The Hall–Condition Index of a Graph and the Overfull Conjecture

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Abstract

Let $s'(G)$ denote the Hall–condition index of a graph G. Hilton and Johnson recently introduced this parameter and proved that $\Delta(G) \leq s'(G) \leq \Delta(G)+1$. A graph G is s'–Class 1 if $s'(G) = \Delta(G)$ and is s'–Class 2 otherwise. A graph G is s'–critical if G is connected, s'–Class 2, and, for every edge e, $s'(G-e) < s'(G)$. We use the concept of the fractional chromatic index of a graph to classify s'–Class 2 in terms of overfull subgraphs, and similarly to classify s'–critical graphs. We apply these results to show that the following variation of the Overfull Conjecture is true;
A graph G is s'–Class 2 if and only if G contains an overfull subgraph H with $\Delta(G) = \Delta(H)$.

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1 Introduction

We consider finite simple graphs, and for terminology and notation not defined here we refer to [1].

An edge list assignment to G is a function from $E(G)$ into some collection of finite sets. If L is an edge list assignment to G, *a proper L-colouring* of G is a function ϕ from $E(G)$ into $\mathcal{L} = \cup_{e \in E(G)} L(e)$ such that $\phi(e) \in L(e)$ and $\phi(e_1) \neq \phi(e_2)$ for every pair e_1 and e_2 of adjacent edges. For an edge list assignment L to G and a symbol $\sigma \in \mathcal{L}$, let $t_G(\sigma, L)$ denote the maximum number of independent edges of G such that each edge contains σ in its list.

Hilton and Johnson introduced a necessary condition, called Hall's condition, for the existence of a proper L-colouring [8].

Definition. A graph G with a list assignment L satisfies *Hall's condition* if, for each subgraph H of G, $|E(H)| \leq \sum\limits_{\sigma \in \mathcal{L}} t_H(\sigma, L)$.

They also introduced the Hall-condition index (or edge-Hall condition number) of a graph [9].

Definition. For a graph G, *the Hall-condition index*, denoted by $s'(G)$, is the smallest integer k such that if $|L(e)| \geq k$ for every $e \in E(G)$, then the edge list assignment L of G satisfies Hall's condition.

Let $i(G)$ denote the edge independence number (or matching number) of G. They proved the following theorem.

Theorem A. [9] *For every graph G we have*

a) $\Delta(G) \leq s'(G) \leq \chi'(G)$,

b) $s'(G) = \max\{\lceil \frac{|E(H)|}{i(H)} \rceil : H \leq G,\ |E(H)|, \neq 0\}$.

By the well–known theorem of Vizing [16] we have $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. Thus $\Delta(G) \leq s'(G) \leq \Delta(G)+1$. We shall call G χ'*–Class 1* if $\chi'(G) = \Delta(G)$ and χ'*–Class 2* if $\chi'(G) = \Delta(G)+1$. Similarly, we shall call G s'*–Class* 1 if $s'(G) = \Delta(G)$ and s'*–Class* 2 if $s'(G) = \Delta(G)+1$.

Recall that a graph G is χ'*–critical* if it is connected, χ'–Class 2, and $\chi'(G-e) < \chi'(G)$ $(\forall e \in E(G))$. We shall similarly call G s'*–critical* if it is connected, s'–Class 2 and $s'(G-e) < s'(G)$ $(\forall e \in E(G))$. K_3 and $K_5 - e$ are example of graphs that are both s'–critical and χ'–critical.

In the second section of this paper we make some preliminary observations about s'–critical graphs. Some of these results are very similar to known results about χ'–critical graphs. This section also contains our first theorem characterizing s'–critical graphs.

A graph G is called *overfull* if $|E(G)| > \Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor$, and *just overfull* if $|E(G)| = \Delta(G)\lfloor\frac{|V(G)|}{2}\rfloor + 1$. An overfull graph is s'–Class 2, and therefore χ'–Class 2. In Section 3 we characterize s'–Class 2 graphs in terms of overfull subgraphs, and we also give two further characterizations of s'–critical graphs. The discussion in this section depends on the formula for $\chi'_f(G)$, the fractional chromatic index of G, derived by Seymour [13] and Stahl [14] independently from Edmonds' Matching Polytope Theorem [4].

In the final section we show that the Overfull Conjecture [2] can be reformulated in terms of $s'(G)$.

2 Some preliminary remarks about s'–critical graphs

Let P denote the Petersen graph, let P^+ denote the Petersen graph with one vertex removed, and let P^* denote the Petersen graph with one edge subdivided. It is shown in [9] that P^+ is s'–Class 1, and it is well–known that P^+ is χ'–critical. Thus there are χ'–critical graphs that are not s'–critical. We show later in this section that P^* is s'–critical, and yet is not χ'–critical. The main connection between s'–critical graphs and χ'–critical graphs is given in Proposition 1.

Proposition 1. *A χ'-critical graph is either s'-critical or s'-Class 1.*

Proof. Suppose that G is χ'-critical and s'-Class 2. By definition, G is connected. Also, since G is χ'-critical, G is χ'-Class 2 and, for each edge e, $\chi'(G-e) < \chi'(G) = \Delta(G)+1$, so $s'(G-e) \leq \chi'(G-e) \leq \Delta(G)$. Since G is s'-Class 2, $s'(G) = \Delta(G)+1$, and so, for each edge e, $s'(G-e) < s'(G)$. Therefore G is s'-critical. □

Our next few results demonstrate similarities between s'-critical graphs and χ'-critical graphs.

Proposition 2. *Any s'-Class 2 graph contains an s'-critical graph of the same maximum degree.*

Proof. Let G be an s'-Class 2 graph. Remove edges and isolated vertices from G until a connected graph H is obtained such that $s'(G) = s'(H)$, but $s'(H-e) < s'(G)$ for all $e \in E(H)$. Then

$$\Delta(H)+1 \geq s'(H) = s'(G) = \Delta(G)+1 \geq \Delta(H)+1,$$

so $\Delta(H) = \Delta(G)$. Then H is s'-critical. □

Proposition 2 was to be expected. However it can be generalized to the following theorem, which could not be anticipated so readily, in view of the fact that, whilst the analogue of it for the parameter $\chi'(G)$ is true (see [5]or [17]), the analogue for the parameter $\chi_T(G)$, the total chromatic number of G, is false (see [7]).

Theorem 3.*Let G be an s'-Class 2 graph, and let $2 \leq d \leq \Delta(G)$. Then G has an s'-critical subgraph H with $\Delta(H) = d$.*

The proof of Theorem 3 is postponed until Section 4.

Proposition 4. *Let G be an s'-critical graph. Then G has at least three vertices of maximum degree. Moreover, if G has exactly three vertices of maximum degree, then G is of the form $K_{2n+1} - M$, where M is a matching with $n-1$ edges.*

Proof. The assertion of Proposition 4 is known to be true for χ'-critical graphs (see [3]). Since $s'(G) \leq \chi'(G)$ for all graphs G, Proposition 4 follows if we show that $K_{2n+1} - M$ is s'-Class 2. But this is true since $K_{2n+1} - M$ is just overfull. □

Proposition 5. *Let G be a graph of s'-Class 1, and let u and v be two non-adjacent vertices in G such that $d_G(u) + d_G(v) < \Delta(G)$. Then $G+uv$ is also of s'-Class 1.*

Proof. Let H' be a subgraph of $G + uv$. We must show that $\frac{|E(H')|}{i(H')} \leq \Delta(G)$. If the edge uv is not in H', then H' is a subgraph of G and we have $\frac{|E(H')|}{i(H')} \leq \Delta(G)$, by Theorem A(b). Now suppose uv is an edge in H' and $H = H' - uv$.

Case 1. $i(H') = i(H) + 1$. Then $\frac{|E(H')|}{i(H')} = \frac{|E(H)|+1}{i(H)+1} \leq \frac{|E(H)|}{i(H)} \leq \Delta(G)$.

Case 2. $i(H') = i(H)$. Suppose to the contrary that $\frac{|E(H')|}{i(H')} > \Delta(G)$. We have then that

$$\Delta(G)i(H) + 1 \geq |E(H)| + 1 = |E(H')| \geq \Delta(G)i(H') + 1 = \Delta(G)i(H) + 1$$

so that $E(H) = i(H)\Delta(G)$. Let $K = H - \{u, v\}$. Then

$$\begin{aligned}|E(K)| = |E(H)| - (d_H(u) + d_H(v)) &\geq \Delta(G)i(H) - (\Delta(G) - 1)\\ &= \Delta(G)(i(H) - 1) + 1.\end{aligned}$$

But since $s'(G) = \Delta(G)$ it follows that $|E(K)| \leq \Delta(G)i(K)$, so $\Delta(G)(i(H)-1) + 1 \leq \Delta(G)i(K)$, so $i(K) > i(H) - 1$. But K is an induced subgraph of H, so $i(K) \leq i(H)$. Therefore $i(H) \geq i(K) > i(H) - 1$, so $i(K) = i(H) = i(H')$. But uv is an edge of H' that is independent of any edge of K, and so $i(K) < i(H')$. This contradiction shows that $\Delta(G) \geq \frac{|E(H')|}{i(H')}$ in this case also. □

Corollary 6. *For every s'-critical graph G and every edge $uv \in E(G)$, $d(u) + d(v) \geq \Delta(G) + 2$.*

Proof. Let G be s'-critical. Then by Proposition 4 G has at least three vertices of maximum degree. Suppose G has an edge uv such that $d_G(u) + d_G(v) \leq \Delta(G) + 1$. Let $G' = G - uv$. Then $\Delta(G) = \Delta(G')$ and

$d_{G'}(u) + d_{G'}(v) \le \Delta(G) - 1$. By Proposition 5, G is Class 1, a contradiction. Therefore $d_G(u) + d_G(v) \ge \Delta(G) + 2$ for all edges uv. □

In the following theorem we show that no s'–critical graph has a cut vertex.

Theorem 7. *Every s'-critical graph is 2–connected.*

Proof. Suppose x is a cut vertex of an s'–critical graph G and $G - x = H_1 \cup K_1$ where H_1 and K_1 are disjoint . Let $H = G[V(H_1) \cup \{x\}]$ and $K = G[V(K_1) \cup \{x\}]$. Since G is s'-critical we obtain $s'(H) \le \Delta(G)$ and $s'(K) \le \Delta(G)$. Let L be a subgraph of G. If $x \notin V(L)$ then L is a union of two subgraphs H' and K', where H' is a subgraph of H_1 and K' is a subgraph of K_1. We have $|E(H')| \le \Delta(G)i(H')$, $|E(K')| \le \Delta(G)i(K')$, and $i(H')+i(K') = i(L)$. But $|E(H')|+|E(K')| = |E(L)|$, so that $\frac{|E(L)|}{i(L)} \le \Delta(G)$.

Now suppose that $x \in V(L)$. In this case L is the union of two subgraphs H'' and K'', where H'' is a subgraph of H and K'' is a subgraph of K and $x \in V(H'') \cap V(K'')$. Let U be a maximum independent set of edges of L. If U has no edge incident with x, define $U_1 = U \cap E(H'')$ and $U_2 = U \cap E(K'')$. Then $i(L) = |U_1| + |U_2| = i(H'') + i(K'')$, and so $|E(L)| \le \Delta(G)i(L)$. If U has an edge incident with x, define $U_1 = U \cap (H''-x)$ and $U_2 = U \cap (K''-x)$. (Either $i(H''-x)+i(K''-x) = |U_1|+|U_2|$ or there is a maximum matching in L which does not saturated x, a possibility dealt with just above.) Then $i(L) = |U| = |U_1|+|U_2|+1$ and $|E(L)| \le |E(H''-x)|+|E(K''-x)|+\Delta(G)$, and therefore $|E(L)| \le \Delta(G)i(H''-x)+\Delta(G)i(K''-x)+\Delta(G) = \Delta(G)i(L)$. It follows that $\max\{\frac{\Delta(H)}{i(H)} : H \le G\} \le \Delta(G)$, a contradiction. Therefore every s'–critical graph is 2–connected. □

We remark that Proposition 5, Corollary 6 and Theorem 7 can all be derived without much difficulty from Theorem 18 , which is independent of them. However the proofs we have given are elementary, whereas our proof of Theorem 18 is relatively sophisticated; so it seems reasonable to give these elementary direct proofs.

We now give our first result characterizing s'–critical graphs.

Theorem 8. *A graph G is s'-critical if and only if*

a) $|E(G)| = \Delta(G)i(G) + 1$;

b) for every non–empty, proper subset F of $E(G)$, $i(G-F) \geq i(G) - \frac{|F|-1}{\Delta(G)}$.

Proof.
Necessity. Let G be an s'–critical graph. Then, by Theorem A there exists a subgraph H of G such that $\frac{|E(H)|}{i(H)} > \Delta(G)$. But since G is s'-critical, H is not a proper subgraph of G. Therefore $G = H$. Thus $|E(G)| > \Delta(G)i(G)$. Now there exists an edge e in G such that $i(G-e) = i(G)$, because if there is no such edge e, then $i(G) = |E(G)|$ and $\Delta(G) = 1$, so $\frac{|E(G)|}{i(G)} \leq \Delta(G)$, a contradiction. By the s'-criticality of G we have $\frac{|E(G-e)|}{i(G-e)} \leq \Delta(G)$, so $|E(G)| - 1 \leq \Delta(G)i(G)$. Therefore we have $\Delta(G)i(G) < |E(G)| \leq \Delta(G)i(G) + 1$. Hence part (a) holds.

Now let F be a non–empty proper subset of the edges of G. Then, since G is s'-critical, we have $\frac{|E(G-F)|}{i(G-F)} \leq \Delta(G)$. Note that F must be non empty and be a proper subset of $E(G)$ for this to hold. We then have that $|E(G)| - |F| \leq \Delta(G)i(G-F)$. Therefore, using ($a$),

$$\Delta(G)i(G) - (|F|-1) = |E(G)| - 1 - |F| + 1 = |E(G-F)| \leq \Delta(G)i(G-F),$$

so that $i(G) - \frac{|F|-1}{\Delta(G)} \leq i(G-F)$, as required for (b).

Sufficiency. Suppose that G satisfies (a) and (b). By (a) and Theorem A, G is s'–Class 2. Suppose that G is not s'–critical. Then G has a proper subgraph H such that $\Delta(G) = \Delta(H)$ and H is s'–critical. Then, by the necessity, $|E(H)| = \Delta(G)i(H) + 1$. Clearly $|E(H)| \neq 0$. Therefore

$$|E(G) - E(H)| = \Delta(G)(i(G) - i(H)) \qquad (*)$$

and $0 < |E(G)| - |E(H)| < |E(G)|$. But by (b) we have

$$i(H) \geq i(G) - \frac{|E(G) - E(H)| - 1}{\Delta(G)}$$

so that $|E(G) - E(H)| - 1 \geq \Delta(G)(i(G) - i(H))$, contradicting $(*)$. Therefore G is s'–critical. □

Finally in this section we show that P^*, the Petersen graph with one edge subdivided, is s'–critical but not χ'–critical (see Figure 1). Since P^+

is χ'-critical and is a proper subgraph of P^*, P^* is not χ'-critical. Let $G = P^*$. Then $\frac{|E(G)|}{i(G)} = \frac{16}{5} > \Delta(G) = 3$. Hence G is s'-Class 2. We shall show that G contains no s'-critical proper subgraph H with maximum degree 3. If G does contain such a subgraph H then, by Theorem 8(a), $|E(H)| = 3i(H) + 1$. Note that G has no K_3 or C_4 as a subgraph. Also note that, by Theorem 7, H has no vertex of degree 1.

Let x be a vertex of H of degree 3. Let y_1, y_2, y_3 be the neighbours of x. Since G has no K_3 or C_4 as a subgraph, y_1, y_2, y_3 have distinct neighbours, say z_1, z_2, z_3 respectively. Then H has the three independent edges y_1z_1, y_2z_2 and y_3z_3, so $i(H) = 3$ or 4.

Suppose $i(H) = 3$. There cannot be a further vertex in H joined to one of z_1, z_2, z_3, for then $i(H) = 4$. But since there are no vertices of degree 1 in H, and since H has no K_3 or C_4, z_1, z_2 and z_3 are connected by a path; thus without loss of generality, we may suppose that H contains the edges z_1z_2 and z_2z_3. But since there is no K_3 or C_4 in G, no further edge can join any pair of $y_1, y_2, y_3, z_1, z_2, z_3$, and so $|E(H)| = 8$, a contradiction.

Finally suppose that $i(H) = 4$. Then H is obtained by deleting 3 edges from G. We may suppose that $V(G) = \{1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5'\}$ and that A is the independent set $A = \{11', 22', 33', 44', 55'\}$ and the remaining edges of G are as shown in Figure 1. An edge of A must be deleted in forming H, otherwise H has five independent edges. Therefore of most two edges are deleted from $G - A = C_5 \cup C_6$. It is easy to check that the only way of deleting up to two edges from $C_5 \cup C_6$ without leaving five independent edges is to delete two edges from C_6, these being either adjacent or opposite. Suppose these edges are adjacent. Since H has no vertex of degree 1, H must be obtained by deleting either a vertex of degree 3 of G, or edges 46, 56 and one of the edges 11′, 22′, 33′ of G. In each case we can check that H has 5 independent edges. If the deleted edges are not adjacent then we can't delete either 46 or 56 (otherwise there would be a vertex of degree 1), and since we must delete an edge of each of A, B, C, we must delete the edges 15, 34, and 22′. But the resulting graph again has 5 independent edges. So in every case $i(H) = 5$, a contradiction. Therefore G is s'-critical.

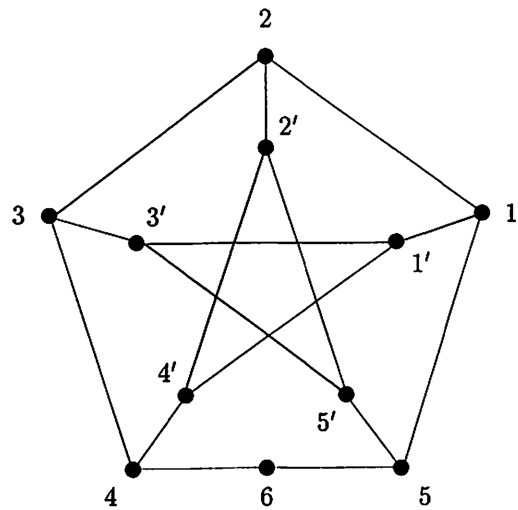

Figure 1.

3 s'–critical graphs and just overfull graphs

In this section we characterize s'–critical graphs in terms of just overfull graphs. We also characterize s'–Class 2 graphs. The discussion depends in a surprising way on a formula for the fractional chromatic index of a graph. We first need some definitions.

Definition. (i) A *k–set edge colouring* of G is an assignment of k–sets to the edges of G so that any two edges that have a common vertex are assigned disjoint k–sets.

(ii) The *k–edge chromatic number* of G, $\chi'^{(k)}(G)$, is the least integer j such that edges of G can be properly k–set coloured, with k–subsets of $\{1, 2, \cdots j\}$.

(iii) The *fractional edge chromatic number* of G, $\chi'_f(G)$, is defined by:

$$\chi'_f(G) = \lim_{k\to\infty} \frac{1}{k}\chi'^{(k)}(G).$$

It is well known that this limit exists and it is equal to the minimum of the numbers $\frac{1}{k}\chi'^{(k)}(G), k = 1, 2, \cdots$. Obviously $\chi'_f(G) \le \chi'(G)$. In the following theorem a formula for $\chi'_f(G)$ was derived from Edmonds' description of the matching polytope [4]; this derivation was made independently by Seymour [13] and Stahl [14]. An account of this may be found in [11].

Theorem B. *For every graph G,*

$$\chi'_f(G) = \max\{\Delta(G), \max\{\frac{2|E(H)|}{(|V(H)|-1)} : H \le G,\ |V(H)| \ge 3 \textit{ is odd}\}\}.$$

Using Theorem B we first prove the following lemma.

Lemma 9. *For every graph G,*

$$max\{\frac{|E(H)|}{i(H)} : H \le G \textit{ and } |E(H)| \ne 0\} =$$

$$\max\{\Delta(G), \max\{\frac{2|E(H)|}{(|V(H)|-1)} : H \le G \textit{ and } |V(H)| \ge 3 \textit{ is odd}\}\}.$$

Proof. Let v be a vertex of maximum degree, and let H_v be the subgraph of G induced by the edges incident with v. Then $\frac{|E(H_v)|}{i(H_v)} = \Delta(G)$. If H is any subgraph of G with $|V(H)| \ge 3$, odd, then $\frac{2|E(H)|}{(|V(H)|-1)} \le \frac{|E(H)|}{i(H)}$. Therefore

$$max\{\frac{|E(H)|}{i(H)} : H \le G \textit{ and } |E(H)| \ne 0\} \ge$$

$$\max\{\Delta(G), \max\{\frac{2|E(H)|}{(|V(H)|-1)} : H \le G \textit{ and } |V(H)| \ge 3 \textit{ is odd}\}\},$$

and so, using Theorem B,

$$max\{\frac{|E(H)|}{i(H)} : H \le G \textit{ and } |E(H)| \ne 0\} \ge \chi'_f(G).$$

Now let G_k be the multigraph obtained from G by replacing each edge by k parallel edges. Then $\chi'^{(k)}(G) = \chi'(G_k)$. If H is a subgraph of G with $|E(H)| \neq 0$ and H_k the corresponding subgraph (with each pair of vertices either not joined or joined by k parallel edges) of H, then

$$\chi'(G_k) \geq \chi'(H_k) \geq \frac{|E(H_k)|}{i(H_k)} = \frac{k|E(H)|}{i(H)}.$$

Thus $\frac{1}{k}\chi'(G_k) \geq \frac{|E(H)|}{i(H)}$. Since this holds for all $k \geq 1$, it follows that $\chi'_f(G) \geq \frac{|E(H)|}{i(H)}$. Since H was an arbitrary subgraph of G with $|E(H)| \neq 0$, it follows that

$$\chi'_f(G) \geq max\{\frac{|E(H)|}{i(H)} : H \leq G \text{ and } |E(H)| \neq 0\}.$$

Therefore

$$\chi'_f(G) = max\{\frac{|E(H)|}{i(H)} : H \leq G \text{ and } |E(H)| \neq 0\}.$$

The result now follows from Theorem B. □

An immediate corollary is:

Corollary 10. *For any graph G,*

$$s'(G) = \lceil \chi'_f(G) \rceil.$$

Putting this another way:

Corollary 11. *For any graph G,*

$$s'(G) = \max\{\Delta(G), \max\{\lceil \frac{2|E(H)|}{(|V(H)|-1)} \rceil : H \leq G \text{ and } |V(H)| \geq 3 \text{ is odd}\}\}.$$

We can now use Corollary 11 to characterize s'–Class 2 graphs.

Theorem 12. *A graph G is s'-Class 2 if and only if G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

Proof. From Corollary 11 we see that a graph G is s'–Class 2 if and only if, for some subgraph H with $|V(H)| \geq 3$, $\frac{2|E(H)|}{(|V(H)|-1)} > \Delta(G)$, i.e.

$|E(H)| > \Delta(G)\lfloor\frac{|V(H)|}{2}\rfloor$ (note that if H satisfies this last inequality, then $|V(H)|$ is necessarily odd). But if G is s'–Class 2 then $\Delta(G)+1 \geq \Delta(H)+1 \geq s'(H) \geq \Delta(G)+1$, so that $\Delta(H) = \Delta(G)$. Theorem 11 now follows easily. □

Before stating our next result, let us remark that if G is overfull it does not necessarily follow that $i(G) = \frac{(|V(G)|-1)}{2}$. An example of an overfull graph G with $i(G) < \frac{(|V(G)|-1)}{2}$ is given in Figure 2.

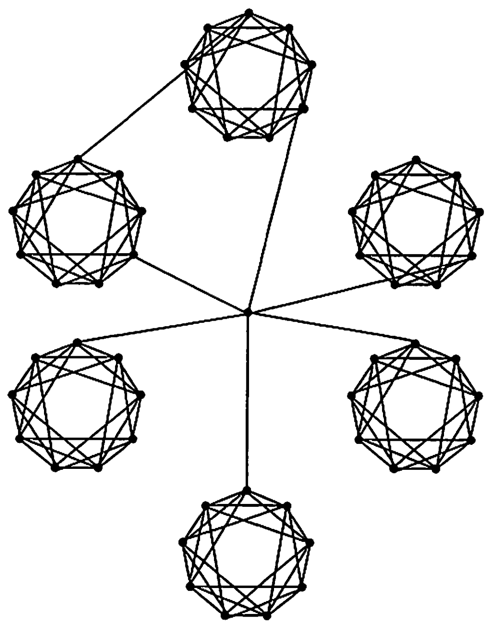

Figure 2.

Theorem 13. *Every s'–critical graph G*

a) is just overfull,

b) satisfies $i(G) = \frac{(|V(G)|-1)}{2}$.

Proof. Let G be an s'–critical graph. Then, since G is s'–Class 2, for some subgraph H of G we have $\frac{|E(H)|}{i(H)} > \Delta(G)$, and so, by Lemma 9, for some subgraph H^* of G with $|V(H^*)|$ odd, we have $\frac{2|E(H^*)|}{(|V(H^*)|-1)} > \Delta(G)$. We shall show that $H^* = G$. Suppose that this is not the case, and that H^* is a proper subgraph of G. Since G is s'–critical,

$$max\{\frac{|E(H')|}{i(H')} : H' \leq H^* \text{ and } |E(H')| \neq 0\} \leq \Delta(G),$$

and so, by Lemma 9, $\frac{2|E(H^*)|}{(|V(H^*)|-1)} \leq \Delta(G)$, a contradiction. Thus $H^* = G$, so that $|V(G)|$ is odd and $\frac{2|E(G)|}{(|V(G)|-1)} > \Delta(G)$, i.e. G is overfull. It follows that

$$|E(G)| \geq \Delta(G)\frac{(|V(G)|-1)}{2} + 1 \geq \Delta(G)i(G) + 1.$$

But, by Theorem 8, $|E(G)| = \Delta(G)i(G) + 1$. Therefore $i(G) = \frac{(|V(G)|-1)}{2}$. It now follows that G is just overfull. □

In 1973 Jakobsen [10] conjectured that every χ'–criticall graph has odd order. In 1981 this was shown by Goldberg [6] to be untrue, but it is interesting to note that the conjecture is true if s' substituted for χ'.

Corollary 14. *Every s'–critical graph has odd order.*

The example of Figure 3 shows that the converse of Theorem 13 is not true. Here $|E(G)| = 21$, $|V(G)| = 11$, $i(G) = 5$ and $\Delta(G) = 4$, so G is just overfull and satisfies $i(G) = \frac{(|V(G)|-1)}{2}$. But G contains $H = K_5 - e$ as a subgraph, and $\frac{|E(H)|}{i(H)} = \frac{9}{2} > 4 = \Delta(G)$, so H is s'–Class 2, and so G is not s'–critical.

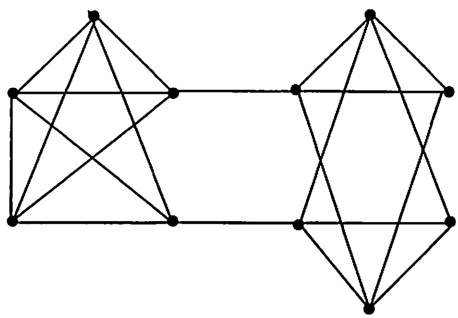

Figure 3.

The graph G of Figure 2 is just overfull but $i(G) < \frac{(|V(G)|-1)}{2}$, so, by Theorem 13, it is not s'–critical. Without quoting Theorem 13, though, it is easy to see that G is not s'–critical, since it has the six overfull (and therefore s'–Class 2) subgraphs of order 9 with the same maximum degree. These two examples (of Figure 2 and 3) suggest our characterization of s'–critical graphs in terms of just overfull subgraphs.

Theorem 15. *A graph G is s'–critical if and only if G is just overfull and contains no proper overfull subgraph of the same maximum degree.*

Proof. If G is s'–critical then, by Theorem 13, G is just overfull ; and clearly G has no proper overfull subgraph of the same maximum degree.

Now suppose that G is just overfull and contains no proper overfull subgraph of the same maximum degree. It is easy to see that G is connected. Also G has odd order and $\frac{|E(G)|}{i(G)} \geq \frac{2|E(G)|}{(|V(G)|-1)} > \Delta(G)$. Thus G is s'–Class 2. If G were not s'–critical, then, by Proposition 2, G has an s'–critical subgraph G' with $\Delta(G) = \Delta(G')$. By Theorem 13, this would contradict the assumption that G has no proper overfull subgraph with the same maximum degree, Therefore G is s'–critical. □

If $\Delta(G) \geq \frac{1}{2}|V(G)|$ then Theorem 15 can be simplified using a theorem of Niessen [12].

Theorem C. *Let G be an overfull graph with $\Delta(G) \geq \frac{1}{2}|V(G)|$. Then G has no overfull subgraph H with $\Delta(H) = \Delta(G)$ and $|V(H)| < |V(G)|$.*

The simplified characterization is:

Theorem 16. *Let G be a graph with $\Delta(G) \geq \frac{1}{2}|V(G)|$. Then G is s'-critical if and only if G is just overfull.*

Proof. This follows from Theorem C and Theorem 15. □

In Figure 2 the overfull proper subgraphs are each connected to the rest of the graph by one or two edges. In general, if an overfull graph G has an overfull subgraph H with $\Delta(H) = \Delta(G)$, then the edge connectivity is low. This feature can be used to give a slightly different characterization of s'–critical graphs.

First some more notation. Let the *deficiency*, $def(G)$, of a graph G be defined by:

$$def(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v)).$$

Proposition 17. *A graph G is overfull if and only if $V(G)$ is odd and $def(G) \leq \Delta(G) - 2$.*

Proof. If G is overfull then $V(G)$ is odd and $|E(G)| > \Delta(G)(\frac{|V(G)|-1}{2})$, so

$$\sum_{v \in V(G)} d_G(v) \geq \Delta(G)(|V(G)| - 1) + 2 = (\sum_{v \in V(G)} \Delta(G)) - (\Delta(G) - 2),$$

so $def(G) \leq \Delta(G) - 2$. The converse follows similarly. □

If $V(G) = W_1 \cup W_2$ where $W_1 \cap W_2 = \emptyset$, let $def_{W_1}(G)$ be defined by:

$$def_{W_1}(G) = \sum_{v \in W_1} (\Delta(G) - d_G(v)).$$

We can now state our third characterization theorem for s'–critical graphs.

Theorem 18. *A graph G is s'-critical if and only if G is just overfull (so $def(G) = \Delta(G) - 2$) and G contains no cut set F of edges such that $G - F$ is the disjoint union of two graphs G_1 and G_2, where $|V(G_1)|$ is odd and $def_{V(G_1)}(G) + |F| \leq \Delta(G) - 2$.*

Proof.
Necessity. Suppose G is s'–critical. Then by Theorem 15 G is just overfull and G has no overfull subgraph of the same maximum degree. Suppose G contains an edge cut F such that $G - F$ is the disjoint union of G_1 and G_2, where $|V(G_1)|$ is odd and $def_{V(G_1)}(G) + |F| \leq \Delta(G) - 2$. Then $def(G_1) \leq \Delta(G) - 2$. Since G_1 is not an overfull subgraph of the same maximum degree, it follows that $\Delta(G) = \Delta(G_1) + k$ for some $k \geq 1$. The inequality $def(G_1) \leq \Delta(G) - 2$ can be rewritten

$$\sum_{v \in V(G_1)} (\Delta(G) - d_{G_1}(v)) \leq \Delta(G) - 2.$$

Therefore

$$\sum_{v \in V(G_1)} (\Delta(G_1) + k - d_{G_1}(v)) \leq \Delta(G_1) + k - 2,$$

so that

$$k(|V(G_1)| - 1) + def(G_1) \leq \Delta(G_1) - 2 \leq |V(G_1)|) - 3,$$

which is impossible. Therefore G contain no such edge–cut F.

Sufficiency. Suppose that G is just overfull and contains no cut set F of edges such that $G - F$ is the disjoint union of two graphs G_1 and G_2, where $|V(G_1)|$ is odd and $def_{V(G_1)}(G) + |F| \leq \Delta(G) - 2$. We need to show that G has no proper overfull subgraph (for then it follows from Theorem 11 that G is s'–critical). Suppose G does contain a proper overfull subgraph G_1 with $\Delta(G_1) = \Delta(G)$. Then $|V(G_1)| < |V(G)|$. Let F be the set of edges of G joining $V(G_1)$ to $V(G) - V(G_1)$. Since G_1 is overfull and $\Delta(G_1) = \Delta(G)$, $def(G_1) \leq \Delta(G) - 2$. Since $def_{V(G_1)}(G) = def(G_1) - |F|$ it follows that $def_{V(G_1)}(G) + |F| \leq \Delta(G) - 2$. But this is a contradiction. Therefore G is s'–critical, as required. □

4 s'–critical subgraphs

In this section we prove Theorem 3 showing that if G is an s'–Class 2 graph, and if $2 \leq d \leq \Delta(G)$, then G has an s'–Class 2 subgraph H with $\Delta(H) = d$. Our proof depends on Theorem 15, and so it seems to be inherently more complicated than the other results in Section 2.

Lemma 19. *Let G be a just overfull graph that does not contain a proper overfull subgraph at the same maximum degree, and let $G \neq K_3$. Then G has a near 1–factor that contains all vertices of maximum degree.*

Proof. Since $G \neq K_3$ and G is just overfull, G is not a complete graph. Let $a \in V(G)$ be a vertex with $d_G(a) < \Delta(G)$. Form a graph G^+ of even order $|V(G^+)| = |V(G)| + 1$ and maximum degree $\Delta(G)$ by attaching a pendant edge at a. Clearly G has a near 1–factor that includes all vertices of maximum degree if G^+ has a 1–factor. Note that $|E(G^+)| = \frac{1}{2}\Delta(G)(|V(G)| - 1) + 2$.

Suppose that G^+ has no 1–factor. Then by Tutte's theorem [15], there is a set $S \subset V(G^+)$ such that $G^+ - S$ contains at least $s+1$ odd components $C_1, C_2, \cdots, C_{s+1}$, where $|S| = s$. Since $|V(G^+)|$ is even, we can say that $G^+ - S$ is the disjoint union of $C_1, C_2, \cdots, C_{s+1}$ and H, where $|V(H)|$ is odd. By assumption, none of $C_1, C_2, \cdots, C_{s+1}, H$ are overfull with maximum degree $\Delta(G)$. Therefore $\frac{1}{2}\Delta(G)(|V(G)| - 1) + 2 = |E(G^+)|$, but

$$\begin{aligned}
|E(G^+)| &\leq s\Delta(G) + |E(C_1)| + |E(C_2)| + \cdots + |E(C_{s+1})| + |E(H)| \\
&\leq s\Delta(G) + \\
&\quad \tfrac{1}{2}\Delta(G)(|V(C_1)| + |V(C_2)| + \cdots + \\
&\quad |V(C_{s+1})| + |V(H)| - (s+2)) \\
&= s\Delta(G) + \tfrac{1}{2}\Delta(G)(|V(G^+)| - s - (s+2)) \\
&= s\Delta(G) + \tfrac{1}{2}\Delta(G)(|V(G)| - 2s - 1) \\
&= \tfrac{1}{2}\Delta(G)(|V(G)| - 1),
\end{aligned}$$

a contradiction. Therefore G^+ does have a 1–factor, and so G has a near 1–factor that includes all vertices of maximum degree. □

We are now in a position to prove Theorem 3.

Proof of Theorem 3. If $d = \Delta(G)$ then this is Proposition 2.

Suppose now that $d = \Delta(G) - 1 \geq 2$. Let G^* be an s'–critical subgraph of G with $\Delta(G^*) = \Delta(G)$. Then by Theorem 15, G^* is just overfull and contains no proper overfull subgraph at the same maximum degree. By Lemma 19, G^* contains a matching J of size $\frac{1}{2}(|V(G^*)| - 1)$ that contains all vertices in G^* of degree $\Delta(G^*)$. Then $\Delta(G^* - J) = \Delta(G) - 1$. Also

$$|E(G^* - J)| = |E(G^*)| - \frac{1}{2}(|V(G^*)| - 1)$$

$$= \frac{1}{2}\Delta(G^*)(|V(G^*)| - 1) + 1 - \frac{1}{2}(|V(G^*)| - 1)$$

$$= \frac{1}{2}\Delta(G^* - J)(|V(G^* - J)| - 1) + 1,$$

so $(G^* - J)$ is s'–Class 2 and has maximum degree $\Delta(G) - 1$. By Proposition 2, $(G^* - J)$ has an s'–critical subgraph H of the same maximum degree. Then H is the required s'–critical subgraph of G of maximum degree $d = \Delta(G) - 1$.

If $d < \Delta(G) - 1$, then we just repeat this argument until we obtain an s'–critical subgraph of degree d. □

5 The Overfull Conjecture

The Overfull Conjecture, due to Chetwynd and Hilton [2], was an attempt to find a simple characterization of graphs of χ'–Class 2 in the case when $\Delta(G) > \frac{|V(G)|}{3}$. It states:

The Overfull Conjecture 1. *Let G be a simple graph with $\Delta(G) > \frac{|V(G)|}{3}$. Then $\chi'(G) = \Delta(G) + 1$ if and only if it has an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

The Overfull Conjecture can be reformulated in the following way.

The Overfull Conjecture 2. *Let G be a simple graph with $\Delta(G) >$*

$\frac{|V(G)|}{3}$. *Then*

$$\chi'(G) = s'(G).$$

Theorem 20. *The two versions of the Overfull Conjecture are equivalent.*

Proof. Let $\Delta(G) > \frac{|V(G)|}{3}$. The equivalence of Conjecture 1 and 2 follows immediately from Theorem 12. □

We remark finally that if the Overfull Conjecture is untrue then there is some graph G with $\Delta(G) > \frac{|V(G)|}{3}$, $s'(G) = \Delta(G)$ and $\chi'(G) = \Delta(G) + 1$. By the definition of $s'(G)$, any edge list assignment to G in which all lists have size $\Delta(G)$ will satisfy Hall's Condition, in particular the edge list assignment where $L(e) = \{c_1, c_2, \cdots, c_{\Delta(G)}\}$ $(\forall e \in E(G))$. It is possible that this observation might help eventually to find a solution to the Overfull Conjecture.

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