+1] \setminus \{r-2, r-1\}$, $[r-2, 2r-1] \setminus \{r, r+1\}$, $[2r, 3r-1]$, $[3r, 4r-1]$, $\cdots$, $[(n-1)r, nr-1]$.

Theorem 2. *Let $r \geq 3$, $n \geq 2$ and $m = rn+1$. Suppose that:*

$$K_m = F_1 \cup F_2 \cup \cdots \cup F_n$$

is the factorization determined by the partition P_1. Then the factors F_i are all r-regular and r-connected. There are at least 2 distinct isomorphism classes which contain factors in this factorization. If, however, m is even and $r > 3$, then there are precisely 3 distinct isomorphism classes. If $n \geq 4$, then $F_4, F_5, \ldots, F_n$ are all isomorphic to F_3.

Theorem 3. *Let $r \geq 3$, $n \geq 2$ and $m = rn+1$. Suppose that:*

$$K_m = F_1 \cup F_2 \cup \cdots \cup F_n$$

is the factorization determined by the partition P_2. Then the factors F_i are all r-regular and r-connected, except when $n = 2$ and $r = 4$, in which case the two factors are both only 3-connected. Except when $n = 2$ and $r = 4$, there are at least 2 distinct isomorphism classes which contain factors in this factorization. If m is even, then F_1 and F_2 are isomorphic. If, however, m is odd, $n > 2$ and $r > 4$, then there are precisely 3 distinct isomorphism classes. If $n \geq 4$, then $F_4, F_5, \ldots, F_n$ are all isomorphic to F_3. If m is even, then F_1 and F_2 are isomorphic.

Culling information from both these theorems and [4] and, in one case, modifying the underlying partition, we have the following corollary:

Corollary. *Let $r \geq 3$, $n \geq 2$, $m = rn + 1$ and $x \in \{1, 2, 3\}$. Then K_m has a factorization into r-connected r-regular factors. The factorization can be chosen so that the factors lie in x isomorphism classes in the following cases:*

(i) $x = 1$: $n \geq 2$ and $r \geq 3$;

(ii) $x = 2$: n even or m even;

(iii) $x = 3$: $n \geq 3$ and $r \geq 5$.

Note that the case $x = 1$ is dealt with in [4] and all the other parts of the corollary are in Theorems 2 and 3, except, in the case $x = 2$, n even, we use a further partition, P_3:

$$\begin{aligned}
&[0, r+1] \setminus \{r-1, r\},\ [r-1, 2r-1] \setminus \{r+1\},\\
&[2r, 3r+1] \setminus \{3r-1, 3r\},\ [3r-1, 4r-1] \setminus \{3r+1\}, \cdots,\\
&[(n-2)r, (n-1)r+1] \setminus \{(n-1)r-1, (n-1)r\},\\
&[(n-1)r-1, nr-1] \setminus \{(n-1)r+1\},
\end{aligned}$$

for which F_k is isomorphic to F_1 when k is odd and is isomorphic to F_2 when k is even (here F_1 and F_2 are the same as in P_1).

In this paper we only use a rather crude method to compare and contrast the isomorphism classes; we consider only the number of triangles in the graphs. It is quite possible that our techniques may be extended to distinguish the isomorphism classes of some or all those graphs for which our method fails to do so.

2 Non-isomorphic 2-factorizations

In this section we prove Theorem 1.

We shall need a preliminary result which is of some interest in its own right. It generalizes a result in [5]. It concerns a K_r edge-coloured with n

colours $c_1, \ldots, c_n$ in such a way that no colour is used on more than two edges at any vertex. Let e_i be the number of edges of colour c_i $(1 \le i \le n)$ and let p_i be the number of paths of colour c_i $(1 \le i \le n)$; here a vertex with no edges of colour c_i on it counts as a path (of length 0). It is easy to see that $e_i + p_i = r$ $(1 \le i \le n)$. Let P_i be the subgraph of K_r induced by the vertices of K_r that do not lie in cycles coloured c_i. We give a necessary and sufficient condition for such an edge-colouring of K_r to be extendible to an edge-colouring of K_{2n+1} in which each colour class is a 2-factor of K_{2n+1} and such that, for $1 \le i \le n$, all the vertices of P_i and all vertices of $K_{2n+1} \backslash K_r$ lie in just one further cycle of colour c_i. Thus the $i-th$ 2-factor F_i of K_{2n+1} contains just one more cycle than does $(F_i|K_r)$, the restriction of this 2-factor to the K_r. In [5] the case considered was when there were no cycles coloured c_i in the edge-coloured K_r; in that case the 2-factors of K_{2n+1} were all Hamilton cycles.

Theorem 4. *Let $1 \le r < 2n+1$. Let K_r be edge-coloured with n colours $c_1, \ldots, c_n$ in such a way that no colour is used on more than two edges at any vertex. Then this edge-colouring of K_r can be extended to a 2-factorization of K_{2n+1} with 2-factors $F_1, \ldots, F_n$, where the edges coloured c_i in K_r all lie in F_i $(1 \le i \le n)$ and where F_i contains just one more cycle than $(F_i|K_r)$, if and only if* (i) $e_i \ge 2r - 2n - 1$ $(1 \le i \le n)$.

Note. Let (ii) be:

(ii) $p_i \le 2n + 1 - r$ $(1 \le i \le n)$.

Conditions (i) and (ii) are easily seen to be equivalent. The result in [5] about Hamilton cycles was stated in terms of condition (ii).

Proof. Necessity. Each further vertex can link together two disjoint paths. Therefore the number p_i of distinct paths in the colour class c_i must not be greater than $2n+1-r$, the number of further vertices. This proves (ii), and (i) is equivalent to (ii).

Before embarking on the proof of sufficiency we need to state a useful result due to de Werra [7, 8, 9] on edge-colouring bipartite multigraphs. A proof of this may also be found in [2]. Given an edge-colouring of a loopless graph G with colours $c_1, \ldots, c_n$, for each $v \in V(G)$, let $C_i(v)$ be the set of edges on v of colour c_i, and for each $u, v \in V(G)$, $u \ne v$, let $C_i(u,v)$ be the set of edges joining u and v of colour c_i. An edge-colouring is *equitable* if, for all $v \in V(G)$,

$$\max_{1 \le i < j \le n} ||C_i(v)| - |C_j(v)|| \le 1,$$

and it is *balanced* if, in addition, for all $u, v \in V(G)$, $u \ne v$,

$$\max_{1 \le i < j \le n} ||C_i(u,v)| - |C_j(u,v)|| \le 1.$$

Thus an edge-colouring is balanced if the colours occur as uniformly as possible at each vertex and if the colours are shared out as uniformly as possible on each multiple edge.

Proposition 5 (de Werra). *For each $n \geq 1$, any finite bipartite multigraph has a balanced edge-colouring with n colours.*

Sufficiency. We shall show that, if $r < 2n$, then our edge-colouring of K_r can be extended to an edge-colouring of K_{r+1} in such a way that the property (*i*) is satisfied (with r replaced by $r + 1$) and with no further cycles of colour c_i ($1 \leq i \leq n$). Also, if $r = 2n$, then we shall show that our edge-colouring of K_r can be extended to a 2-factorization of K_{2n+1} (in this case, F_i necessarily has exactly one more cycle than $(F_i|K_{2n})$).

If $r = 2n$, then, by condition (i), the number e_i of edges of colour c_i is at least $2n - 1$, and so $e_i \in \{2n-1, 2n\}$. But since K_{2n} contains exactly $n(2n-1)$ edges and each of the n colour classes has at least $2n-1$ edges, in fact each colour class has exactly $2n-1$ edges. Thus $e_i = 2n-1$. Therefore there is exactly one path coloured c_i. It is not possible that this path is the trivial one consisting of a single vertex. For in K_{2n}, a vertex v has degree $2n - 1$. Since v has on it at most two edges of each colour, it follows that $n-1$ colours occur on two edges at v, and exactly one colour occurs on one edge incident with v. There are two vertices at which colour c_i occurs on only one edge. Thus it follows that the required 2-factorization of K_{2n+1} can be found by adjoining a further vertex, say v_{2n+1}, to the K_{2n}, joining it to each vertex of K_{2n}, and colouring each edge $v_i v_{2n+1}$ ($1 \leq i \leq 2n$) with the colour that occurs on only one edge of K_{2n} incident with v_i.

Now consider the case when $r < 2n$. We start by constructing a bipartite graph B with vertex sets $\{c_1, \ldots, c_n\}$ and $\{v_1, \ldots, v_r\}$. Join vertices c_i and v_j by x edges (where $x = 0, 1$ or 2) if there are $2 - x$ edges of colour c_i incident with v_j in the K_r. Then

(a) $d_B(v_j) = 2n -$ (the number of edges incident with v_j in the K_r)
$= 2n - (r - 1)$
$= 2n + 1 - r$;

(b) $d_B(c_i)$ is even since it is joined to each vertex that is the end of a c_i-path in K_r; moreover $0 \leq d_B(c_i) \leq 2(2n + 1 - r)$. The case $d_B(c_i) = 0$ occurs if each vertex of K_r is in a cycle coloured c_i; the case $d_B(c_i) = 2(2n + 1 - r)$ occurs if $e_i = 2r - 2n - 1$ (so that $p_i = 2n + 1 - r$).

We first give B a balanced edge-colouring with $2n + 1 - r$ colours, $\kappa_1, \ldots, \kappa_{2n+1-r}$. Let B^* be the subgraph induced by the edges coloured

κ_1 and κ_2. Then

$$d_{B^*}(v_j) = 2 \qquad (1 \le j \le r),$$
$$d_{B^*}(c_i) \le 4 \qquad (1 \le i \le n),$$
$$|E(B^*)| = 2r.$$

For each i, $1 \le i \le n$, if $d_{B^*}(c_i) = 2$ or 3 we pair together two of the vertices joined to c_i, and if $d_{B^*}(c_i) = 4$ we pair together all four vertices joined to c_i, forming what we shall call *i-pairs*, as follows: First, if there are two edges joining c_i to v_j, then we form an i-pair of v_j with itself. If c_i is joined in B^* to v_j and v_j^*, say, and in the K_r v_j and v_j^* are the two end vertices of a path coloured c_i, then v_j and v_j^* form an i-pair. If there is more than one vertex joined to c_i in B^* still not paired off in an i-pair, then the vertices that remain are paired off arbitrarily (with possibly one vertex left over).

From B^* we form a further bipartite graph B^+ as follows. The vertex set of B^+ is $\{v_1, \ldots, v_r\} \cup \{c_{11}, c_{12}, c_{21}, c_{22}, \ldots, c_{n1}, c_{n2}\}$. Each vertex c_i of B^* is split into two vertices c_{i1} and c_{i2}. We assign the edges that were incident with c_i to c_{i1} and c_{i2} so that $d_{B^+}(c_{il}) \le 2$ $(1 \le i \le n,\ 1 \le l \le 2)$ and so that whenever two edges incident with c_i are joined to vertices v_j and v_j^* that form an i-pair then the two edges are both assigned to c_{i1} or are both assigned to c_{i2}.

We then give B^+ a balanced 2-edge-colouring with colours, say α and β. Then we transfer this edge-colouring to B^*. We then have a balanced 2-edge-colouring of B^* with α and β. Let B_α^* be the subgraph of B^* induced by the edges coloured α. Then

$$d_{B_\alpha^*}(v_j) = 1 \qquad (1 \le j \le r),$$
$$d_{B_\alpha^*}(c_i) \le 2 \qquad (1 \le i \le n),$$
$$|E(B_\alpha^*)| = r.$$

Moreover, B_α^* has the following property P.

P: If v_j and v_j^* form an i-pair, then exactly one of the edges v_jc_i and $v_j^*c_i$ is in B_α^*.

We use B_α^* to determine how to extend the original edge-colouring of K_r to an edge-colouring of the K_{r+1} obtained by adding one vertex v to K_r and one edge between v and each vertex of the K_r. If c_iv_j is an edge of B_α^*, then we colour the edge vv_j in our K_{r+1} with the colour c_i. When this is done for each j $(1 \le j \le r)$, the K_r is extended to an edge-coloured K_{r+1} in which the i-th colour class induces a subgraph with no vertices of degree greater than two and with no more cycles than it had in K_r; moreover condition (i) is satisfied (with r replaced by $r + 1$). To see this, first note that each of the new edges does actually receive a colour; this is because $d_{B^*_\alpha}(v_j) = 1$ $(1 \le j \le r)$. In our construction of the bipartite graph B, c_i and v_j were joined by x edges if there were $2-x$ edges of colour c_i incident with v_j in the K_r. Thus a vertex in the K_{r+1} will have degree at most two in the subgraph induced by the i-th colour class. This is also true of the vertex v, since $d_{B^*_\alpha}(c_i) \le 2$. By property P, if there are two vertices v_j and v_j^* joined to c_i in B^*_α, v_j and v_j^* do not form an i-pair; therefore, if $d_{B^*_\alpha}(c_i) = 2$, the two vertices v_j and v_j^* joined in K_{r+1} to v by edges coloured c_i are not the end vertices of the same path in P_i. Thus in the K_{r+1} the i-th colour class has no more cycles than it had in K_r.

Clearly (i) is satisfied with r replaced by $r+1$ for any i with $e_i \ge 2r-2n+1 = 2(r+1)-2n-1$. But consider the possibility that $e_i = 2r-2n$ or $e_i = 2r-2n-1$. Then $d_{B^*}(c_i) \in \{2,3,4\}$ if $e_i = 2r-2n$, or $d_{B^*}(c_i) = 4$ if $e_i = 2r-2n-1$. Therefore $d_{B^*_\alpha}(c_i) \in \{1,2\}$ if $e_i = 2r-2n$, and $d_{B^*_\alpha}(c_i) = 2$ if $e_i = 2r-2n-1$. Therefore at least one edge on v is coloured c_i in the K_{r+1} if $e_i = 2r-2n$, and two edges on v are coloured c_i in K_{r+1} if $e_i = 2r-2n-1$. Therefore (i) (with r replaced by $r+1$) is satisfied by our edge-coloured K_{r+1}.

Repeating this argument a finite number of times leads to the case $r-2n$, which we have already dealt with.

□

Now we turn to the proof of Theorem 1.

Proof (Theorem 1). We first give suitable 2-factorizations in cases when $2n+1 = 9$ and 11. We take the vertex sets to be $\{1,2,\dots,2n+1\}$. We denote a cycle $(a_1,a_2,\dots,a_r,a_1)$ by $\langle a_1,a_2,\dots,a_r\rangle$ (thus $\langle a_1,a_2,\dots,a_r\rangle = \langle a_2,a_3,\dots,a_r,a_1\rangle$, etc.). Then the 2-factorizations are given by

$2n+1 = 9$:
- H_1: $\langle 1,2,3\rangle$, $\langle 4,5,6\rangle$, $\langle 7,8,9\rangle$;
- H_2: $\langle 2,5,9\rangle$, $\langle 1,4,3,8,6,7\rangle$;
- H_3: $\langle 1,6,3,9\rangle$, $\langle 2,4,8,5,7\rangle$;
- H_4: $\langle 1,5,3,7,4,9,6,2,8\rangle$.

$2n+1 = 11$:
- H_1: $\langle 1,2,3\rangle$, $\langle 9,10,11\rangle$, $\langle 4,5,6,7,8\rangle$;
- H_2: $\langle 6,8,9\rangle$, $\langle 1,11,3,4,2,5,7,10\rangle$;
- H_3: $\langle 3,5,8,10\rangle$, $\langle 1,4,9,7,2,11,6\rangle$;
- H_4: $\langle 3,8,11,4,7\rangle$, $\langle 1,5,10,6,2,9\rangle$;
- H_5: $\langle 1,7,11,5,9,3,6,4,10,2,8\rangle$.

Now we give two similar recursive constructions, both of which use Theorem 4.

Recursion 1. *Let $n \ge 5$. From any 2-factorization of K_{2n+1} into 2-factors $J(2n+1;3,3,2n-5)$, $J(2n+1;i,2n+1-i)$ $(3 \le i \le n)$ and $J(2n+1;2n+1)$,* *we can construct a 2-factorization of K_{4n-3} into 2-factors $J(4n-3;3,3,4n-9)$, $J(4n-3;i,4n-3-i)$ $(3 \leq i \leq 2n-2)$ and $J(4n-3;4n-3)$.*

In the case when $n = 4$, there is a 2-factorization of K_{2n+1} of the former kind from which a 2-factorization of K_{4n-3} of the latter kind can be constructed by a similar process.

Construction for Recursion 1. Let $n \geq 5$, and suppose we have a 2-factorization of K_{2n+1} of the type described. Let C_1 be the set of edges in the cycles of $J(2n+1;3,3,2n-5)$. For $3 \leq i \leq n$, let C_{i-1} be the set of edges in the i-cycle and let C_{2n-i} be the set of edges in the $(2n+1-i)$-cycle of $J(2n+1;i,2n+1-i)$. Finally let C_{2n-2} be the set of edges in the $(2n+1)$-cycle in $J(2n+1;2n+1)$. We then have $2n-2$ disjoint sets $C_1, \ldots, C_{2n-2}$ whose union is the edge-set of K_{2n+1}.

We shall modify this 2-factorization of K_{2n+1} slightly in order that the desired 2-factorization of K_{4n-3} can be obtained by an application of Theorem 4. Note that condition (i) in Theorem 4 becomes here $e_i \geq 2(2n+1)-(4n-3)$ $(1 \leq i \leq 2n-2)$,i.e. $e_i \geq 5$ $(1 \leq i \leq 2n-2)$. We have that $|C_i| \geq 5$ $(i \in \{1,2,\ldots,2n-2\} \setminus \{2,3\})$, but $|C_2| = 3$ and $|C_3| = 4$. Thus we need to increase the size of C_2 and C_3 by 2 and 1 respectively. We also need to re-assign at least one edge of the $(2n-5)$-cycle in C_1 so that the paths remaining can become part of the $(4n-9)$-cycle in K_{4n-3}, and we need to re-assign at least one edge of the $(2n+1)$-cycle of edges in C_{2n-2} so that the paths remaining can become part of a $(4n-3)$-cycle in K_{4n-3}.

Let the $(2n-5)$-cycle whose edges are in C_1 be denoted by X. Let v_1, v_2, v_3 be the three vertices of the 3-cycle with edges in C_2. We show that there is an edge e of X that is not incident with any of v_1, v_2 or v_3. Then we colour e with colour c_2, colouring the edges of $C_1 \setminus \{e\}$ with colour c_1. If $n \geq 6$ then the $(2n-5)$-cycle X has at least seven vertices, so the edge e exists in this case. If $n = 5$ then the 5-cycle X is incident with at most two of v_1, v_2 and v_3, since v_1, v_2 and v_3 induce a 3-cycle that is edge-disjoint from X. Therefore the edge e exists in this case also.

We colour the edges of C_2 with colour c_2. Let the $(2n+1)$-cycle with edges in C_{2n-2} be denoted by Y. Since $n \geq 5$, $2n+1 \geq 11$, so there is an edge e^* of Y that is not incident with any of y_1, y_2, y_3. Since $e \in X$, $e \neq e^*$. We colour e^* with colour c_2 also. Since e is also coloured c_2, there are now five edges coloured c_2.

Let the four vertices of the 4-cycle of edges in C_3 be w_1, w_2, w_3, w_4. Since $2n+1 \geq 11$, there are at least three edges of C_{2n-2} that are not incident with any of w_1, w_2, w_3, w_4, and so there is an edge e^{**} of $C_{2n-2} \setminus \{e^*\}$ that is not incident with any of w_1, w_2, w_3, w_4. We colour the edge e^{**} with colour c_3, and we also colour the edges of C_3 with colour c_3. Then there are five edges coloured c_3.

For $4 \le i \le 2n-3$ we colour the edges of C_i with colour c_i. We colour the edges of $C_{2n-2} \setminus \{e^*, e^{**}\}$ with colour c_{2n-5}. Each colour is now used on at least five edges of the K_{2n+1}, so Condition (i) of Theorem 4 is satisfied. Each of X and Y have at least two colours used on them.

We now apply Theorem 4; this yields a 2-factorization of K_{4n-3} in which the 2-factor coloured c_1 has two 3-cycles and a $(4n-9)$-cycle, for $2 \le i \le 2n-3$, the 2-factor coloured c_i has an $(i+1)$-cycle and a $(4n-i-4)$-cycle, and the 2-factor coloured c_{2n-2} is a Hamilton cycle. This establishes the first recursion when $n \ge 5$.

In the special case when $n = 4$ (so that $2n+1 = 9$), we have to modify the 2-factorization given at the beginning of this proof in a slightly different way. In the K_9 we colour the edges as follows (where the notation $\langle a, b, \ldots, c\rangle$ means all edges of the cycle $(a, b, \ldots, c)$):

c_1 :	$\langle 1,2,3\rangle$, $\langle 7,8,9\rangle$, 45, 56;
c_2:	$\langle 2,5,9\rangle$, 46, 37;
c_3:	$\langle 1,6,3,9\rangle$, 47;
c_4:	$\langle 2,4,8,5,7\rangle$;
c_5:	$\langle 1,4,3,8,6,7\rangle$;
c_6:	$\langle 1,5,3,7,4,9,6,2,8\rangle \setminus \{37,47\}$.

We then apply Theorem 4 to obtain a 2-factorization of the desired kind.

Recursion 2. *Let $n \ge 7$. From any 2-factorization of K_{2n+1} into 2-factors $J(2n+1; 3, 3, 2n-5)$, $J(2n+1; i, 2n+1-i)$ $(3 \le i \le n)$ and $J(2n+1; 2n+1)$, we can construct a 2-factorization of K_{4n-5} into $J(4n-5; 3, 3, 4n-1)$, $J(4n-5; i, 4n-5-i)$ $(3 \le i \le 2n-3)$ and $J(4n-5; 4n-5)$.*

Moreover, in the case when $n = 5$, from the 2-factorization of K_{11}, given at the beginning of the proof, we can construct by a similar process such a 2-factorization of K_{15}. If $n = 6$, we give a 2-factorization of K_{13} from which we construct similarly a 2-factorization of K_{19} of the desired kind.

Construction for Recursion 2. Let $n \ge 7$ and suppose we have a 2-factorization of K_{2n+1} of the type described. Let D_1 be the set of edges in the cycles of $J(2n+1; 3, 3, 2n-5)$ and let D_2 be the set of edges in the cycles of $J(2n+1; 3, 2n-2)$. For $4 \le i \le n$, let D_{i-1} be the set of edges in the i-cycle and let D_{2n-i} be the set of edges in the $(2n+1-i)$-cycle of $J(2n+1; i, 2n+1-i)$. Finally let D_{2n-3} be the set of edges in the $(2n+1)$-cycle in $J(2n+1; 2n+1)$. We then have $2n-3$ disjoint sets $D_1, \ldots, D_{2n-3}$ whose union is the edge-set of K_{2n+1}.

We shall modify this 2-factorization of K_{2n+1} slightly in order that the desired 2-factorization of K_{4n-5} can be obtained by an application of

Theorem 4. Here Condition (i) in Theorem 4 becomes $e_i \geq 2(2n+1) - (4n-5)$ $(1 \leq i \leq 2n-3)$,i.e. $e_i \geq 7$ $(1 \leq i \leq 2n-3)$. We have that $|D_i| \geq 7$ $(i \in \{1, 2, \ldots, 2n-3\} \setminus \{2, 3\})$, but $|D_3| = 4$, $|D_4| = 5$ and $|D_5| = 6$. Thus we need to increase the size of D_3, D_4 and D_5 by 3, 2 and 1 respectively. We also need to re-assign at least one edge of the $(2n-5)$-cycle in D_1 so that the paths remaining can become part of the $(4n-11)$-cycle in K_{4n-5}, we need to re-assign at least one edge of the $(2n-2)$-cycle in D_2 so that the paths remaining can become part of a $(4n-8)$-cycle in K_{4n-5} and we need to re-assign at least one edge of the $(2n+1)$-cycle of edges in D_{2n-3} so that the paths remaining can become part of a $(4n-5)$-cycle in K_{4n-5}.

Let the $(2n-5)$-cycle whose edges are in D_1 be denoted by X. Let w_1, w_2, w_3, w_4 be the four vertices of the 4-cycle with edges in D_3. Since $n \geq 7$ the $(2n-5)$-cycle has at least 9 vertices, so there is an edge e of X that is not incident with any of w_1, w_2, w_3, w_4. Then we colour e with colour c_3 and colour the edges of $C_1 \setminus \{e\}$ with colour c_1.

Now let the $(2n-2)$-cycle whose edges are in D_2 be denoted by Y. Since $(2n-2) \geq 12$ there are at least four edges that are not incident with any of w_1, w_2, w_3, w_4. Let e^* and e^{**} be two such edges. Colour e^* and e^{**} with colour c_3, and colour the remaining edges of $D_2 \setminus \{e^*, e^{**}\}$ with colour c_3 (the edges of the 4-cycle (w_1, w_2, w_3, w_4) and e, e^*, e^{**}).

Let the $(2n+1)$-cycle with edges in D_{2n-3} be denoted by Z. Let y_1, y_2, y_3, y_4, y_5 be the five vertices of the 5-cycle with edges in D_4, and let $z_1, z_2, \ldots, z_6$ be the six vertices of the 6-cycle in D_5. Since $2n+1 \geq 15$ there are at least three edges of Z that are not incident with any of $z_1, \ldots, z_6$. Let f be one such edge, and colour it c_5. There are at least five edges of Z that are not incident with any of $y_1, \ldots, y_5$. Let f^* and f^{**} be two such edges that are distinct from f. Colour f^* and f^{**} with colour c_4. There are now seven edges coloured c_4 (the edges of the 5-cycle $(y_1, y_2, \ldots, y_5, y_1)$ together with f^* and f^{**}) and seven edges coloured c_5 (the edges of the 6-cycle $(z_1, z_2, \ldots, z_6, z_1)$ together with f).

For $6 \leq i \leq 2n-4$ we colour the edges of D_i with colour c_i and we colour the edges of $D_{2n-3} \setminus \{f, f^*, f^{**}\}$ with colour c_{2n-3}. Each colour is now used on at least seven edges of the K_{2n+1}, so Condition (i) of Theorem 4 is satisfied. Each of X, Y and Z have at least two colours used on them.

We now apply Theorem 4; this yields a 2-factorization of K_{4n-5} in which the 2-factor coloured c_1 has two 3-cycles and a $(4n-11)$-cycle, for $2 \leq i \leq 2n-4$, the 2-factor coloured c_i has an $(i+1)$-cycle and a $(4n-i-6)$-cycle, and the 2-factor coloured c_{2n-3} is a Hamilton cycle. This establishes the second recursion when $n \geq 7$.

In the special case when $n = 6$, let us consider the following 2-factorization of K_{13}:

H_1 : $\langle 1,2,3 \rangle$, $\langle 11,12,13 \rangle$, $\langle 4,5,6,7,8,9,10 \rangle$;

H_2: $\langle 5,7,9\rangle$, $\langle 1,6,12,8,11,10,3,13,2,4\rangle$;
H_3: $\langle 1,7,10,12\rangle$, $\langle 2,6,11,3,8,4,9,13,5\rangle$;
H_4: $\langle 1,5,8,2,10\rangle$, $\langle 3,9,6,13,4,11,7,12\rangle$;
H_5: $\langle 3,4,6,8,10,5\rangle$, $\langle 1,13,7,2,12,9,11\rangle$;
H_6: $\langle 1,9,2,11,5,12,4,7,3,6,10,13,8\rangle$.

We now modify this 2-factorization in a way suitable for the application of Theorem 4. The modification is similar to, but not the same as that described above for the general case. The colour classes corresponding to the colours $c_1, \ldots, c_9$ are as follows:

c_1 : $\langle 1,2,3\rangle$, $\langle 11,12,13\rangle$, $\langle 4,5,6,7,8,9,10\rangle \setminus \{45,56,67,89\}$;
c_2: $\langle 5,7,9\rangle$, $\langle 1,6,12,8,11,10,3,13,2,4\rangle \setminus \{6-12\}$;
c_3: $\langle 1,7,10,12\rangle$, $\{45,56,89\}$;
c_4: $\langle 1,5,8,2,10\rangle$, $\{67, 6-12\}$;
c_5: $\langle 3,4,6,8,10,5\rangle$, $\{2-11\}$;
c_6: $\langle 1,13,7,2,12,9,11\rangle$;
c_7: $\langle 3,9,6,13,4,11,7,12\rangle$;
c_8: $\langle 2,6,11,3,8,4,9,13,5\rangle$;
c_9: $\langle 1,9,2,11,5,12,4,7,3,6,10,13,8\rangle \setminus \{2-11\}$.

Here $6-12$ is used to denote the edge joining vertices 6 and 12. We use similar notation whenever a vertex is denoted by a 2-digit number.

In the special case when $n = 5$ we modify the 2-factorization of K_{11} given at the beginning of this proof. Again the modification is similar to, but not the same as, the modification we described in the general case. The colour classes corresponding to the colours $c_1, \ldots, c_7$ are as follows:

c_1 : $\langle 1,2,3\rangle$, $\langle 9,10,11\rangle$, $\langle 4,5,6,7,8\rangle \setminus \{56,67,78\}$;
c_2: $\langle 8,9,6\rangle$, $\langle 1,11,3,4,2,5,7,10\rangle \setminus \{1-11\}$;
c_3: $\langle 3,5,8,10\rangle$, $\{17,67,1-11\}$;
c_4: $\langle 11,4,7,3,8\rangle$, $\{56, 2-10\}$;
c_5: $\langle 1,5,10,6,2,9\rangle$, $\{78\}$;
c_6: $\langle 1,4,9,7,2,11,6\rangle$;
c_7: $\langle 2,7,11,5,9,3,6,4,10,2,8\rangle \setminus \{17, 2-10\}$.

This establishes Recursion 2.

Theorem 1 now follows easily. First we note that we have given particular decompositions which establish the theorem for K_9 and K_{11}. Using Recursion 1 we obtain a decomposition for K_{13} (we actually give another in the course of the proof). For $n \geq 5$ we obtain a decomposition for K_{4n-5} from Recursion 2 (either a special or the general case), and we obtain a decomposition for K_{4n-3} from Recursion 1 (either the special or the general case). This proves Theorem 1. □

3 Non-isomorphic r-connected r-factorizations

To prove Theorems 2 and 3 it is convenient to introduce the following notation. Firstly, if X is a subset of the index set I, we define:

$$G(X) = \bigcup_{i \in X} G_i,$$

and then, in particular, if t is a positive integer and $p, q, i_1, i_2, \ldots, i_t$ are elements of I such that:

$$p \leq i_1 < i_2 < \cdots < i_t \leq q,$$

we define:

$$G(p, q; i_1, i_2, \ldots, i_t) = G([p, q] \setminus \{i_j \mid 1 \leq j \leq t\}),$$

and:

$$G(p, q) = G([p, q]).$$

Then the factorization of Theorem 2 becomes:

$$K_m = G(0, r+1; r-1, r) \cup G(r-1, 2r-1; r+1) \cup G(2r, 3r-1) \\ \cup G(3r, 4r-1) \cup \cdots \cup G((n-1)r, nr-1),$$

and the factorization of Theorem 3 becomes:

$$K_m = G(0, r+1; r-2, r-1) \cup G(r-2, 2r-1; r, r+1) \cup G(2r, 3r-1) \\ \cup G(3r, 4r-1) \cup \cdots \cup G((n-1)r, nr-1).$$

The proof now separates into two parts, the isomorphism and non-isomorphism of the r-factors, and their connectivity.

3.1 Isomorphism and non-isomorphism

It was shown, in [6], that, for any integer k and integer s such that $2 \leq s < m-1$ and sm is even, $G(k, k+s-1)$ is isomorphic to $G(0, s-1)$. With $k = (i-1)r$ and $s = r$ we see that, for $2 \leq i \leq n$, $G((i-1)r, ir-1)$ is isomorphic to $G(0, r-1)$. Also, if n is even (r even and m odd), putting $s = 2r$, then, for $1 \leq j < \frac{n}{2}$, $G(2jr, 2(j+1)r-1)$ is isomorphic to $G(0, 2r-1)$. Moreover, for $1 \leq j < \frac{n}{2}$, the isomorphism sends $G(2jr, (2j+1)r+1; (2j+1)r-1, (2j+1)r)$ onto $G(0, r+1; r-1, r)$ and $G((2j+1)r-1, 2(j+1)r-1; (2j+1)r+1)$ onto $G(r-1, 2r-1; r+1)$ (Recall that these graphs arise from the partition P_3). Using a similar method, the graphs $G(0, r+1; r-2, r-1)$ and $G(r-2, 2r-1; r, r+1)$ are isomorphic when m is even. To be precise it is easy to see that, when m is even, the mapping which sends ∞ to ∞ and the vertex x $(0 \leq x \leq m-2)$ to $\frac{m}{2} + r - 1 - x$ is an isomorphism of $G(0, r+1; r-2, r-1)$ onto $G(r-2, 2r-1; r, r+1)$. Indeed it maps the edge-set E_k bijectively onto the edge-set E_{2r-1-k}. Moreover the same graphs are isomorphic when $r = 4$ (and m is odd). In this case the required isomorphism maps ∞ to ∞ and the vertex x $(0 \leq x \leq m-2)$ to $x+1$. To deal with the non-isomorphisms of Theorems 2 and 3 it suffices to consider the number of triangles in each r-factor. These triangles are separated into three distinct types. Those triangles with vertices in V will be designated as type-1 triangles. Those triangles which are not type-1 besides having ∞ as a vertex will have two distinct vertices, x, y from V. The edges then belong to the graphs G_a, G_b, G_c, where:

$$a \equiv 2x; \quad b \equiv 2y; \quad c \equiv x + y \mod (m-1).$$

Necessarily a, c are distinct modulo $(m-1)$ as are b, c. However a, b need not be distinct. It is easy to see that $a \equiv b \mod (m-1)$ if and only if m is odd and $x - y \equiv k \mod (m-1)$, where $m - 1 = 2k$. Therefore these triangles separate into two cases; those for which a, b are distinct, designated type-2, and those for which $a \equiv b \mod (m-1)$, designated type-3. Note that type-3 triangles only occur when m is odd. Note also that type-1 and type-2 triangles occur in subgraphs of the form $G_a \cup G_b \cup G_c$, where a, b, c are distinct modulo $(m-1)$, whilst type-3 triangles occur in subgraphs of the form $G_a \cup G_b$, where a, b are distinct modulo $(m-1)$. The existence of triangles of the various types is dealt with in the following three lemmas.

Lemma 6. *Let a, b, c be distinct integers modulo $(m-1)$. If m is even; say $m = 2k$ for some integer k; then the system of congruences:*

$$x + y \equiv a; \quad y + z \equiv b; \quad z + x \equiv c \mod (m-1)$$

has the unique solution:

$$x \equiv k(a - b + c); \quad y \equiv k(a + b - c); \quad z \equiv k(-a + b + c) \mod (m-1).$$

On the other hand, if m is odd; say $m = 2k+1$; then the system has a solution if and only if $a + b + c \equiv 2d \mod (m-1)$ for some integer d, in which case there are precisely two solutions; namely:

$$x \equiv d - b; \quad y \equiv d - c; \quad z \equiv d - a \mod (m-1)$$

and:

$$x \equiv k + d - b; \quad y \equiv k + d - c; \quad z \equiv k + d - a \mod (m-1).$$

Proof. Suppose that $m = 2k$ for some integer k. Let x, y, z be integers such that:

$$x + y \equiv a; \quad y + z \equiv b; \quad z + x \equiv c \mod (m-1).$$

Then:

$$2x \equiv (x+y) - (y+z) + (z+x) \equiv a - b + c \mod (m-1).$$

Since $2k = m \equiv 1 \mod (m-1)$,

$$x \equiv 2kx \equiv k(a-b+c) \mod (m-1).$$

Similarly $y \equiv k(a+b-c) \mod (m-1)$ and $z \equiv k(-a+b+c) \mod (m-1)$. Conversely it is easy to see that these values for x, y, z satisfy the system of congruences.

Now suppose that $m = 2k+1$ for some integer k and that x, y, z are integers which satisfy the system of congruences. Then, adding the congruences we obtain:

$$a + b + c \equiv 2(x+y+z) \mod (m-1).$$

Thus $a + b + c \equiv 2d \mod (m-1)$ for some integer d. Conversely suppose that $a + b + c \equiv 2d$ for some integer d and suppose that x, y, z are integers which satisfy the system of congruences. Then:

$$\begin{aligned} 2x &\equiv (x+y) - (y+z) + (z+x) && \mod (m-1) \\ &\equiv a - b + c && \mod (m-1) \\ &\equiv a + b + c - 2b && \mod (m-1) \\ &\equiv 2(d-b) && \mod (m-1) \end{aligned}$$

It follows that $x - (d-b)$ is divisible by k. Thus either (i) $x - (d-b) = k(2h+1)$ for some integer h, or (ii) $x - (d-b) = k(2h)$ for some integer h. If (i), then $x \equiv k + d - b \mod (m-1)$. If (ii), then $x \equiv d - b \mod (m-1)$. In case (i) it follows that, since $-k \equiv k \mod (m-1)$ and $a + b - d \equiv d - c \mod (m-1)$:

$$\begin{aligned} y &\equiv a - x && \mod (m-1) \\ &\equiv a - k - d + b && \mod (m-1) \\ &\equiv k + d - c && \mod (m-1). \end{aligned}$$

Similarly $z \equiv k + d - a \mod (m-1)$. In case (ii) it follows, in the same way, that $y \equiv d - c \mod (m-1)$ and $z \equiv d - a \mod (m-1)$. It is easy to check that these two possibilities are indeed solutions of the system of congruences. □

Lemma 7. *Let a, b, c be integers which are distinct modulo $(m-1)$. If the system of congruences:*

$$2x \equiv a; \quad 2y \equiv b; \quad x + y \equiv c \mod (m-1)$$

has a solution, then $a + b \equiv 2c \mod (m-1)$. If $a + b \equiv 2c \mod (m-1)$ and $m = 2k$ for some integer k, then the system has the unique solution:

$$x \equiv ka; \quad y \equiv kb \mod (m-1).$$

If $a + b \equiv 2c \mod (m-1)$ and $m = 2k + 1$ for some integer k, then the system is solvable if and only if a, b are even; in which case, if $a = 2\alpha$ and $b = 2\beta$ for some integers α and β, then there are precisely two solutions; namely:

$$\begin{aligned} x &\equiv \alpha; & y &\equiv \beta & &\mod (m-1); \\ x &\equiv k + \alpha; & y &\equiv k + \beta & &\mod (m-1), \end{aligned}$$

when $c \equiv \alpha + \beta \mod (m-1)$, or:

$$\begin{aligned} x &\equiv \alpha; & y &\equiv k + \beta & &\mod (m-1); \\ x &\equiv k + \alpha; & y &\equiv \beta & &\mod (m-1), \end{aligned}$$

when $c \equiv k + \alpha + \beta \mod (m-1)$.

Proof. The first statement is trivial. Now suppose that $a + b \equiv 2c \mod (m-1)$ and $m = 2k$ for some integer k. Then $2k \equiv 1 \mod (m-1)$ and hence, if x, y, z is a solution, then:

$$\begin{aligned} x &\equiv 2kx \equiv ka \mod (m-1); \\ y &\equiv 2ky \equiv kb \mod (m-1), \end{aligned}$$

and, conversely, these values for x and y are solutions:

$$2(ka) \equiv a; \quad 2(kb) \equiv b; \quad (ka) + (kb) \equiv k(a+b) \equiv 2kc \equiv c \mod (m-1).$$

Next suppose that $m = 2k+1$ for some integer k, that $a+b \equiv 2c \mod (m-1)$ and that x, y are solutions of the system. Clearly, since $(m-1)$ is even and $a \equiv 2x \mod (m-1)$ and $b \equiv 2y \mod (m-1)$, then a and b are even. Conversely suppose that $a = 2\alpha$ and $b = 2\beta$ for some integers α and β. Then $2(\alpha + \beta) \equiv 2c \mod (m-1)$. It follows that either (i) $c \equiv \alpha + \beta \mod (m-1)$ or (ii) $c \equiv k + \alpha + \beta \mod (m-1)$. Suppose that x, y are solutions. Then, in particular, $2x \equiv 2\alpha \mod (m-1)$ so that either $x \equiv \alpha \mod (m-1)$ or $x \equiv k + \alpha \mod (m-1)$. Consider the case (i). If $x \equiv \alpha \mod (m-1)$, then:

$$y \equiv c - x \equiv c - \alpha \equiv \beta \mod (m-1);$$

but, if $x \equiv k + \alpha \mod (m-1)$, then, since $-k \equiv k \mod (m-1)$:

$$y \equiv c - x \equiv c - k - \alpha \equiv k + \beta \mod (m-1).$$

It is easy to see that these are indeed two distinct solutions of the system of congruences. Similarly in case (*ii*), if $x \equiv \alpha \mod (m-1)$, then:

$$y \equiv c - x \equiv c - \alpha \equiv k + \beta \mod (m-1);$$

but, if $x \equiv k + \alpha \mod (m-1)$, then:

$$y \equiv c - x \equiv c - k - \alpha \equiv \beta \mod (m-1),$$

and again we have two distinct solutions of the system. □

Lemma 8. *Suppose that $m = 2k+1$ for some integer k. Let a, b be distinct integers modulo $(m-1)$. Then the system of congruences:*

$$2x \equiv a; \quad 2y \equiv a; \quad x + y \equiv b \mod (m-1)$$

have a solution if and only if a is even; say $a = 2\alpha$ for some integer α; and $b \equiv a + k \mod (m-1)$, in which case there are two distinct solutions; namely:

$$x \equiv \alpha; \quad y \equiv \alpha + k \mod (m-1)$$

and:

$$x \equiv \alpha + k; \quad y \equiv \alpha \mod (m-1).$$

Proof. Suppose that the integers x, y satisfy the system of congruences. Then, since $(m-1)$ is even and $a \equiv 2x \mod (m-1)$, a is even. Moreover:

$$2b \equiv 2(x+y) \equiv 2a \mod (m-1),$$

and hence, since a, b are distinct modulo $(m-1)$, $b \equiv a + k \mod (m-1)$. Now suppose that $a = 2\alpha$ for some integer α and that $b \equiv a+k \mod (m-1)$. If $x \equiv \alpha \mod (m-1)$ and $y \equiv \alpha + k \mod (m-1)$, then:

$$\begin{aligned} 2x &\equiv 2\alpha \equiv a && \mod (m-1); \\ 2y &\equiv 2(\alpha + k) \equiv 2\alpha \equiv a && \mod (m-1); \\ x + y &\equiv 2\alpha + k \equiv a + k \equiv b && \mod (m-1). \end{aligned}$$

This yields a solution of the system. By symmetry, $x \equiv \alpha + k \mod (m-1)$, $y \equiv \alpha \mod (m-1)$ is also a solution. Suppose that x, y is any solution.

Then $2x \equiv a \equiv 2\alpha \mod (m-1)$. Therefore either $x \equiv \alpha \mod (m-1)$ or $x \equiv \alpha + k \mod (m-1)$. If $x \equiv \alpha \mod (m-1)$, then:

$$y \equiv b - x \equiv (a+k) - \alpha \equiv \alpha + k \mod (m-1).$$

If $x \equiv \alpha + k \mod (m-1)$, then:

$$y \equiv b - x \equiv (a+k) - (\alpha + k) \equiv \alpha \mod (m-1).$$

Therefore there are precisely two distinct solutions. □

Let s be an integer such that $3 \leq s \leq m-1$ and suppose that X be any subset of I with cardinality s. Then $G(X)$ contains $\binom{s}{3}$ subgraphs of the form $G_a \cup G_b \cup G_c$ where a, b, c are distinct integers modulo $(m-1)$. It follows, from Lemma 6, that, if m is even, then there are precisely $\binom{s}{3}$ type-1 triangles in $G(X)$. On the other hand, if m is odd, then type-1 triangles only occur in subgraphs of the form $G_a \cup G_b \cup G_c$ where the integers a, b, c are distinct modulo $(m-1)$ and such that $a+b+c$ is even. In this case a, b, c are all even or one is even and the other two are odd. Suppose that s is even and X contains $s/2$ odd numbers and $s/2$ even numbers (in this case we say that X is *parity balanced*). Then the number of subgraphs in which a, b, c are all even is:

$$\binom{s/2}{3} = \frac{1}{48}s(s-2)(s-4).$$

The number of subgraphs in which one is even and the other two are odd is:

$$\frac{s}{2}\binom{s/2}{2} = \frac{1}{16}s^2(s-2).$$

Therefore the number of triangles in $G(X)$ is:

$$\begin{aligned} 2[\frac{1}{48}s(s-2)(s-4) + \frac{1}{16}s^2(s-2)] \\ &= \frac{1}{24}s(s-2)[(s-4)+3s] \\ &= \frac{1}{6}s(s-1)(s-2) \\ &= \binom{s}{3}. \end{aligned}$$

In either case the number of triangles is $\binom{s}{3}$. This completes the proof of

Proposition 9. *Let s be an integer such that $3 \leq s \leq m-1$. Let X be a subset of I with cardinality s and suppose that, if m is odd, then X is parity balanced. Then the graph $G(X)$ contains precisely $\binom{s}{3}$ type-1 triangles.*

Note that the sets occurring in each of the partitions P_1 and P_2 satisfy the conditions of this proposition. Hence the associated graphs all have the same number of type-1 triangles.

Given a subset X of I, let $T(X)$ denoted the set of all triples $\{a, b, c\}$, where a, b, c are (distinct) elements of X such that $a + b \equiv 2c \pmod{m-1}$ and, whenever m is odd a and b are both even. Then, by Lemma 7, the number of type-2 triangles in $G(X)$ is given by $|T(X)|$, when m is even, and $2|T(X)|$, when m is odd. As for graphs, we write $T(p, q; i_1, i_2, \ldots, i_t) = T(X)$, when $X = [p, q] \setminus \{i_j \mid 1 \leq j \leq t\}$, and, in particular, $T(p, q) = T([p, q])$.

Let p, q be integers such that $0 \leq p, q \leq m - 2$ and $0 \leq q - p \leq r + 1$. Put $X = [p, q]$. Let a, b, c be distinct integers belonging to X such that $a + b \equiv 2c \mod (m-1)$. Moreover assume that, when m is odd, a and b are both even. Then $a + b - 2c = t(m-1)$ for some integer t. Now:

$$-2(m-1) < -2(q-p) \leq a + b - 2c \leq 2(q-p) < 2(m-1).$$

Thus $t = 0, 1$ or -1. We may assume that $a < b$.

If $t = 0$, then $a + b = 2c$ and hence a and b are both odd or both even. Moreover $a < c = \frac{1}{2}(a + b) < b$.

If $t = 1$, then $a + b - 2c = m - 1 = nr$. Hence:

$$nr \leq 2(q-p) \leq 2(r+1) < 3r.$$

Therefore $n = 2$, so that m is odd and r is even. In this case a and b are even. Then $2r = a + b - 2c < 2(b - c)$ so that $r < b - c \leq q - p \leq r + 1$. It follows that $b - c = q - p = r + 1$ so that $b = q$ and $c = p$. Then:

$$2r = a + b - 2c = a + q - 2p = a - p + r + 1$$

and hence $a = p + r - 1 = q - 2$. In particular, $p = c < a < b = q$, but $a \neq \frac{1}{2}(c + b)$ (otherwise $q - 2 = \frac{1}{2}(p + q)$ so that $r = q - p - 1 = 3$; contradiction). Also q is even and p is odd. Conversely, if $n = 2$ and if q is even and p is odd with $q - p = r + 1$, then, putting $a = q - 2$, $b = q$ and $c = p$, a and b are both even and $a + b - 2c = 2(q - p) - 2 = 2r = m - 1$; i.e. $t = 1$.

If $t = -1$, then $a + b - 2c = -nr$ and hence again:

$$nr \leq 2(q-p) < 3r$$

so that $n = 2$ and hence again m is odd and r is even, so that a and b are odd. Moreover $2r = 2c - a - b < 2(c - a)$. Then $r < c - a \leq q - p \leq r + 1$ and hence $c - a = q - p = r + 1$. It follows that $a = p$, $c = q$ and:

$$b = 2c - a - 2r = p + 2 = q - r - 1.$$

In particular, $p = a < b < c = q$ and $b \neq \frac{1}{2}(a+c)$ (otherwise we again obtain the contradiction, $r = 3$). However, in this case, p is even and q is odd. Conversely, if $n = 2$ and if p is even and q is odd with $q - p = r + 1$, then, putting $a = p$, $b = p + 2$ and $c = q$, a and b are both even and $a + b - 2c = 2(p - q) + 2 = -2r = -(m - 1)$; i.e. $t = -1$.

These arguments complete the proof of

Lemma 10. *Let a, b, c be integers in I such that $a < b < c$ and $c - a \leq r + 1$. Then $\{a, b, c\} \in T(I)$ if and only if precisely one of the following conditions hold:*

(i) $b = \frac{1}{2}(a+c)$, and a and b are both even or both odd if m is even, or both just even if m is odd;

(ii) $n = 2$, $c - a = r + 1$, a is odd, c is even and $b = c - 2$;

(iii) $n = 2$, $c - a = r + 1$, a is even, c is odd and $b = a + 2$.

We now apply Lemma 10 to determine the number of type-2 triangles in $G(p, q)$. Suppose that m is even (so that r and n are both odd). In this case the number of type-2 triangles is equal to $|T(p, q)|$. By the lemma $|T(p, q)|$ is the number of pairs of distinct even integers or distinct odd integers in $[p, q]$. If p and q are both odd, then the number of even integers in $[p, q]$ is $\frac{1}{2}(q - p)$ and the number of odd integers in $[p, q]$ is $\frac{1}{2}(q - p) + 1$. If p and q are both even, then the number of even integers is $\frac{1}{2}(q - p) + 1$ and the number of odd is $\frac{1}{2}(q - p)$. Therefore, in either case:

$$|T(p, q)| = \binom{\frac{1}{2}(q - p) + 1}{2} + \binom{\frac{1}{2}(q - p)}{2} = \frac{1}{4}(q - p)^2.$$

If one of p and q is odd and the other even, then the number of even integers in $[p, q]$ is $\frac{1}{2}(q - p + 1)$ as is the number of odd integers. Therefore:

$$|T(p, q)| = 2\binom{\frac{1}{2}(q - p + 1)}{2} = \frac{1}{4}[(q - p)^2 - 1].$$

Now suppose that m is odd and $n > 2$. In this case the number of type-2 triangles is equal to $2|T(p, q)|$ and $|T(p, q)|$ is the number of pairs of distinct even integers in $[p, q]$. Thus, if p and q are both odd, then:

$$|T(p, q)| = \binom{\frac{1}{2}(q - p)}{2} = \frac{1}{8}(q - p)(q - p - 2);$$

if one of p and q is odd and the other even, then:

$$|T(p, q)| = \binom{\frac{1}{2}(q - p + 1)}{2} = \frac{1}{8}[(q - p)^2 - 1];$$

and otherwise, if p and q are both even, then:

$$|T(p,q)| = \binom{\frac{1}{2}(q-p)+1}{2} = \frac{1}{8}(q-p)(q-p+2).$$

If m is odd, $n=2$, $q-p=r+1$, and either p is even and q is odd or p is odd and q is even, we have in addition 1 further triple; thus:

$$|T(p,q)| = \frac{1}{8}[(q-p)^2+7].$$

This completes the proof of

Proposition 11. *Let p,q be integers such that $0 \le p,q \le m-2$ and $0 \le q-p \le r+1$. Let N be the number of type-2 triangles in the graph $G(p,q)$. If p and q are both odd, then $N = \frac{1}{4}(q-p)^2$ when m is even and $N = \frac{1}{4}(q-p)(q-p-2)$ when m is odd. If p and q are both even, then $N = \frac{1}{4}(q-p)^2$ when m is even and $N = \frac{1}{4}(q-p)(q-p+2)$ when m is odd. If p is odd and q is even, or p is even and q is odd, then $N = \frac{1}{4}[(q-p)^2-1]$ unless m is odd, $n=2$ and $q-p=r+1$, in which case $N = \frac{1}{4}[(r+1)^2+7]$.*

In particular, by Proposition 11, the number of type-2 triangles in $G(0,r-1)$ is equal to $\frac{1}{4}(r-1)^2$ when m is even and $\frac{1}{4}[(r-1)^2-1] = \frac{1}{4}r(r-2)$ when m is odd.

To deal with the other graphs $G(0,r+1;r-1,r)$, $G(r-1,2r-1;r+1)$, $G(0,r+1;r-2,r-1)$ and $G(r-2,2r-1;r,r+1)$, we first note that, if m is even, then:

$$|T(0,r+1)| = |T(r-2,2r-1)| = \frac{1}{4}(r+1)^2;$$
$$|T(r-1,2r-1)| = \frac{1}{4}[r^2-1],$$

if m is odd and $n>2$, then:

$$|T(0,r+1)| = |T(r-2,2r-1)| = \frac{1}{8}[(r+1)^2-1];$$
$$|T(r-1,2r-1)| = \frac{1}{8}r(r-2),$$

and, if m is odd and $n=2$, then:

$$|T(0,r+1)| = |T(r-2,2r-1)| = \frac{1}{8}[(r+1)^2+7];$$
$$|T(r-1,2r-1)| = \frac{1}{8}r(r-2).$$

Next we remove triples from the appropriate sets $T(0,r+1)$, $T(r-1,2r-1)$ and $T(r-2,2r-1)$ to obtain the sets $T(0,r+1;r-1,r)$, $T(r-1,2r-1;r+1)$, $T(0,r+1;r-2,r-1)$ and $T(r-2,2r-1;r,r+1)$.

First consider the case when m is even. By Lemma 10 we obtain $r+1$ triples $\{a,b,c\}$ in $T(0,r+1)$ which contain $r-1$ or r; namely:

$\{0,\frac{r-1}{2},r-1\},\{2,\frac{r+1}{2},r-1\},\dots,\{r-3,r-2,r-1\},\{r-1,r,r+1\},$
$\{1,\frac{r+1}{2},r\},\{3,\frac{r+3}{2},r\},\dots,\{r-2,r-1,r\},\{r-3,r-1,r+1\}.$

Therefore:

$$|T(0,r+1;r-1,r)| = |T(0,r+1)| - (r+1) = \frac{1}{4}(r-1)^2 - 1.$$

Similarly, if $r > 3$, there are $\frac{1}{2}(r+3)$ triples in $T(r-1,2r-1)$ which contain $r+1$:

$\{r+1,r+2,r+3\},\{r+1,r+3,r+5\},\dots,\{r+1,r+\frac{1}{2}(r-1),2r-2\},$
$\{r-1,r,r+1\},\{r,r+1,r+2\},\{r-1,r+1,r+3\};$

but, if $r=3$, there are only 2:

$\{2,3,4\},\{3,4,5\}.$

Therefore:

$$|T(r-1,2r-1;r+1)| = \begin{cases} |T(r-1,2r-1)| - \frac{1}{2}(r+3) & (r>3) \\ |T(2,5)| - 2 & (r=3) \end{cases}$$
$$= \begin{cases} \frac{1}{4}(r-1)^2 - 2 & (r>3) \\ 0 & (r=3). \end{cases}$$

There are $r+3$ triples in $T(0,r+1)$ which contain $r-2$ and $r-1$:

$\{0,\frac{r-1}{2},r-1\},\{2,\frac{r+1}{2},r-1\},\dots,\{r-3,r-2,r-1\},\{r-1,r,r+1\},$
$\{1,\frac{r-1}{2},r-2\},\{3,\frac{r+1}{2},r-2\},\dots,\{r-4,r-3,r-2\},\{r-2,r-1,r\},$
$\{r-5,r-2,r+1\},\{r-4,r-2,r\},\{r-3,r-2,r+1\};$

if $r>3$, but, if $r=3$ there are 4:

$\{0,1,2\},\{1,2,3\},\{2,3,4\},\{0,2,4\}.$

Therefore:

$$|T(0,r+1;r-2,r-1)| = \begin{cases} |T(0,r+1)| - (r+3) & (r>3) \\ |T(0,4)| - 4 & (r=3) \end{cases}$$
$$= \begin{cases} \frac{1}{4}(r-1)^2 - 3 & (r>3) \\ 0 & (r=3). \end{cases}$$

Now consider the case when m is odd. Any additional triples occurring in the case $n=2$ are listed in brackets with a similar modification to the total number. There are $\frac{r}{2}$ triples in $T(0,r+1)$ which contain $r-1$ or r:

$$\{0,\tfrac{r}{2},r\},\{2,\tfrac{r+2}{2},r\},\dots,\{r-2,r-1,r\}.$$

Hence:

$$|T(0,r+1;r-1,r)| = |T(0,r+1)| - \frac{r}{2}$$
$$= \begin{cases} \frac{1}{8}r(r-2) & (n\neq 2) \\ \frac{1}{8}r(r-2)+1 & (n=2). \end{cases}$$

There is just one triple in $T(r-1,2r-1)$ which contain $r+1$:

$$\{r,r+1,r+2\}.$$

Hence:

$$|T(r-1,2r-1;r+1)| = |T(r-1,2r-1)| - 1$$
$$= \frac{1}{8}r(r-2) - 1.$$

There are $\frac{1}{2}(r+2)$ triples in $T(0,r+1)$ which contain $r-2$ and $r-1$:

$$\{0,\tfrac{r-2}{2},r-2\},\{2,\tfrac{r-4}{2},r-2\},\dots,\{r-4,r-3,r-2\},$$
$$\{r-2,r-1,r\},\{r-4,r-2,r\};$$
$$(\{0,2,5\} \text{ when } r=4 \text{ and } n=2).$$

Therefore:

$$|T(0,r+1;r-2,r-1)| = \begin{cases} |T(0,r+1)| - \frac{r}{2} - 1 & (n>2 \text{ or } r>4) \\ |T(0,r+1)| - \frac{r}{2} - 2 & (n=2 \text{ and } r=4) \end{cases}$$
$$= \begin{cases} \frac{1}{8}r(r-2) - 1 & (n\neq 2 \text{ or } r=4) \\ \frac{1}{8}r(r-2) & (n=2 \text{ and } r>4). \end{cases}$$

If $r>4$, then there are $\frac{r}{2}+2(+1)$ triples in $T(r-2,2r-1)$ which contain r and $r+1$:

$\{r, r+1, r+2\}, \{r, r+2, r+4\}, \ldots, \{r, r+\frac{r-2}{2}, 2r-2\},$
$\{r-2, r-1, r\}, \{r-2, r, r+2\}, \{r-2, r+1, r+4\},$
$(\{r-2, r, 2r-1\}).$

Note that, in the case $n = 2$, the additional triple added to $T(r-2, 2r-1)$ is then removed in obtaining $T(r-2, 2r-1; r, r+1)$. Therefore, when $r > 4$:

$$|T(r-2, 2r-1; r, r+1)| = \begin{cases} |T(r-2, 2r-1)| - \frac{r}{2} - 2 & (n \neq 2) \\ |T(r-2, 2r-1)| - \frac{r}{2} - 3 & (n = 2) \end{cases}$$
$$= \frac{1}{8}r(r-2) - 2.$$

Recall that, when $r = 4$, $G(r-2, 2r-1; r, r+1)$ is isomorphic to $G(0, r+1; r-2, r-1)$ and hence:

$$\begin{aligned} |T(r-2, 2r-1; r, r+1)| &= |T(0, r+1; r-2, r-1)| \\ &= \frac{1}{8}r(r-2) - 1 \\ &= 0. \end{aligned}$$

It follows that the number of type-2 triangles in the given graphs are as stated in Table 1. Note that the table shows no distinction between

graph	m is even	m is odd
$G(0, r-1)$	$\frac{1}{4}(r-1)^2$	$\frac{1}{4}r(r-2)$
$G(0, r+1; r-1, r)$	$\frac{1}{4}(r-1)^2 - 1$	$\begin{cases} \frac{1}{4}r(r-2) & (n > 2) \\ \frac{1}{4}r(r-2) + 2 & (n = 2) \end{cases}$
$G(r-1, 2r-1; r+1)$	$\begin{cases} \frac{1}{4}(r-1)^2 - 2 & (r > 3) \\ 0 & (r = 3) \end{cases}$	$\frac{1}{4}r(r-2) - 2$
$G(0, r+1; r-2, r-1)$	$\begin{cases} \frac{1}{4}(r-1)^2 - 3 & (r > 3) \\ 0 & (r = 3) \end{cases}$	$\begin{cases} \frac{1}{4}r(r-2) - 2 & (n > 2 \text{ or } r = 4) \\ \frac{1}{4}r(r-2) & (n = 2 \text{ and } r > 4) \end{cases}$
$G(r-2, 2r-1; r, r+1)$	$\begin{cases} \frac{1}{4}(r-1)^2 - 3 & (r > 3) \\ 0 & (r = 3) \end{cases}$	$\begin{cases} \frac{1}{4}r(r-2) - 4 & (r > 4) \\ 0 & (r = 4) \end{cases}$

Table 1: Number of type-2 triangles

the graphs $G(0, 4; 2, 3)$ and $G(2, 5; 4)$ when m is even. By Lemma 6, each of these graphs has a unique (type-1) triangle. Indeed it is easy to see, by simply drawing the graphs when $n = 3$ (highlighting the symmetry relative to the triangle), that the graphs are isomorphic. However, when $n \geq 5$, again drawing the graphs in a sufficiently close neighbourhood of the triangle, it is clearly evident that they are now non-isomorphic. The table also fails to distinguish between the graphs $G(0, r-1)$ and $G(0, r+1; r-1, r)$ when m is odd and $n > 2$. In this case further study is necessary.

Next we use Lemma 8 to determine the number of type-3 triangles. Such triangles exist only when m is odd and hence r is even. In this case, given distinct integers a, b modulo $(m-1)$, the subgraph $G_a \cup G_b$ has two type-3 triangles if and only if either a is even or b is even, and $a - b \equiv \frac{1}{2}(m-1) \pmod{m-1}$. If $n \geq 3$, then:

$$2(r+1) < 3r \leq m - 1;$$

i.e. $r + 1 < \frac{1}{2}(m-1)$, and hence $G(0, r+1)$ and $G(r-2, 2r-1)$ have no type-3 triangle. If $n = 2$, then $r = \frac{1}{2}(m-1)$. In this case Table 2 details the number of such triangles.

$G(0, r-1)$	0
$G(0, r+1; r-1, r)$	0
$G(r-1, 2r-1; r+1)$	0
$G(0, r+1; r-2, r-1)$	2
$G(r-2, 2r-1; r, r+1)$	2

Table 2: Number of type-3 triangles ($n = 2$)

It is easy to see directly that the graphs $G(0, 5; 2, 3)$ and $G(2, 7; 4, 5)$ are isomorphic. These are special cases which we make note of later. For our purpose we do not need to make any further study of these graphs in order to establish any additional comparisons between the isomorphism classes of these graphs. It is now clear that we have verified the assertions of Theorems 2 and 3 related to this question.

3.2 Connectivity

We now consider the connectedness of the graphs. Our aim is to show that for each of the given graphs, G, if S is a vertex cut of minimum cardinality, then $|S| = r$. Since such a G is r-regular, $|S| \leq r$. Now $G \backslash S = A \cup B$, where $|V(A)| \geq 1$, $|V(B)| \geq 1$ and $V(A) \cap V(B) = \emptyset$. Note that $G \backslash S$ contains no edges joining $V(A)$ to $V(B)$, and each vertex of S is joined to a vertex of A and to a vertex of B. It suffices to show that for each of the graphs $|S| \geq r$. We will use the methods of [6]. Recall that $V = \{0, 1, 2, \ldots, m-2\}$. If $V \cap V(A) = \emptyset$, then $V(A) = \{\infty\}$ and hence all r vertices which are adjacent to ∞ are in S. Hence we may assume that there exists a vertex i in $V(A) \cap V$ and a vertex j in $V(B) \cap V$. Choose the integers i and j so that $|i - j|$ is minimal. Moreover we may assume, without loss of generality, that i and j are chosen so that $i < j$ and $0 \leq i < m - 1$. By the minimality of $j - i$, the vertex q belongs to S whenever $i < q < j$. Thus $|S| \geq j - i - 1$. Hence we may assume that $j - i \leq r$. Note that also $m - 1 \geq r + 4$; for if $r \geq 4$, then $m - 1 \geq 2r \geq r + 4$ and, if $r = 3$, then $n \geq 3$ and, since $m - 1 = rn \geq 9$, the inequality is trivial. Note also that, since $n - 1$ and r cannot both be odd, $m - 1 - r = (n-1)r$ must be even. Recall that, if $G = G(X)$ and $x, y \in V$ then x is adjacent to y in G if and only if $x + y \equiv z \pmod{m-1}$ for some

$z \in X$ and x is adjacent to ∞ in G if and only if $2x \equiv z \pmod{m-1}$ for some $z \in X$. We now consider each graph individually:

$G(0, r-1)$. This case has already been dealt with in [4, 6].

$G = G(0, r+1; r-1, r)$. Since i and j are not adjacent, $i+j \equiv k \pmod{m-1}$, where $r+2 \le k < m-1$ or $k = r-1$ or $k = r$. Then $i+j-k+(m-1) = s(m-1)$ for some integer s. In dealing with this graph we make much use of the following identities:

$$2i \equiv k-(j-i) \pmod{m-1}; \qquad 2j \equiv k+(j-1) \pmod{m-1}.$$

The argument is now divided into several cases:

(A) $1 \le j-i \le r-2$: Then:

$$j < j-k+(m-1) \le i-k+r-2+(m-1) < i+(m-1),$$

and moreover, whenever $j-k+(m-1) \le q \le i-k+r-2+(m-1)$, q is adjacent to both i and j:

$$i+j-k+(m-1) \le i+q < j+q \le i+j-k+r-2+(m-1);$$

i.e.

$$s(m-1) \le i+q < j+q \le s(m-1)+r-2.$$

Since these $(r-1)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we now have $r-2$ distinct vertices in S. To determine 2 further vertices in S we consider the following subcases:

(a) $3 \le j-i \le r-2$: Then $r \ge 5$. It follows that $m-1 \ge r+6$.

(i) $r+2 \le k < m-1$: Then, since $i-k+r+1+(m-1) \le i-1+(m-1)$, the vertex $i-k+r+1$ is distinct from the above $r-2$ vertices of S and is clearly adjacent to i and j and hence belongs to S. If $k \le m-3$, then $j-k-1+(m-1) \ge j+1$ and hence $(i, i-k+r, j+1, j-k-1, j)$ is a path in G. This path yields a further vertex in S. Note that, if $k = m-3$, then $j-k-1 \equiv j+1 \pmod{m-1}$. In this case we make the convention that the repeated vertex is not present in the path so that we really mean the path $(i, i-k+r, j+1, j)$. We will use this convention throughout the paper. If $k = m-2$, then $i-k+r-2+(m-1) = i+r-1$ and hence, since $i+r+3 \ge i-3+(m-1)$ $(i, i+r, i-1, i+r+3, i-2, i+r+4, i-3, \dots)$ is a path which eventually leads to a vertex which is adjacent to ∞. Each vertex of this path is labeled by an integer between $i+r$ and $i-1+(m-1)$. Such a path is called a *zigzag*. Also, since $2j \equiv j-i-1 \pmod{m-1}$, j is adjacent to ∞ and hence this zigzag extends to a path from i to j.

(ii) $k = r-1$: Then $(i, j+2, j-k-2, j+1, j-k-1, j)$ is a path in G and, since $j-k-3+(m-1) \geq j+4$, $(j, j-k-3, j+3, j-k-4, j+4, \ldots)$ is a zigzag. Since $2i \equiv r-1-(j-i) \pmod{m-1}$, i is adjacent to ∞ and this yields a further 2 vertices in S.

(iii) $k = r$: Then $i-k+r-2+(m-1) = i-2+(m-1)$ and $j-k-2+(m-1) \geq j+2$, and hence the two paths $(i, i-1, j+2, j-k-2, j)$ and $(i, j+1, j-k-1, j)$ yield the required 2 vertices of S.

(b) $j-i=1$: Then $2i \equiv k-1 \pmod{m-1}$ and $2j \equiv k+1 \pmod{m-1}$.

(i) $k = r-1$: Then $i-k+r-2+(m-1) = i-1+(m-1)$ and i is adjacent to ∞.

$r \geq 4$: Since $m-1 \geq r+4$, $j-k-2+(m-1) \geq j+3$ and we have the zigzag $(j, j+1, j-k-2, j+3, j-k-3, \ldots)$ and the path $(i, j+2, j-k-1, j)$.

$r = 3$: Then $k = 2$, $n \geq 3$ and $j-k+(m-1) = i-k+r-2+(m-1) = i-1+(m-1)$. If $n \geq 4$, then $m-1 = rn \geq 12$ and hence $i-6+(m-1) \geq j+5$. Hence we have the path $(i, j+2, i-3, j+1, j)$ and the zigzag $(j, i-2, j+4, i-6, j+5, i-7, \ldots)$. If $n = 3$, then $m = 10$ and $2i \equiv 1 \pmod 9$. Therefore $i = 5$ and $j = 6$. In this case we have the paths $(5, \infty, 2, 7, 6)$ and $(5, 8, 1, 3, 6)$.

(ii) $k = r$: Then $i-k+r-2 = i-2$ and $j-k-1+(m-1) > j+1$. In this case we have the paths $(i, i-1, \infty, j)$ and $(i, j+1, j-k-1, j)$.

(iii) $k = r+2$: Then $i-k+r-2+(m-1) = i-4+(m-1)$ and hence $(i, i-3, i-1, j)$ is a path. Now i and $i-2$ are adjacent to ∞. Since $j-k-1+(m-1) \geq j+1$, $(i, \infty, i-2, j+1, j-k-1, j)$ is a path.

(iv) $k = m-2 \geq r+3$: Then j is adjacent to ∞. Now $i-k+r-2+(m-1) = i+r-1$. The argument depends on whether $i-k+r+2+(m-1) \leq i-4+(m-1)$; i.e. $m-1 = k+1 \geq r+7$; i.e. $(n-1)r \geq 7$. This clearly holds if $n = 2$ and $r \geq 7$ or if $n = 3$ and $r \geq 4$ or if $n \geq 4$. In this case we have the zigzag $(i, i+r+3, i-4, i+r+4, i-5, \ldots)$. Also we have the path $(i, i+r, i-1, i+r+1, i-2, i+r+2, j)$. We must now consider the special cases $n = 2$, $r = 4, 6$ and $n = 3$, $r = 3$:

$n = 2$, $r = 4$: In this case the required paths are $(i, i+4, i+6, j)$ and $(i, i+7, i+5, \infty, j)$.

$n = 2$, $r = 6$: In this case the paths are $(i, i+9, \infty, j)$ and $(i, i+6, i+8, j)$.

$n = 3$, $r = 3$: The paths are $(i, i+3, i+8, i+4, i+7, i+5, j)$ and $(i, i+6, \infty, j)$.

(v) $k = r+3 \leq m-3$: Then $m-1 \geq r+5$. For the general case we need $j-k-2+(m-1) \geq j+2$; i.e. $m-1 \geq r+7$. The case $m-1 = r+5$ cannot occur since then $r=5$ and $n=2$ (if r is odd then n must also be odd). Therefore the exceptions only arise when $rn = m-1 = r+6$; i.e. $n=2$, $r=6$ or $n=3$, $r=3$. In the general case we have the paths $(i, i-4, j+2, j-k-2, j+1, j-k-1, j)$ and $(i, i-1, i-3, i-2, j)$.

$n=2$, $r=6$: The paths are $(i, i+11, i+9, i+10, j)$ and $(i, i+8, i+3, j)$.

$n=3$, $r=3$: The paths are $(i, i+8, i+6, i+7, j)$ and $(i, i+5, i+3, j)$.

(vi) $k = m-3 \geq r+4$: Then $m-1 \geq r+6$ and $i-k+r-2+(m-1) = i+r$. For the general case we need $i+r+2 \leq i-4+(m-1)$; i.e. $m-1 = k+2 \geq r+8$. The case $m-1 = r+7$ cannot occur since then $r=7$ and $n=2$. Therefore the exceptions only arise when $rn = r+6$; i.e. $n=2$, $r=6$ or $n=3$, $r=3$. In the general case we have the path $(i, i+r+1, i-1, i+r+2, i-2, i+r+3, j)$ and the zigzag $(i, i+r+4, i-4, i+r+5, i-5, \dots)$, which may be adjoined to the path $(\infty, j+1, j)$.

$n=2$, $r=6$: The paths are $(i, i+7, i+11, \infty, i+2, j)$ and $(i, i+10, i+9, j)$.

$n=3$, $r=3$: The paths are $(i, i+4, i+8, \infty, i+2, j)$ and $(i, i+7, i+6, j)$.

(vii) $k = r+4 \leq m-4$: Then $m-1 \geq r+7$. Again we cannot have $m-1 = r+7$. Therefore $m-1 \geq r+8 = k+4$. Then $i-k+r-2 = i-6$ and $j-k-2+(m-1) \geq j+2$. Hence we have paths $(i, i-5, j+2, j-k-2, j+1, j-k-1, j)$ and $(i, i-2, i-3, j)$.

(viii) $k = m-4 \geq r+5$: Then $i-k+r+2+(m-1) \leq i-3+(m-1)$ and $j-k+(m-1) = j+3$. In this case we have the two paths $(i, i-k+r+2, i-3, i-k+r+1, j)$ and $(i, i-k+r-1, j+2, j)$.

(ix) $m-5 \geq k \geq r+5$: Then $j-k-2+(m-1) \geq j+2$ and $i-k+r+2+(m-1) \leq i-3+(m-1)$. In this case we have the paths $(i, i-k+r-1, j+2, j-k-2, j+1, j-k-1, j)$ and $(i, i-k+r+2, i-3, i-k+r+1, j)$.

(c) $j-i = 2 \leq r-2$: Then $r \geq 4$. Also $2i \equiv k-2 \pmod{m-1}$ and $2j \equiv k+1 \pmod{m-1}$.

(i) $k = r-1$: Then $i-k+r-2+(m-1) = i-1+(m-1)$ and i and j are both adjacent to ∞. Thus $\infty \in S$. Moreover $j-k+(m-1) > j+3$. Hence we have the paths $(i, j+2, j-k-1, j)$.

(ii) $k = r$: Then $i-k+r-2 = i-2$ and $j-k-2+(m-1) \geq j+2$. In this case we have the paths $(i, i-1, j+2, j-k-2, j)$ and

$(i, j+1, j-k-1, j)$.

(iii) $r+2 \leq k \leq m-5$. Now $i-k+r+1+(m-1) \leq i-1+(m-1)$ and $j-k-2+(m-1) \geq j+2$. In that case the paths are $(i, i-k+r-1, j+2, j-k-2, j+1, j-k-1, j)$ and $(i, i-k+r, i-1, i-k+r+1, j)$.

(iv) $k=m-2 \geq r+2$: Then $i-k+r-2+(m-1) = i+r-1$ and $j-k+(m-1) = j+1$. Since in this case $2j \equiv 1 \pmod{m-1}$, j is adjacent to ∞ and also r must be odd. Therefore $m-1 \geq r+6$. Then $i+r+3 \leq i-3+(m-1)$ and we have the zigzag $(i, i+r+1, i-2, i+r+3, i-3, \ldots)$ and the path $(i, i+r, i-1, i+r+2, j)$.

(v) $k=m-3 \geq r+2$: Then $i-k+r-2+(m-1) = i+r$ and $j-k+(m-1) = j+2$. Also $i+r+3 \leq i-1+(m-1)$ and hence the paths $(i, i+r+1, i-1, i+r+3, j)$ and $(i, i+r+2, j+1, j)$.

(vi) $k=m-4 \geq r+2$: Then $i-k+r-2+(m-1) = i+r+1$ and $j-k+(m-1) = j+3$. Since $m-1 \geq r+5$, $i+r+3 \leq i-2+(m-1)$ and hence we have the paths $(i, i+r+2, j+2, j)$ and $(i, i+r+3, j+1, j)$.

(B) $j-i = r-1$: Then we already have $r-2$ vertices in S between i and j. We need a further 2 vertices in S. Now $2i \equiv k+1-r \pmod{m-1}$ and $2j \equiv k+r-1 \pmod{m-1}$.

(i) $k=r-1$: Now i is adjacent to ∞. Since $m-1 \geq r+4$, $j+2 = i+r+1 \leq i-3+(m-1)$ and hence we have the path $(i, j+2, i-3, j+1, i-2, j)$. If $m-1 \geq r+7$, then $j+3 = i+r+2 \leq i-5+(m-1)$ and hence we have the zigzag $(j, i-1, j+3, i-5, j+4, \ldots)$. If $r+4 \leq m-1 = rn \leq r+6$, then either $n=2$ and $r=4,6$, or $n=3$ and $r=3$.

$n=2$, $r=4$: Then $m-1=8$, $k=3$, $j=i+3$ and the paths are $(i, \infty, i+4, i+6, j)$ and $(i, i+5, j)$.

$n=2$, $r=6$: Then $m-1=12$, $k=5$, $j=i+5$ and the paths are $(i, \infty, i+6, i+10, j)$ and $(i, i+7, j)$.

$n=3$, $r=3$: Then $m-1=9$, $k=2$, $j=i+2$ and the paths are (i, ∞, j) and $(i, i+4, i+5, i+8, j)$.

(ii) $k=r$: Now $2i \equiv 1 \pmod{m-1}$ and hence r must be odd. If $n=3$, $r=3$, then we have paths $(i, i+3, i+6, j)$ and $(i, i+8, i+4, i+5, i+7, j)$. Otherwise $m-1 \geq r+10$. In this case $j+3 = i+r+2 \leq i-8+(m-1)$ and hence we have the paths $(i, i-1, j+2, i-5, j+3, i-2, j)$ and $(i, j+1, i-3, j)$.

(iii) $r+2 \leq k \leq m-6$: Now $i-k+r+1+(m-1) \leq i-1+(m-1)$ and $i-k+r-4+(m-1) \geq i+r+1 = j+2$. In this case we have paths $(i, i-k+r, i-1, i-k+r+1, j)$ and $(i, i-k+r-1, j+2, i-k+r-4, j+1, i-k+r-3, j)$.

(iv) $k = m-5 \geq r+2$: Then $m-1 \geq r+6$ and hence $j+5 = i+r+4 \leq i-2+(m-1)$. Therefore we have the paths $(i, j+5, i-1, j+6, j)$ and $(i, j+4, j+2, j)$.

(v) $k = m-4 \geq r+2$: Then $m-1 \geq r+5$ and hence $j+4 = i+r+3 \leq i-2+(m-1)$. Therefore we have the paths $(i, j+3, j+2, j)$ and $(i, j+4, j+1, j)$.

(vi) $k = m-3$: Then $j+3 = i+r+2 \leq i-2+(m-1)$, and hence we have the paths $(i, j+2, \infty, j)$ and $(i, j+3, j+1, j)$.

(vii) $k = m-2$: Then j is adjacent to ∞. If $r \geq 4$, then, since $j+4 = i+r+3 \leq i-1+(m-1)$, we have the path $(i, j+3, j)$ and the zigzag $(i, j+1, i-1, j+4, i-2, \dots)$. If $r = 3$, then either $n = 3$ or $n \geq 5$. If $r = 3$ and $n = 3$, then $m = 10$, $k = 8$, $i = 3$ and $j = 5$. In this case we have the paths $(3, 7, 2, \infty, 5)$ and $(3, 1, 8, 5)$ joining i to j. If $r = 3$ and $n \geq 5$, then $m-1 \geq 15$ and hence $i+8 \leq i-7+(m-1)$. In this case, since $j = i+r-1 = i+2$ we obtain the path $(i, i+4, i-1, i+5, j)$ and the zigzag $(i, i+7, i-4, i+8, i-5, \dots)$.

(C) $j - i = r$: In this case we already have $r-1$ vertices in S and hence we require just one more. Now $2i \equiv k - r \pmod{m-1}$ and $2j \equiv k + r \pmod{m-1}$.

(i) $k = r-1$: Then $2i+1 = (i+j)-(j-i)+1 \equiv 0 \pmod{m-1}$. Hence $m-1$ is odd and therefore r and n are odd. Thus $r \geq 3$ and $n \geq 3$. Then $m-1 \geq r+6$. Hence $j+2 = i+r+2 < i-3+(m-1)$ and we have the path $(i, j+2, i-3, j+1, i-2, j)$.

(ii) $k = r$: Now $j+1 = i+r+1 \leq i-3+(m-1)$. Then we have the path $(i, j+1, i-3, j)$.

(iii) $r+2 \leq k < m-1$: Then:

$$j = i+r < i-k+r+1+(m-1) \leq i-1+(m-1).$$

Hence, since $j-k+1 = i-k+r+1$ is adjacent to both i and j, $i-k+r+1 \in S$.

$G = G(r-1, 2r-1; r+1)$. Since i and j are not adjacent, $i+j-r+1 \equiv k \pmod{m-1}$, where $r+1 \leq k < m-1$ or $k = 2$. Then $i+j-k+(m-1) = r-1+s(m-1)$ for some integer s. In this case we consider the identities $2i \equiv r-1+k-(j-i) \pmod{m-1}$ and $2j \equiv r-1+k+(j-i) \pmod{m-1}$. The main argument is divided into the two cases, $r+1 \leq k < m-1$ and $k = 2$:

(I) $r+1 \leq k < m-1$: This case is again separated into various special cases:

(A) $1 \leq j-i \leq r-3$: Then $r \geq 4$ and:

$$j < j-k+3+(m-1) \leq i-k+r+(m-1) < i+(m-1),$$

and moreover, whenever $j-k+3+(m-1) \leq q \leq i-k+r+(m-1)$, q is adjacent to both i and j:

$$i+j-k+3+(m-1) \leq i+q < j+q \leq i+j-k+r+(m-1);$$

i.e.

$$r+2+s(m-1) \leq i+q < j+q \leq s(m-1)+2r-1.$$

Since these $(r-2)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we now have $r-3$ distinct vertices in S. To determine 3 further vertices in S we consider the following subcases:

(a) $3 \leq j-i \leq r-3$: The vertices $j-k+(m-1)$ and $j-k+1+(m-1)$ are distinct from the above $r-3$ vertices of S and are clearly adjacent to both i and j and hence both belong to S. Moreover, if $k \geq r+2$, then $i-k+r+1+(m-1) \leq i-1+(m-1)$, and hence $(i, i-k+r+1, i-1, j-k+2, j)$ is a path and hence at least one of its vertices must be a further vertex in S. If $k = r+1$, then $(i, i-1, j-k+2, j)$ is a path in G.

(b) $j-i = 1 \leq r-3$: Then $r \geq 4$.

(i) $k = r+1$. Then $i-k+r+(m-1) = i-1+(m-1)$ and i is adjacent to ∞. Now $m-1 \geq r+4$. If $m-1 = r+4$, then $n=2$ and $r=4$; otherwise $m-1 \geq r+6$.

$n=2$, $r=4$: In this case we have the paths $(i, \infty, i+2, i+6, j)$, $(i, i+7, j)$ and $(i, i+5, i+3, j)$.

$m-1 \geq r+6$: Then $m-1 \geq k+5$ and hence $j-k-1+(m-1) \geq j+4$. Hence $(j, j-k-1, j+4, j-k-2, j+5, \ldots)$ is a zigzag. Also we have paths $(i, j-k+1, j+1, j-k+2, j)$ and $(i, j-k, j)$.

(ii) $r+2 \leq k \leq m-3$. Then $i-k+r+1+(m-1) \leq i-1+(m-1)$ and $j-k-1+(m-1) \geq j+1$. Therefore we have the paths $(i, i-k+r+1, i-1, j-k+2, j)$, $(i, j-k, j)$ and $(i, j-k+1, j+1, j-k-1, j)$.

(iii) $k = m-2$: Then $i-k+r+(m-1) = i+r+1$, $j-k+3+(m-1) = j+4$ and $j+r+1 = i+r+2 \leq i-2+(m-1)$. Hence we have the paths $(i, j+r+1, i-1, j+3, j)$, $(i, j+2, \infty, j)$ and $(i, j+1, j)$.

(c) $j-i=2\geq r-3$: Then $r\geq 5$. It follows that $m-1\geq r+6$.

(i) $k=r+1$: Then $m-5\geq k+5$. Moreover $i-k+r+(m-1)=i-1+(m-1)$, i is adjacent to ∞ and $j-k-2+(m-1)\geq j+3$ and hence we have the zigzag $(j,j-k-1,j+3,j-k-2,j+4,\dots)$ and the paths $(i,j-k+1,j)$ and $(i,j-k,j+1,j-k+2,j)$.

(ii) $r+2\leq k\leq m-4$: Then $i-k+r+1+(m-1)\leq i-1+(m-1)$ and $j-k-2+(m-1)\geq j+1$. Hence we have the paths $(i,i-k+r+1,i-1,j-k+2,j)$, $(i,j-k,j+1,j-k-2,j)$ and $(i,j-k+1,j)$.

(iii) $k=m-3$, then $i-k+r+(m-1)=i+r+2$, $j-k+3+(m-1)=j+5$ and $j+r+1=i+r+3\leq i-1+(m-1)$. Hence we have the paths $(i,j+r+1,i-1,j+4,j)$, $(i,j+3,j)$ and $(i,j+2,j+1,j)$.

(iv) $k=m-2$: Then $i-k+r+(m-1)=i+r+1$, $j-k+3+(m-1)=j+4$ and $j+r+1=i+r+3\leq i-1+(m-1)$. Hence j is adjacent to ∞ and we have the zigzag $(i,i+r+2,i-1,i+r+3,\dots)$. Also we have the paths $(i,j+1,j+3,j)$ and $(i,j+2,j)$.

(B) $j-i=r-2$: Then there are $r-3$ vertices in S.

(a) $r\geq 5$: Then $m-1\geq r+6$. Hence $j+1=i+r-1\leq i-7+(m-1)$.

(i) $k=r+1$: Then we have paths $(i,\infty,i-1,j)$, $(i,i-3,j)$ and $(i,i-2,j)$.

(ii) $k=r+2$: We have paths $(i,i-1,i-2,j)$, $(i,i-3,j)$ and $(i,i-4,j)$.

(iii) $k\geq r+3$: Then $j-k+3+(m-1)\leq i-2+(m-1)$ and we have paths $(i,j-k+1,j)$, $(i,j-k+3,i-2,j-k+2,j)$ and $(i,j-k,j)$.

(b) $r=4$: Then $j-i=2$ and m is odd. Now $2i-1\equiv k \pmod{m-1}$. Thus k is odd.

$k=5$: Then i and $i-1$ are both adjacent to ∞ and, since $j+1=i+3\leq i-5+(m-1)$, we have paths $(i,\infty,i-1,j)$, $(i,i-2,j)$ and $(i,i-3,j+1,i-5,j)$.

$7\leq k\neq m-4$: Then $j-k+6+(m-1)\leq i-1+(m-1)$ and $j-k-2+(m-1)\geq j+1$ and hence we have paths $(i,j-k+4,i-1,j-k+2,j)$, $(i,j-k+1,j)$ and $(i,j-k,j+1,j-k-2,j)$.

$k=m-2$: Then $j+5=i+7\leq i-1+(m-1)$ and hence we have paths $(i,j+1,\infty,j)$, $(i,j+2,j)$ and $(i,j+5,i-1,j+3,j)$.

(c) $r=3$: Then $j-i=1$ and $m-1\geq 9$. Now $4\leq k<m-1$.

(i) $k=4$: Since $i-5+(m-1)\geq i+4=j+3$, we have paths $(i,\infty,i-1,j)$, $(i,i-2,i+2,i-4,j)$ and $(i,i-3,j)$.

(ii) $k=5$: Again, since $i-5+(m-1)\geq j+3$, we have paths $(i,i-1,i-2,j)$, $(i,i-3,j+1,i-5,j)$ and $(i,i-4,j)$.

(iii) $6 \le k \le m-3$: Then $j-k+3+(m-1) \le i-2+(m-1)$ and $j-k-1+(m-1) \ge j+1$ and hence we have paths ; $(i, j-k+1, j+1, j-k-1, j)$, $(i, j-k+3, i-2, j-k+2, j)$ and $(i, j-k, j)$.

(iv) $k = m-2$: Then j is adjacent to ∞. Since $i-3+(m-1) \ge i+6$, we have the zigzag $(i, i+5, i-3, i+6, \dots)$. Moreover we have paths $(i, i+3, i-1, i+4, j)$ and $(i, i+2, j)$.

(C) $j-i = r-1$: Then we have $r-2$ vertices in S.

(a) $r \ge 4$: Then $j-k+1+(m-1) = i-k+r+(m-1) \le i-1+(m-1)$. Hence $j-k$ and $j-k+1$ are both adjacent to i and to j.

(b) $r = 3$: Then $j = i+2$ and $m-1 \ge 9$. Hence $j-k+1+(m-1) < i-1+(m-1)$ and $i-3+(m-1) \ge i+6$.

(i) $k = m-2$: Then $j+2 = i+4$ is adjacent to both i and j and hence belongs to S. Now j is adjacent to ∞. Also we have the zigzag $(i, i+6, i-3, i+7, \dots)$.

(ii) $k = m-3$: Then $j+3 = i+5$ is adjacent to both i and j. Moreover we have the path $(i, i+4, i+3, j)$.

(iii) $k \le m-4$: Then $j-k-2+(m-1) \ge j+1$. In this case we have the paths $(i, j-k+1, j)$ and $(i, j-k, j+1, j-k-2, j)$.

(D) $j-i = r$: In this case we already have $r-1$ vertices in S. Note that $j < j-k+(m-1) = i+r-k+(m-1) \le i-1+(m-1)$ and moreover that $j-k$ is adjacent to both i and j.

(II) $k = 2$: Again the argument is separated into various special cases:

(A) $1 \le j-i \le r-3$: Now $i+j \equiv r+1 \pmod{m-1}$. Whenever $j+1 \le q \le i-2+r$, q is adjacent to both i and j:

$$i+j+1 \le i+q < j+q \le i+j-2+r;$$

Since these $(r-2)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we have $r-3$ distinct vertices in S. Note that, since:

$$2j = (i+j)+(j-i) \equiv (j-i)+r+1 \pmod{m-1}$$

and $1 \le j-i \le r-3$, j is adjacent to ∞. To determine 3 further vertices in S we consider the following subcases:

(a) $3 \le j-i \le r-3$: Since $r \ge 6$, $m-1 \ge r+6$. Hence $i+r+2 \le i-4+(m-1)$. Then we have paths $(i, i+r, i-2, j)$, $(i, i+r-1, i-1, j)$ and the zigzag $(i, i+r+1, i-3, i+r+2, \dots)$.

(b) $j - i = 1 \leq r - 3$: Then $r \geq 4$ and i is adjacent to ∞. Thus $\infty \in S$. Now $i+r-1 \leq i-2+(m-1)$ and hence, provided $r \geq 5$, we have the paths $(i, i-1, j)$ and $(i, i+r-1, i-2, j)$. If, however, $r = 4$, then $m - 1 \geq 8$ and hence $i + 5 \leq i - 3 + (m - 1)$ so that we have the paths $(i, i - 1, j)$ and $(i, i + 3, i - 3, i + 5, i - 2, j)$.

(c) $j-i = 2 \geq r-3$: Then $r \geq 5$ and i is adjacent to ∞. Thus $\infty \in S$. Now $i+r \leq i-2+(m-1)$ and we have the paths $(i, i+r, i-2, j)$ and $(i, i + r - 1, i - 1, j)$.

(B) $j - i = r - 2$: Then there are $r - 3$ vertices in S. Moreover j is adjacent to ∞.

(a) $r \geq 5$: Then $m - 1 \geq r + 6$. Hence $j + 5 \leq i - 3 + (m - 1)$. Therefore we have the zigzag $(i, j + 1, i - 3, j + 5, i - 4, \dots)$ and the paths $(i, j + 2, i - 1, j)$ and $(i, j + 3, i - 2, j)$.

(b) $r = 4$: Then $j - i = 2$ and m is odd. Now i is adjacent to ∞. Thus $\infty \in S$. Also $m - 1 \geq 8$ and hence $i + 5 \leq i - 3 + (m - 1)$. Thus we have paths $(i, i + 4, i - 1, j)$ and $(i, i + 3, i - 2, j)$.

(c) $r = 3$: Then $j - i = 1$ and m is even. Hence $m - 1 \geq 9$. Again i is adjacent to ∞ so that $\infty \in S$. In this case we have paths $(i, i + 2, i - 2, j)$ and $(i, i - 1, j)$.

(C) $j-i = r-1$: Then we have $r-2$ vertices in S. Note that $j = i+r-1$.

(a) $r \geq 4$: Then $j + 2 = i + r + 1 \leq i - 3 + (m - 1)$. Hence we have paths $(i, j + 2, i - 1, j)$ and $(i, j + 1, i - 2, j)$.]

(b) $r = 3$: Then $j = i + 2$ and hence, since $m - 1 \geq 9$, $j + 3 \leq i-4+(m-1)$, we have the path $(i, j+1, i-2, j)$. Also i is adjacent to ∞ and we have the zigzag $(j, i - 1, j + 2, i - 4, j + 3, \dots)$.

(D) $j-i = r$: In this case we already have $r-1$ vertices in S. Note that $j \leq i - 2 + (m - 1)$. Hence we have the path $(i, j + 1, i - 2, j)$.

<u>$G = G(0, r + 1; r - 2, r - 1)$</u>. In the case $r = 4$, $n = 2$ the graph G ($G = G(0, 5; 2, 3)$) is not r-connected, i.e. 4-connected, but only 3-connected ($\{1, 5, \infty\}$ is a vertex cut). We exclude this case in the argument which follows. Now, if $n = 2$, then, since m is odd and hence r is even, $r \geq 6$ and therefore $m-1 = 2r \geq r+6$. If $n \geq 3$, then, since $r \geq 3$, $m-1 \geq 3r \geq r+6$. Therefore with the exclusion we have in general $m - 1 \geq r + 6$.

Since i and j are not adjacent, $i + j \equiv k \pmod{m - 1}$, where $r + 2 \leq k < m - 1$ or $k = r - 2$ or $k = r - 1$. Then $i + j - k + (m - 1) = s(m - 1)$ for some integer s. In this case we consider the identities $2i \equiv k - (j - i) \pmod{m - 1}$ and $2j \equiv k + (j - i) \pmod{m - 1}$. The argument is now divided into several cases:

(A) $1 \leq j - i \leq r - 3$: Then:

$$j < j - k + (m - 1) \leq i - k + r - 3 + (m - 1) < i + (m - 1),$$

and moreover, whenever $j-k+(m-1) \leq q \leq i-k+r-3+(m-1)$, q is adjacent to both i and j:

$$i+j-k+(m-1) \leq i+q < j+q \leq i+j-k+r-3+(m-1);$$

i.e.

$$s(m-1) \leq i+q < j+q \leq s(m-1)+r-3.$$

Since these $(r-2)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we now have $r-3$ distinct vertices in S. To determine 3 further vertices in s we consider the following subcases:

(a) $4 \leq j-i \leq r-3$: Then $r \geq 7$ and hence $(m-1) \geq 2r \geq r+7$.

(i) $r+2 \leq k < m-1$: Then $i-k+r+1+(m-1) \leq i-1+(m-1)$ and hence the vertices $i-k+r+(m-1)$ and $i-k+r+1+(m-1)$ are distinct from the above $r-3$ vertices of S and are clearly adjacent to i and j and hence both belong to S. Moreover, when $k \leq m-3$, $j-k-1+(m-1) \geq j+1$ and therefore we have the path $(i, i-k+r-1, j+1, j-k-1, j)$, and, when $k=m-2$, since $2j \equiv j-i-1 \pmod{m-1}$ and $i-k+r+2+(m-1) = i+r+3 \leq i-4+(m-1)$, j is adjacent to ∞ and $(i, i+r-1, i-1, i+r+3, i-2, i+r+4, \dots)$ is a zigzag.

(ii) $k=r-2$: Then $j-k-4+(m-1) = j-r-2+(m-1) \geq j+5$. Hence $(i, j+2, j-k-2, j)$ and $(i, j+3, j-k-3, j)$ are paths. Moreover i is adjacent to ∞ and we have the zigzag $(j, j-k-4, j+4, j-k-5, j+5, \dots)$.

(iii) $k=r-1$: Then, $j-k-3+(m-1) = j-r-2+(m-1) \geq j+5$. Hence we have the paths $(i, j+1, j-k-1, j)$, $(i, j+2, j-k-2, j)$ and $(i, i-1, j+3, j-k-3, j)$.

(b) $j-i=1 \leq r-3$: Then $r \geq 4$.

(i) $k=r-2$: Then $i-k+r-3+(m-1) = i-1+(m-1)$ and i is adjacent to ∞. Note that r cannot be even. For otherwise $m-1$ and k are both even but this contradicts the fact that $2i = i+j-1 \equiv k-1 \pmod{m-1}$.

$r \geq 7$: Then $j-k-2+(m-1) > j+4$ and we have the zigzag $(j, j+1, j-k-2, j+4, j-k-3, \dots)$ and the paths $(i, j+3, j-k-1, j)$ and $(i, j+2, j)$.

$r=5$: Then $j-k+(m-1) = i-2+(m-1)$ and, since $m-1 = 5n \geq 15$, $i-4+(m-1) \geq i+11$. In this case we have paths $(i, i+3, j)$, $(i, \infty, i+2, j)$ and $(i, i+4, i-4, i+7, i-3, j)$.

(ii) $k=r-1$: Then $i-k+r-3+(m-1) = i-2+m-1$ and $j-k-1+(m-1) = i-r+1+(m-1)$.

$r \geq 5$: Then, since $(m-1) > r+5$, $i-r+1+(m-1) > i+6 = j+5$. In this case we have the paths $(i, i-1, \infty, j)$, $(i, j+2, j-k-1, j)$ and $(i, j+1, j)$.

$r = 4$: Then, since n cannot be equal to 2, $m-1 = 4n \geq 12$. Hence $i-4+(m-1) \geq i+8 = j+7$ and we have the paths (i, ∞, j), $(i, j+2, i-4, j+5, i-3, j)$ and $(i, j+1, j)$.

(iii) $k = r+2$: Then $i-k+r-3 = i-5$. Therefore, since $m-1 \geq r+6 = k+4$, $j-k-2+(m-1) \geq j+2$. Hence we have paths $(i, i-4, j+2, j-k-2, j+1, j-k-1, j)$, $(i, i-1, j)$ and $(i, \infty, i-2, j)$.

(iv) $k = r+3$: Then $i-k+r-3 = i-6$. Since $m-1 \geq r+6 = k+3$, $j-k-1+(m-1) \geq j+2$. Hence we have paths $(i, i-2, j)$, $(i, i-5, j+2, i-4, j+1, j-k-1, j)$ and $(i, i-1, \infty, i-3, j)$.

(v) $k = r+4$: Then $i-k+r-3 = i-7$. Since $m-1 \geq r+6 = k+2$, $j-k-1+(m-1) \geq j+1$. Hence we have paths $(i, i-3, j)$, $(i, i-2, i-4, j)$ and the path $(i, i-6, i-1, \infty)$ together with the zigzag $(j, j-k-1, j+1, j-k-2, \ldots)$.

(vi) $k = r+5 \leq m-3$: Then $i-k+r-3 = i-8$ and $m-1 \geq k+2$. Hence $j-k-1+(m-1) \geq j+1$ and therefore we have paths $(i, i-7, i-1, i-6, j+1, j-k-1, j)$, $(i, i-3, i-5, j)$ and $(i, i-4, j)$.

(vii) $k = r+6 \leq m-3$: Then $i-k+r-3 = i-9$ and $m-1 \geq k+2$. Then $j-k-1+(m-1) \geq j+1$ and hence we have paths $(i, i-4, \infty, i-2, i-6, j)$, $(i, i-8, i-1, i-7, j+1, j-k-1, j)$ and $(i, i-5, j)$.

(viii) $r+5 \leq k \leq m-3$: Then $j-k-1+(m-1) \geq j+1$ and $i-k+r+2+(m-1) \leq i-3+(m-1)$. In this case we have the paths $(i, i-k+r+1, j)$, $(i, i-k+r-2, i-2, i-k+r, j)$ and $(i, i-k+r+2, i-1, i-k+r-1, j+1, j-k-1, j)$.

(ix) $k = m-2$: In this case j is adjacent to ∞. The argument depends on whether $i-k+r+2+(m-1) \leq i-5+(m-1)$; i.e. $m-1 = k+1 \geq r+8$; i.e. $(n-1)r \geq 8$. This clearly holds if $n = 2$ and $r \geq 8$ or if $n = 3$ and $r \geq 4$ or if $n \geq 5$. We therefore must consider the special case $n = 2$, $r = 6$:

$m-1 \geq r+8$: We have the zigzag $(i, i+r+3, i-5, i+r+4, \ldots)$ and also the paths $(i, i+r-1, i-1, i+r, i-2, i+r+1, j)$ and $(i, i+r+2, j)$.

$n = 2$, $r = 6$: In this case the paths are $(i, i+5, i+10, \infty, j)$, $(i, i+9, i+7, j)$ and $(i, i+8, j)$.

(c) $j-i = 2 \leq r-3$: Then $r \geq 5$.

(i) $k = r-2$: Then $i-k+r-3+(m-1) = i-1+(m-1)$. Moreover i and j are both adjacent to ∞. Thus $\infty \in S$. Also

$j-k+(m-1)=i-r+4+(m-1)$ and, since $m-1\geq r+6$, $i-r+1+(m-1)\geq i+7=j+5$. Hence we have the paths $(i,j+2,i-r+2,j)$ and $(i,j+3,i-r+1,j+1,j)$.

(ii) $k=r-1$: Then again i and j are both adjacent to ∞ and hence $\infty\in S$. Now $i-k+r-3+(m-1)=i-2+m-1$ and $j-k+(m-1)=i-r+3+(m-1)$. Again $i-r+1+(m-1)\geq j+5$. In this case we have the paths $(i,j+1,i-r+2,j)$ and $(i,j+2,j-r+1,j)$.

(iii) $k=r+2$: Then $i-k+r-3+(m-1)=i-5+(m-1)$. Then, since $m-1\geq r+6=k+4$, $j-k-2+(m-1)\geq j+2$. Then i is adjacent to ∞ and we have paths $(i,i-4,i-1,j)$ and $(i,i-3,i-2,j)$ and the zigzag $(j,j-k-1,j+1,j-k-2,\ldots)$.

(iv) $k=r+3$: Then $i-k+r-3+(m-1)=i-6+(m-1)$. Since $m-1\geq r+6=k+3$, $j-k-1+(m-1)\geq j+2$. Hence the paths are $(i,i-1,i-3,j)$, $(i,\infty,i-2,j)$ and $(i,i-4,j+1,j-k-1,j)$.

(v) $k=r+4$: Then $i-k+r-3+(m-1)=i-7+(m-1)$. Since $m-1\geq r+6=k+2$, $j-k-1+(m-1)\geq j+1$. Hence the paths are $(i,i-1,i-4,j)$, $(i,i-2,i-3,j)$ and $(i,i-5,j+1,j-k-1,j)$.

(vi) $k=m-2\geq r+5$: Then $i-k+r-3+(m-1)=i+r-2$ and $j-k+(m-1)=j+1$. Also j is adjacent to ∞. If $m-1\geq r+8$, then $i+r+4\leq i-4+(m-1)$. Then we have the zigzag $(i,i+r+3,i-4,i+r+4,\ldots)$ and the paths $(i,i+r,i-2,i+r+2,j)$ and $(i,i+r-1,i-1,i+r+1,j)$. Otherwise, since $m-1\geq r+6$ and $r\geq 5$, the only other possibilities occur when $m-1-r=6$; i.e. when $(n-1)r=6$; i.e. when $n=2$ and $r=6$. In this case we have the paths $(i,i+5,\infty,j)$, $(i,i+9,i+8,j)$ and $(i,i+10,i+7,j)$.

(vii) $r+5\leq k\leq m-3$: Now $i-k+r+2+(m-1)\leq i-3+(m-1)$ and $j-k-1+(m-1)\geq j+1$. In that case we have paths $(i,i-k+r-2,i-1,i-k+r,j)$, $(i,i-k+r-1,j+1,j-k-1,j)$ and $(i,i-k+r+2,i-3,i-k+r+1,j)$.

(d) $j-i=3\leq r-3$. Then $r\geq 6$.

(i) $k=r-2$: Then $i-k+r-3+(m-1)=i-1+(m-1)$. Also i and j are adjacent to ∞. Thus $\infty\in S$. Now $j-k+(m-1)=i-r+5+(m-1)$ and, since $m-1\geq r+6$, $i-r+2+(m-1)\geq i+8=j+5$. Hence we have the paths $(i,j+2,i-r+3,j)$ and $(i,j+3,i-r+2,j)$.

(ii) $k=r-1$: Now $i-k+r-3+(m-1)=i-2+m-1$ and $j-k+(m-1)=i-r+4+(m-1)$. Since $(m-1)\geq r+6$, $i-r+1+(m-1)\geq i+7=j+4$. In this case we have the paths $(i,j+1,i-r+3,j)$, $(i,j+2,i-r+2,j)$ and $(i,i-1,j+3,i-r+1,j)$.

(iii) $k=m-2$: Then $i-k+r-3+(m-1)=i+r-2$ and $j-k+(m-1)=j+1$. Also j is adjacent to ∞. If $m-1\geq r+8$, then $i + r + 4 \leq i - 4 + (m - 1)$. Hence we have the zigzag $(i, i+r+4, i-4, i+r+5, \dots)$. Also we have the paths $(i, i+r+1, j)$ and $(i, i+r-1, i-1, i+r+2, j)$. Otherwise, since $m-1 \geq r+6$ and $r \geq 6$, the only other possibilities occur when $m-1-r = 6$; i.e. when $(n-1)r = 6$; i.e. when $n = 2$ and $r = 6$. In this case we have the paths $(i, i+11, \infty, j)$, $(i, i+10, i+8, j)$ and $(i, i+7, j)$.

(iv) $r+2 \leq k \leq m-3$: Now $i-k+r+1+(m-1) \leq i-1+(m-1)$ and $j-k-1+(m-1) \geq j+1$. In that case we have paths $(i, i-k+r, j)$, $(i, i-k+r-1, j+1, j-k-1, j)$ and $(i, i-k+r-2, i-1, i-k+r+1, j)$.

(B) $j - i = r - 2$: Then we have $r - 3$ vertices in S.

(a) $k = r - 2$: Then, since $2i \equiv 0 \pmod{m-1}$, i is adjacent to ∞. Now $m - 1 \geq r + 4$. We will consider the cases $m - 1 = r + 4$ and $m - 1 \geq r + 5$ separately.

$r \geq 5$: It is easy to show that $m - 1 \geq r + 8$ unless $n = 2$, $r = 6$. In the general case $j + 4 = i + r + 2 \leq i - 6 + (m - 1)$ and hence we have paths $(i, j+2, i-3, j)$, $(i, j+3, i-1, j)$ and the zigzag $(j, i-2, j+4, i-6, j+5, \dots)$. If $n = 2$, $r = 6$, then we have paths $(i, i+6, i+8, i+10, j)$, $(i, i+7, i+11, j)$ and $(i, \infty, i+9, j)$.

$r = 4$: Then $m-1 \geq 12$ and hence $j+5 < i-3+(m-1)$. In this case we have paths $(i, j+3, i-1, j)$ and $(i, j+3, i-3, j+5, i-2, j)$. Also, since $2j \equiv 4 \pmod{m-1}$, j is adjacent to ∞ so that $\infty \in S$.

$r = 3$: In this case $2j \equiv 2 \pmod{m-1}$ so that $i+2 = j+1$ is adjacent to ∞. Therefore we have paths $(i, i+3, j)$, $(i, i+4, i-1, j)$ and $(i, \infty, i+2, j)$.

(b) $k = r - 1$: Then $2i \equiv 1 \pmod{m-1}$. It follows that $m - 1$ and hence r is odd.

$r \geq 5$: Then i is adjacent to ∞. Since $m-1 \geq r+6$, $j+3 = i+r+1 \leq i - 5 + (m-1)$ and hence we have the paths $(i, j+1, i-3, j)$ and $(i, i-1, j+3, i-2, j)$. If $(n-1)r = m-1-r = 6$, then $n = 2$, $r = 6$. But r is odd. Hence $m - 1 \geq r + 8$. Then $j + 6 = i + r + 4 \leq i - 4 + (m-1)$ and hence we have the zigzag $(j, i-4, j+6, i-5, \dots)$.

$r = 3$: Then $k = 2$ and $j = i + 1$. In this case j is adjacent to ∞. Now $m - 1 = 3n \geq 9$. Hence $i + 4 \leq i - 5 + (m-1)$ and we have the paths $(i, i-1, i+4, i-2, j)$ and $(i, i+2, j)$. If $n = 3$, then $m-1 = 9$ and we have the path $(i, i+3, i+5, i+6, \infty, j)$. If $n \geq 5$, then $m-1 \geq 15$ and $i+7 < i-4+(m-1)$ and hence in this case we have the zigzag $(i, i+3, i-4, i+8, i-5, \dots)$.

(c) $r + 2 \leq k < m - 1$

(i) $r \geq 5$ Now $i - k + r + 1 + (m-1) \leq i - 1 + (m-1)$.

$k \leq m-5$: Then $i-k+r-3+(m-1) \geq i+r+1 = j+3$. In this case we have paths $(i, i-k+r, j)$, $(i, i-k+r-2, j+3, i-k+r-3, j)$ and $(i, i-k+r-1, i-1, i-k+r+1, j)$.

$k = m-4$: Then $i+r+4 \leq i+k+2 = i-1+(m-1)$, and hence the paths $(i, i+r+3, j)$, $(i, i+r+2, i-1, i+r+4, j)$ and $(i, i+r+1, \infty, j)$

$k = m-3$: Then $i+r+3 \leq i+k+1 = i-1+(m-1)$, and hence the paths $(i, i+r+2, j)$, $(i, i+r+1, i-1, i+r+3, j)$ and $(i, i+r, \infty, j)$

$k = m-2$: Then $i+r+2 \leq i+k = i-1+(m-1)$ and, provided $r \geq 6$, we have the paths $(i, i+r+2, j)$, $(i, i+r+1, j)$ and $(i, i+r, \infty, j)$. If $r = 5$, then $m-1 \geq 15$ and $j = i+3$. In this case $i+10 < i-1+(m-1)$ and we have the paths $(i, i+6, j)$, $(i, i+10, i-1, i+7, j)$ and $(i, i+5, \infty, j)$.

(ii) $r = 4$: Then $j = i+2$ and $6 \leq k < m-1$. Also $m-1$ is even and, since $k \equiv 2j-2 \pmod{m-1}$, k must be even. Now $m-1 = 4n \geq 12$. Hence $i+5 \leq i-7+(m-1)$.

$k = 6$: We have paths $(i, \infty, i-2, j)$, $(i, i-4, i+5, i-5, j)$ and $(i, i-3, i-1, j)$.

$k = 8$: We have paths $(i, i-1, \infty, i-3, j)$, $(i, i-6, i+5, i-7, j)$ and $(i, i-2, i-4, j)$.

$10 \leq k \leq m-5$: Then $i-k+6+(m-1) \leq i-4+(m-1)$ and $i-k+1+(m-1) \geq i+5 = j+3$. In this case we have paths $(i, i-k+3, j+1, i-k+4, j)$, $(i, i-k+2, j+3, i-k+1, j)$ and $(i, i-k+6, i-3, i-k+5, j)$.

$k = m-3$: Then, since $i+8 \leq i-4+(m-1)$, we obtain the paths $(i, i+4, \infty, j)$, $(i, i+5, i-1, i+6, j)$ and $(i, i+8, i-3, i+7, j)$.

(iii) $r = 3$: Then $j = i+1$ and $5 \leq k < m-1$.

$k = 5$: Now $m-1 = 3n \geq 9$. Hence $i+4 \leq i-5+(m-1)$. Therefore we have paths $(i, \infty, i-2, j)$, $(i, i-1, j)$ and $(i, i-4, i+4, i-5, j)$.

$6 \leq k \leq m-5$: Then $i-k+5+(m-1) \leq i-1+(m-1)$ and $i-k+(m-1) \geq i+4 = j+3$. In this case we have paths $(i, i-k+1, j+3, i-k, j)$, $(i, i-k+5, i-1, i-k+2, j+1, i-k+3, j)$ and $(i, i-k+4, j)$.

$k = m-4$: Then, if $n \geq 4$, $i+9 \leq i-3+(m-1)$, and hence we obtain the paths $(i, i+4, i+3, j)$, $(i, i+8, i-1, i+9, i-2, i+6, j)$ and $(i, i+7, j)$. If $n = 3$, then $m = 10$ and, since $2i+1 = i+j \equiv 6 \pmod 9$, $i = 7$ and $j = 8$. In this case we have the paths $(7, 5, 8)$, $(7, 2, 1, 8)$ and $(7, 6, \infty, 0, 4, 8)$.

$k = m-3$: Then, if $n \geq 4$, $i+9 \leq i-3+(m-1)$, and hence we obtain the paths $(i, i+7, i-1, i+4, i+2, j)$ and $(i, i+6, j)$ and the path $(i, i+3, \infty)$ together with the zigzag $(j, i+5, i-2, i+9, i-3, \ldots)$. If $n = 3$, then $m = 10$ and, since $2i+1 = i+j \equiv 7 \pmod 9$, $i = 3$ and $j = 4$. In this case we have the paths $(3, 0, 4)$, $(3, 1, 8, 4)$ and $(3, 6, \infty, 2, 7, 5, 4)$.

$k = m - 2$: Then, since $i + 6 \leq i - 3 + (m - 1)$ we have paths $(i, i+5, j)$, $(i, i+2, i+4, j)$ and $(i, i+6, i-1, i+3, \infty, j)$.

(C) $j - i = r - 1$ Then we have $r - 2$ vertices in S.

(a) $k = r - 2$: Then $2i + 1 \equiv 0 \pmod{m-1}$. Hence $m - 1$ and r are odd.

$r \geq 5$: Since $m-1 \geq r+6$, $j+3 = i+r+2 \leq i-4+(m-1)$ and hence we have the paths $(i, j+3, i-1, j)$ and $(i, j+2, i-3, j+1, i-2, j)$.

$r = 3$: If $n \geq 4$, then j is adjacent to ∞ and $i + 8 \leq i - 4 + (m - 1)$. Hence in this case we have the path $(i, i+5, i-1, j)$ and the zigzag $(i, i+4, i-3, i+8, i-4, \ldots)$. If $r = 3$ and $n = 3$, then, since $2i + 2 = i + j \equiv 1 \pmod 9$, $i = 4$ and $j = 6$. In this case we have the paths $(4, 0, 3, 6)$ and $(4, 8, 1, 2, \infty, 6)$.

(b) $k = r - 1$: Then $2i \equiv 0 \pmod{m-1}$. Hence $m - 1$ and r are even. In particular $r \geq 4$. Since $m - 1 \geq r + 6$, $j + 3 = i + r + 2 \leq i - 4 + (m - 1)$ and hence we have the paths $(i, j+1, i-3, j)$ and $(i, j+2, i-1, j+3, i-2, j)$.

(c) $r + 2 \leq k < m - 1$:

(i) $r \geq 5$: Now $i - k + r + 1 + (m - 1) \leq i - 1 + (m - 1)$ and $i - k + r - 1 + (m-1) \geq i + r = j + 1$. In this case we have paths $(i, i-k+r+1, j)$ and $(i, i-k+r, j)$.

(ii) $r = 4$: Then $j = i + 3$ and $6 \leq k < m - 1$. If $k \geq 9$, then $i - k + 8 + (m - 1) \leq i - 1 + (m - 1)$, and hence we obtain the paths $(i, i-k+4, j)$ and $(i, i-k+8, i-1, i-k+5, j)$. Since $m - 1$ is even and $k \equiv i + j = 2i + 3 \pmod{m-1}$, k must be odd. Therefore we need to consider the case $k = 7$. In this case we have paths $(i, i-3, j)$ and $(i, \infty, i-2, j)$.

(iii) $r = 3$: Then $m - 1$ is odd, $j = i + 2$ and $5 \leq k < m - 1$.

$k = 5$: Then i is adjacent to ∞. Now $m - 1 = 3n$, where n is odd. If $n \geq 5$, then $i + 6 \leq i - 9 + (m - 1)$, and we have the path $(i, i-3, i+3, i-2, j)$ and the zigzag $(j, i-5, i+6, i-6, \ldots)$. If $n = 3$, then, since $2i \equiv 3 \pmod 9$, $i = 6$, $j = 8$ and we have the paths $(6, \infty, 0, 4, 8)$ and $(6, 3, 1, 8)$.

$k = 6$: Then i and $i - 2$ are adjacent to ∞ and we have the paths $(i, i-1, i-3, j)$ and $(i, \infty, i-2, j)$.

$k = 7$: Then $i + 3 \leq i - 6 + (m - 1)$ and hence the paths $(i, i-2, i-3, j)$ and $(i, i-5, i+3, i-4, j)$.

$k = 8$: Then $i-1$ and $i-3$ are adjacent to ∞. Now $m-1 \geq 9$ so that $j+1 \leq i-6+(m-1)$ and hence we have the paths $(i, i-2, i-4, j)$ and $(i, i-3, \infty, i-1, i-5, j)$.

$k = 9$: Then $m-1 \geq 11$ so that $j+2 \leq i-7+(m-1)$ and hence the paths $(i, i-3, i-1, i-6, j)$ and $(j, i-5, i-2, \infty)$ together with the zigzag $(i, i-7, j+2, i-8, \dots)$.

$k = 10$: Then $i-2$ and $i-4$ are adjacent to ∞. Now $m-1 \geq 11$ so that $j+1 \leq i-8+(m-1)$ and hence we have the paths $(i, i-5, i-3, i-1, i-7, j)$ and $(i, i-4, \infty, i-2, i-6, j)$.

$k \geq 11$: Then $i-k+8+(m-1) \leq i-3+(m-1)$ and $i-k+2+(m-1) \geq j+1$, and hence we obtain the paths $(i, i-k+6, i-1, i-k+3, j)$ and $(i, i-k+5, i-3, i-k+8, i-2, i-k+4, j)$.

(D) $j-i = r$: In this case we already have $r-1$ vertices in S and hence we require just one more.

(a) $k = r-1$: Then $2i+1 \equiv 0 \pmod{m-1}$ and hence r and n are odd. Thus $r \geq 3$ and $n \geq 3$. Since $m-1 \geq r+6$, $j+3 = i+r+3 \leq i-3+(m-1)$ and we have the path $(i, j+2, i-1, j+3, i-2, j)$.

(b) $k = r-2$: Again $j+3 \leq i-3+(m-1)$. Then we have the path $(i, j+3, i-1, j)$.

(c) $r+2 \leq k < m-1$: Then

$$j = i+r < i-k+r+(m-1) \leq i-2+(m-1).$$

Hence, since $j-k = i-k+r$ is adjacent to both i and j, $i-k+r \in S$.

$G = G(r-2, 2r-1; r, r+1)$. In the case it suffices to assume that m is odd, r is even and $r \geq 6$. Then $m-1 \geq r+6$. Since i and j are not adjacent, $i+j-r+2 \equiv k \pmod{m-1}$, where $r+2 \leq k < m-1$ or $k = 2$ or $k = 3$. Then $i+j-k+(m-1) = r-2+s(m-1)$ for some integer s. In this case we consider the identities $2i \equiv r-2+k-(j-i) \pmod{m-1}$ and $2j \equiv r-2+k+(j-i) \pmod{m-1}$. The main argument is divided into the three cases, $r+2 \leq k < m-1$, $k = 2$ and $k = 3$

(I) $r+2 \leq k < m-1$: This case is again separated into various special cases:

(A) $1 \leq j-i \leq r-2$: Then

$$j < j-k+3+(m-1) \leq i-k+r+1+(m-1) < i+(m-1),$$

and moreover, whenever $j-k+4+(m-1) \leq q \leq i-k+r+1+(m-1)$, q is adjacent to both i and j

$$\begin{aligned} i+j-k+4+(m-1) &\leq i+q \\ &< j+q \leq i+j-k+r+1+(m-1); \end{aligned}$$

i.e.

$$r+2+s(m-1) \leq i+q < j+q \leq 2r-1+s(m-1).$$

Since these $(r-2)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we now have $r-3$ distinct vertices in S. To determine 3 further vertices in S we consider the following subcases:

(a) $4 \leq j-i \leq r-2$: Then the vertices $j-k$ and $j-k+1$ are distinct from the above $r-3$ vertices of S and both are clearly adjacent to i and j and hence belong to S. To obtain a further vertex in S we consider the following cases separately:

(i) $j-i \leq r-3$:

$k \geq r+3$: Then $i-k+r+2+(m-1) \leq i-1+(m-1)$, and hence we have the path $(i, i-k+r+2, i-1, j-k+2, j)$.

$k = r+2$: Then i is adjacent to ∞ and we have the zigzag $(j, j-k+2, j+1, j-k-1, j+2, \ldots)$.

(ii) $j-i = r-2$:

$k = r+2$: Then i is adjacent to ∞. Since $m-1 \geq r+6$, $j-k-1+(m-1) \geq j+3$. Hence, since $r \geq 6$, we have the zigzag $(j, j-k+2, j+1, j-k-1, j+2, \ldots)$.

$k = r+3$: Then $i-k+r+2+(m-1) = i-1+(m-1)$ and we have the path $(i, i-1, j-k+2, j)$.

$k \geq r+4$: Then $i-k+r+3+(m-1) \leq i-1+(m-1)$ and hence we have the path $(i, i-k+r+3, i-1, j-k+2, j)$.

(b) $j-i=1$: Then $j-k$ is adjacent to both i and j. We need 2 further vertices in S. Note that $2j \equiv k+r-1 \pmod{m-1}$ and hence k is odd. Then $r+3 \leq k \leq m-2$.

$k = m-2$: Then $2j \equiv r-2 \pmod{m-1}$. Now $j-k+(m-1) = j+1$ and $i-k+r+1+(m-1) = i+r+2 \leq i-4+(m-1)$. Hence we have the paths $(i, j+2, \infty, j)$ and $(i, i+r+3, i-1, i+r+4, i-2, j+4, j)$.

$k = r+3$: Then $i-k+r+1+(m-1) = i-2+(m-1)$ and $j-k+(m-1) \geq j+3$. Hence we have the path $(i, j-k+1, j+2, j-k+2, j+1, j-k+3, j)$. Also we have the path $(i, i-1, \infty)$. Now $j-k-1+(m-1) \geq j+6$; i.e. $m-1 \geq k+7 = r+10$, unless $n=2$, $r=6$ or $n=2$, $r=8$ or $n=3$, $r=4$. If $m-1 \geq r+10$, then we have the zigzag $(j, j-k-1, j+5, j-k-2, \ldots)$. If $n=2$, $r=6$, then the path and the zigzag become $(i, j+4, j+2, j)$ and $(i, i-1, \infty, j+1, j+6, j)$ respectively. If $n=2$, $r=8$, we have the paths $(i, j+6, j+4, j)$ and $(i, i-1, \infty, j+2, j+8, j)$. If $n=3$, $r=4$, we have the paths $(i, j+6, j+3, j+1, j+4, j)$ and $(i, i-1, \infty, j+8, j)$.

$k = m-4 \geq r+5$: Then $m-1 \geq r+8$. Now $i-k+r+1+(m-1) = i+r+4$ and $j-k+4+(m-1) = j+7$. Then $i+r+5 \leq i-3+(m-1)$. Hence we have the paths $(i, i+r+5, i-1, j+5, j+1, j+6, j)$ and $(i, j+4, j+2, j)$.

$r+5 \leq k \leq m-5$: Then $i-k+r+3+(m-1) \leq i-2+(m-1)$ and $j-k-2+(m-1) \geq j+2$. Hence we have paths $(i, i-k+r+2, i-1, i-k+r+3, i-2, j-k+3, j)$ and $(i, j-k+1, j+2, j-k-2, j+1, j-k-1, j)$.

(c) $j-i = 2 \leq r-2$: Since $2j \equiv k+r \pmod{m-1}$, k is even. Then $r+2 \leq k \leq m-3$.

$k = r+2$: Then $2i \equiv 2r-2 \pmod{m-1}$. Thus i is adjacent to ∞. Then we have the paths $(i, j-k+1, j+1, j-k+3, j)$ and $(i, j-k, j+2, j-k+2, j)$ Now $m-1 \geq r+6$. If $m-1 \geq r+8 = k+6$, $j-k-1+(m-1) \geq j+5$ and hence we have the zigzag $(j, j-k-1, j+4, j-k-2, \ldots)$. If $m-1 = r+6$, then $n = 2$ and $r = 6$. In this case the zigzag becomes $(j, j-k-1, \infty)$.

$k = m-3$: Then $2j \equiv r-2 \pmod{m-1}$. Thus j is adjacent to ∞. Now $i-k+r+1+(m-1) = i+r+3+(m-1)$ and $j-k+4+(m-1) = j+6$. Since $m-1 \geq r+6$, $i+r+4 \leq i-2+(m-1)$ and we have the paths $(i, j+2, j+4, j)$ and $(i, j+3, j+1, j+5, j)$ and the zigzag $(i, i+r+4, i-1, i+r+5, \ldots)$.

$k = r+4 \leq m-5$: Then $m-1 \geq r+8 = k+4$. Now $i-k+r+1+(m-1) = i-3+(m-1)$ and $j-k+(m-1) \geq j+3$. Hence we have the paths $(i, j-k, j+2, j-k+2, j)$ and $(i, j-k+1, j+1, j-k+3, j)$. Since $2i-2 \equiv 2r-2 \pmod{m-1}$, we have the path $(i, i-1, \infty)$. If $m-1 \geq r+10$, then $j-k-1+(m-1) \geq j+5$, and hence we have the zigzag $(j, j-k-1, j+4, j-k-2, \ldots)$. Otherwise $m-1 = r+8$ and hence $n = 2$, $r = 8$ in which case, since $2j+6 \equiv 10 \pmod{18}$, $j+3$ is adjacent to ∞. In this case the zigzag may be replaced by the path $(j, j+3, \infty)$.

$r+6 \leq k \leq m-5$: Then $i-k+r+3+(m-1) \leq i-2+(m-1)$ and $j-k-2+(m-1) \geq j+2$. Hence we have the paths $(i, i-k+r+3, i-2, j-k+3, j)$, $(i, i-k+r+2, i-1, j-k+2, j)$ and $(i, j-k+1, j+1, j-k-2, j)$.

(d) $j-i = 3 \leq r-2$: Since $2j \equiv k+r+1 \pmod{m-1}$, k is odd. Then $j-k+1$ is adjacent to both i and j so that $j-k+1 \in S$. Moreover we have the path $(i, j-k, j+1, j-k+3, j)$. If $k \geq r+3$, then $i-k+r+2+(m-1) \leq i-1+(m-1)$ and we have the path $(i, i-k+r+2, i-1, j-k+2, j)$. If $k = r+2$, then $2i \equiv 2r-3 \pmod{m-1}$ and hence i is adjacent to ∞. Also r is odd and hence $m-1 \geq r+14 = k+12$. Therefore $j-k-2+(m-1) \geq j+10$.

Hence we have the zigzag $(j, j-k-2, j+4, j-k-3, \ldots)$.

(B) $j-i=r-1$: Then we have $r-2$ vertices in S. Note that $j-k+1+(m-1)=i-k+r+(m-1)\le i-2+(m-1)$. Then $j-k$ and $j-k+1$ are both adjacent to i and to j.

(C) $j-i=r$: In this case we already have $r-1$ vertices in S. Note that $j<j-k+1+(m-1)=i+r-k+1+(m-1)\le i-1+(m-1)$. Then $j-k+1$ is adjacent to both i and j.

(II) $k=2$: Now $i+j\equiv r \pmod{m-1}$. Then $j-i\equiv r-2i \pmod{m-1}$ and hence $j-i$ is even. Again the argument is separated into various special cases:

(a) $2\le j-i\le r-4$: Whenever $j+2\le q\le i-1+r$, q is adjacent to both i and j:

$$i+j+2\le i+q<j+q\le i+j-1+r;$$

Since these $(r-2)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we have $r-3$ distinct vertices in S. Note that, since:

$$2j=(i+j)+(j-i)\equiv (j-i)+r \pmod{m-1}$$

and $2\le j-i\le r-4$, $j+1$ is adjacent to ∞. To determine 3 further vertices in S we consider the following subcases:

(i) $4\le j-i\le r-4$: Then j is adjacent to ∞. Since $m-1\ge r+6$, $i+r+3\le i-3+(m-1)$. Then we have paths $(i, i+r, i-1, j)$, $(i, i+r+1, i-2, j)$ and $(i, i+r+2, i-3, j+1, \infty, j)$.

(ii) $j-i=2\le r-3$: Both i and j are adjacent to ∞. Thus $\infty\in S$. Now $i+r+1\le i-2+(m-1)$ and we have the paths $(i, i+r, i-1, j)$ and $(i, i+r+1, i-2, j)$.

(b) $j-i=r-2$: Then there are $r-3$ vertices in S. Since $m-1\ge r+6$, $j+5\le i-3+(m-1)$. Therefore we have the paths $(i, j+3, i-1, j)$, $(i, j+4, i-2, j)$ and $(i, j+2, i-3, j+1, j)$.

(c) $j-i=r$: In this case we already have $r-1$ vertices in S. Note that, since $m-1\ge r+6$, $j+2=i+r+2\le i-4+(m-1)$. Hence we have the path $(i, j+2, i-3, j+1, i-2, j)$.

(III) $k=3$: Now $i+j\equiv r+1 \pmod{m-1}$. Then $j-i\equiv r+1-2i \pmod{m-1}$ and hence $j-i$ is odd. Again the argument is separated into various special cases:

(a) $1 \le j-i \le r-3$: Whenever $j+1 \le q \le i-2+r$, q is adjacent to both i and j:

$$i+j+1 \le i+q < j+q \le i+j-2+r;$$

Since these $(r-2)-(j-i)$ vertices are distinct from the $(j-i)-1$ vertices between i and j, we have $r-3$ distinct vertices in S. Note that, since:

$$2j = (i+j)+(j-i) \equiv (j-i)+r+1 \pmod{m-1}$$

and $1 \le j-i \le r-3$, j is adjacent to ∞. To determine 3 further vertices in S we consider the following subcases:

(i) $3 \le j-i \le r-3$: Since $m-1 \ge r+6$, $i+r+2 \le i-4+(m-1)$. Then we have paths $(i, i+r, i-2, j)$ and $(i, i+r+1, i-3, j)$ and the zigzag $(i, i+r-1, i-1, i+r+2, i-4, i+r+3, \ldots)$.

(ii) $j-i=1 \le r-3$: Then $r \ge 4$ and both $i-1$ and j are adjacent to ∞. Hence we have the paths $(i, i-2, j)$ and $(i, i-1, \infty, j)$. If $r=4$, then $n \ge 3$ so that $m-1 \ge 12$. Hence $i+6 \le i-6+(m-1)$ and we have the path $(i, i+3, i-4, i+6, i-3, j)$. If $r=5$, then $m-1 \ge 15$ and hence $i+7 \le i-8+(m-1)$ so that we have the path $(i, i+4, i-5, i+7, i-3, j)$. If $r \ge 6$, then, since $i+r-1 \le i-7+(m-1)$, we have the path $(i, i+r-1, i-3, j)$.

(b) $j-i=r-1$: Then we have $r-2$ vertices in S. Since $m-1 \ge r+6$, $j+3 = i+r+2 < i-3+(m-1)$. Hence we have paths $(i, j+2, i-1, j+3, i-2, j)$ and $(i, j+1, i-3, j)$.

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