Rainbow Colourings of Chains

W. R. Johnstone and D. J. White
Department of Mathematics, University of Reading,
Whiteknights, P.O. Box 220, Reading, U.K.
email: w.r.johnstone@reading.ac.uk
email: d.j.white@reading.ac.uk

Abstract

We prove that if m be a positive integer and X is a totally ordered set then there exists a function $\phi : X \to \{1, \dots, m\}$ such that, for every interval I in X and every positive integer $r \leq |I|$, there exist elements $x_1 < x_2 < \cdots < x_r$ of I such that $\phi(x_{i+1}) \equiv \phi(x_i) + 1 \pmod{m}$ for $i = 1, \dots, r-1$.

We show that, given any totally ordered set X and any integer $m \geq 2$, there is an m-colouring of X such that between any two points of the same colour there are points of all the other $m-1$ colours. Indeed we prove a slightly stronger version of this result that is obtained by introducing a periodic m-colouring of $\mathbb{Z}$ which itself may be regarded as determined by a total ordering of the set of m colours. Our proof is obtained by considering suitable colourings of subsets of X. These colourings are partially ordered in a natural way and Zorn's lemma is applied.

Throughout this paper $m \geq 2$ is an integer and $\leqslant$ is a total ordering of a non-empty set X. Let $\mathcal{X} = (X_1, X_2, \dots, X_m)$ be an m-tuple of pairwise disjoint subsets of X with $X_1 \cup X_2 \cup \ldots \cup X_m = X$. We say that a finite increasing sequence $(x_1, x_2, \dots, x_r)$ in X is an *$\mathcal{X}$-rainbow* if there is an integer t such that for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, m\}$

$$x_i \in X_j \text{ whenever } i + t \equiv j \pmod{m}.$$

We say that $\mathcal{X}$ is an *m-rainbow colouring* of X if, for every integer $r \geq 1$ and every interval $I \subseteq X$ that contains at least r elements, there is an r-term $\mathcal{X}$-rainbow in I.

Examples. (i) An m-rainbow colouring $(X_1, \dots, X_m)$ of $\mathbb{Z}$ is given by $X_i = m\mathbb{Z} + i$.

JCMCC 35 (2000), pp. 193-196

(ii) An m-rainbow colouring $(X_1, \ldots, X_m)$ of the set $\mathbb{Q}$ of rational numbers is given by $\frac{a}{b} \in X_i$ if a, b are relatively prime integers with $b \geq 1$ and $b \equiv i \pmod{m}$.

Theorem. Every totally ordered set has an m-rainbow colouring.

An m-rainbow colouring of X is constructed by "combining" certain m-rainbow colourings of "run complete" subsets of X.

Let $A \subseteq X$. By an m-rainbow colouring of A we mean an m-rainbow colouring of the ordered subset A of X. If $\mathcal{A} = (A_1, \ldots, A_m)$ and $\mathcal{B} = (B_1, \ldots, B_m)$ are m-tuples of subsets of X, define $\mathcal{A} \vee \mathcal{B} = (A_1 \cup B_1, \ldots, A_m \cup B_m)$.

Lemma 1. Let $\mathcal{C} = (C_1, \ldots, C_m)$ and $\mathcal{D} = (D_1, \ldots, D_m)$ be m-rainbow colourings of subsets C and D respectively of X and suppose that $C < D$; i.e. $c < d$ for every $c \in C$, $d \in D$. Also suppose that C has no greatest element or D has no least element. Then $\mathcal{C} \vee \mathcal{D}$ is an m-rainbow colouring of $C \cup D$.

Proof. Let I be an interval of $C \cup D$ with $|I| \geq r$, where $|I|$ denotes the cardinality of I. If $I \subseteq C$, then there is an r-term $\mathcal{C}$-rainbow in I and this is also a $\mathcal{C} \vee \mathcal{D}$-rainbow. If $I \subseteq D$ a similar argument applies. If I meets both C and D, then $I \cap C$ or $I \cap D$ is infinite and so certainly $|I \cap C| \geq r$ or $|I \cap D| \geq r$. Since $I \cap C$ and $I \cap D$ are intervals of C and D respectively, there is an r-term $\mathcal{C}$- or $\mathcal{D}$-rainbow in $I \cap C$ or $I \cap D$ and this is also a $\mathcal{C} \vee \mathcal{D}$-rainbow in I.

A subset R of X is a *run* if, for every pair $a \leqslant b$ of elements of R, the interval $\{x \in X : a \leqslant x \leqslant b\}$ is a finite subset of R. All runs are intervals and the empty set is a run. We say that a subset Y of X is *run complete* in X if $R \subseteq Y$ whenever R is a run of X with $R \cap Y \neq \varnothing$. An infinite run has the order type of $\mathbb{Z}$ or $\mathbb{N} = \{1, 2, \ldots\}$ or $-\mathbb{N} = \{-1, -2, \ldots\}$. The set of all maximal runs of X is a partition of X and a subset of X is run complete if and only if it is a union of maximal runs of X.

Lemma 2. Suppose that D is a subset of X that is run complete in X and $x \in X \setminus D$. Let I_x be the union of all the intervals of X that contain x and do not meet D. Then I_x is run complete in X. If I_x is finite, then $D_x = \{d \in D : d \leqslant x\}$ has no greatest element and $D^x = \{d \in D : x \leqslant d\}$ has no least element. If I_x is infinite, then there is a subset B of I_x that has the order type of $\mathbb{Z}$ or $\mathbb{N}$ or $-\mathbb{N}$ and B is run complete in X.

Proof. Suppose that R is a run of X with $R \cap I_x \neq \varnothing$. Then $R \cup I_x$ is an interval of X. Also $R \cup I_x$ does not meet D for otherwise $R \cap D \neq \varnothing$ and $R \subset D$ by the run completeness of D: but this gives the contradiction $R \cap I_x = \varnothing$. Hence $R \cup I_x \subseteq I_x$ by definition of I_x, so $R \subseteq I_x$, which proves that I_x is run complete in X.

Suppose that I_x is finite and let $d \in D_x$. Then d is not the greatest element of D_x, for otherwise $\{d\} \cup I_x$ would be a run of X that meets D, giving $\{d\} \cup I_x \subseteq D$ by the run completeness of D and thus contradicting $I_x \cap D = \varnothing$. Hence D_x has no greatest element and similarly D^x has no least element.

Now suppose that I_x is infinite. If any one of the maximal runs whose union is I_x is infinite, then we may take B to be this run. Therefore suppose that I_x is a union of finite maximal runs. Because I_x is infinite, it contains a subset B_0 of order type $\mathbb{N}$ or $-\mathbb{N}$. Let B be the union of all the maximal runs that meet B_0. Since all the runs of this union are finite, it is easy to see that B has the same order type as B_0. Also B is run complete in X and $B \subseteq I_x$ because I_x is run complete in X.

Proof (of Theorem). Let $\mathfrak{D}$ denote the set of all m-rainbow colourings of subsets of X that are run complete in X. Given $\mathcal{C}$ and $\mathcal{D}$ in $\mathfrak{D}$, write $\mathcal{C} \preccurlyeq \mathcal{D}$ if and only if $\mathcal{C} \vee \mathcal{D} = \mathcal{D}$: equivalently $\mathcal{C} = (C_1, \dots, C_m) \preccurlyeq (D_1, \dots, D_m) = \mathcal{D}$ if and only if $C_1 \subseteq D_1, \dots, C_m \subseteq D_m$. Then $\preccurlyeq$ is a partial ordering on $\mathfrak{D}$. Notice that $\mathfrak{D}$ is not empty since it contains the m-term sequence $(\varnothing, \dots, \varnothing)$.

We show that every chain in $(\mathfrak{D}, \preccurlyeq)$ has an upper bound. Suppose that $\mathfrak{C}$ is a chain in $\mathfrak{D}$, i.e. each $(C_1, \dots, C_m) = \mathcal{C} \in \mathfrak{C}$ is an m-rainbow colouring of $C_1 \cup \dots \cup C_m$, and $C_1 \cup \dots \cup C_m$ is run complete in X and any two elements of $\mathfrak{C}$ are comparable in the $\preccurlyeq$ ordering. We claim that

$$\left(\bigcup_{(C_1,\dots,C_m)\in\mathfrak{C}} C_1, \dots, \bigcup_{(C_1,\dots,C_m)\in\mathfrak{C}} C_m \right) = (C_1^*, \dots, C_m^*),$$

say, is an upper bound in $\mathfrak{D}$ of $\mathfrak{C}$. First $C_1^* \cup \dots \cup C_m^*$ is run complete in X, since it is $\bigcup_{(C_1,\dots,C_m)\in\mathfrak{C}}(C_1 \cup \dots \cup C_m)$ which is a union of sets that are run complete in X. Let $r \in \mathbb{N}$ and I be an interval of $C^* = C_1^* \cup \dots \cup C_m^*$ with $|I| \geq r$. Then, since $\mathfrak{C}$ is a chain in $\mathfrak{D}$, $|I \cap (C_1 \cup \dots \cup C_m)| \geq r$ for some $(C_1, \dots, C_m) \in \mathfrak{C}$ and so there is a $(C_1, \dots, C_m)$-rainbow $(x_1, \dots, x_r)$ in $C_1 \cup \dots \cup C_m$ which is clearly also a $(C_1^*, \dots, C_m^*)$-rainbow in C^*. Hence, by Zorn's lemma, $\mathfrak{D}$ has a maximal element with respect to $\preccurlyeq$. The proof is now completed by showing that any such maximal element is an m-colouring of X.

Take any $\mathcal{D} = (D_1, \dots, D_m) \in \mathfrak{D}$ and suppose that $D = D_1 \cup \dots \cup D_m \neq X$. It is sufficient to show that $\mathcal{D}$ is not maximal in $\mathfrak{D}$. Choose any $x \in X \setminus D$ and let I_x, D_x, D^x be as defined in Lemma 2. If I_x is infinite let B be determined by Lemma 2 and if I_x is finite put $B = I_x$. Define

$$\mathcal{D}_x = (D_1 \cap (\leftarrow, x), \dots, D_m \cap (\leftarrow, x))$$
$$\mathcal{D}^x = (D_1 \cap (x, \rightarrow), \dots, D_m \cap (x, \rightarrow)),$$

where $(\leftarrow, x) = \{x' \in X : x' < x\}$, $(x, \rightarrow) = \{x' \in X : x < x'\}$. Then $\mathcal{D}_x$ and $\mathcal{D}^x$ are in $\mathfrak{D}$. We consider separately the four cases when B is finite or has the order type of $\mathbb{Z}$ or $\mathbb{N}$ or $-\mathbb{N}$.

Suppose first that B is finite and let $\mathcal{B}$ be any rainbow colouring of B. By Lemma 2, D_x has no greatest element and D^x has no least element. Hence two applications of Lemma 1 show successively that $\mathcal{D}_x \vee \mathcal{B}$ is in $\mathfrak{D}$ and then that $\mathcal{D}_x \vee \mathcal{B} \vee \mathcal{D}^x = \mathcal{B} \vee \mathcal{D}$ is in $\mathfrak{D}$. Hence $\mathfrak{D}$ is not maximal in $\mathfrak{D}$.

Suppose next that B has the order type of $\mathbb{Z}$ and again let $\mathcal{B}$ be any m-rainbow colouring of B. Since B has no least nor greatest element two applications of Lemma 1 show, as in the previous paragraph, that $\mathcal{B} \vee \mathcal{D}$ is in $\mathfrak{D}$.

Now suppose that B has the order type of $\mathbb{N}$. If D_x has no greatest element, then we soon see as in the previous cases that $\mathcal{B} \vee \mathcal{D}$ is in $\mathfrak{D}$ when $\mathcal{B}$ is any rainbow colouring of B. Suppose therefore that D_x has a greatest element $d \in D_i$, say, and let b be the least element of B. Let $\mathcal{B} = (B_1, \dots, B_m)$ be the m-rainbow colouring of B such that $b \in B_j$ where $j \equiv i + 1 \pmod{m}$. Then it is easy to check that $\mathcal{D}_x \vee \mathcal{B}$ is an m-rainbow colouring of $D_x \cup B$ and since $D_x \cup B$ is run complete we have $\mathcal{D}_x \vee \mathcal{B} \in \mathfrak{D}$. Since B has no greatest element another application of Lemma 1 shows that $\mathcal{D}_x \vee \mathcal{B} \vee \mathcal{D}^x = \mathcal{B} \vee \mathcal{D}$ is in $\mathfrak{D}$ so again $\mathfrak{D}$ is not maximal.

Finally when B has the order type of $-\mathbb{N}$, we may prove that $\mathfrak{D}$ is not maximal by the obvious variation of the order type $\mathbb{N}$ argument. This completes the proof of the Theorem.

Finally we deduce the "between" result stated in the first sentence of this paper. Take any rainbow colouring $(X_1, \dots, X_m)$ of X and suppose that $x, y \in X_j$ and $x < y$. Let c be the number of the sets $\{X_1, \dots, X_m\} \setminus \{X_j\}$ that meet the interval $[x, y] \setminus \{x, y\}$. Then $[x, y]$ contains at least $c + 2$ elements, so it contains a $(c + 2)$-term rainbow. Hence it meets at least $\min(m, c + 2)$ of the sets $X_1, \dots, X_m$ and therefore $\min(m, c + 2) \leq c + 1$. Hence $m \leq c + 1$ and clearly also $c \leq m - 1$, so that $c = m - 1$ as required.