Antimagic Labelings of Antiprisms

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ABSTRACT. A connected graph $G = (V, E)$ is said to be (a, d)-antimagic if there exist positive integers a, d and a bijection $g\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f_g\colon V \to \mathbf{N}$, defined by $f_g(v) = \sum\{g(u, v)\colon (u, v) \in E(G)\}$, is injective and $f_g(V) = \{a, a+d, \ldots, a+(|V|-1)d\}$. We deal with (a, d)-antimagic labelings of the antiprisms.

1 Introduction

Interest in graph labeling problems began in the mid - 1960s with a conjecture of G. Ringel [7] and a paper by A. Rosa [8]. In the intervening three decades well over 230 papers on this topic have appeared. Despite the large number of papers, there are relatively few general results or methods on graph labelings. Indeed, most of the results focus on particular classes of graphs and utilize ad hoc methods. Thus, it is one objective in the theory of labeling of graphs to find a new kind of labelings giving the chance of discovering a mathematical theory whose definitions, statements and criteria enables graph theorists to develop a complete theory of determining the set of all graphs labeled by the new kind of labeling [3].

In this paper all graphs are finite, simple, undirected and connected. Let G be such a graph with the vertex set $V(G)$ and the edge set $E(G)$, where $|V(G)|$ and $|E(G)|$ are the number of vertices and edges of G. The weight $w(v)$ of a vertex $v \in V(G)$ under an edge labeling g is the sum of values $g(e)$ assigned to all edges incident to a given vertex v.

A connected graph $G = (V, E)$ is said to be (a, d)-antimagic if there exist positive integers $a, d \in N$ and bijection $g\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that the induced mapping $f_g\colon V(G) \to W$ is also a bijection, where $W = \{w(v)\colon v \in V(G)\} = \{a, a+d, \ldots, a+(|V|-1)d\}$ is the set of weights of vertices. If $G = (V, E)$ is (a, d)-antimagic and $g\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$

JCMCC 35 (2000), pp. 217-224

is a corresponding bijective mapping of G then g is said to be an (a, d)-antimagic labeling of G.

N. Hartsfield and G. Ringel [6] introduced the concept of an antimagic graph. In [6], we find the conjecture that every connected graph $G = (V, E)$ of order $|V(G)| \geq 3$ and size $|E(G)| \geq 2$ is antimagic. Since there are no hints that one could be able to prove or disprove this conjecture R. Bodendiek and G. Walther started looking for a new edge labeling arising from the antimagic labeling. This search was successful and led to the new concept of an (a, d)-antimagic labeling which was defined in [2]. (a, d)-antimagic labelings of the special graphs called parachutes are described in [4,5]. In [1] are characterized all (a, d)-antimagic graphs of prisms D_n when n is even and it is shown that if n is odd the prisms D_n are $(\frac{5n+5}{2}, 2)$-antimagic.

The antiprism Q_n, $n \geq 3$, is a regular graph of degree $r = 4$ (Archimedean convex polytope) and for $n = 3$ it is octahedron.

Let $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2\}$ be index sets. We will denote the vertex set of Q_n by $V = \{x_{1,1}, x_{1,2}, \ldots, x_{1,n}, x_{2,1}, x_{2,2}, \ldots, x_{2,n}\}$ and edge set by $E = \{(x_{j,i}x_{j,i+1}) : j \in J \text{ and } i \in I\} \cup \{(x_{1,i}x_{2,i}) : i \in I\} \cup \{(x_{1,i}x_{2,i-1}) : i \in I\}$.

We make the convention that $x_{j,n+1} = x_{j,1}$ for $j \in J$ and $x_{2,0} = x_{2,n}$ to simplify later notations.

This paper describes $(6n + 3, 2)$-antimagic labelings and $(4n + 4, 4)$-antimagic labelings for the antiprisms Q_n, where $n \geq 3$, $n \not\equiv 2 \pmod 4$. Some conjectures are proposed in the last section.

2 Diophantine equation

Assume that Q_n is an (a, d)-antimagic graph of order $|V(Q_n)| = 2n$ and size $|E(Q_n)| = 4n$, $n \geq 3$, and $W = \{w(v) : v \in V(Q_n)\} = \{a, a + d, a + 2d, \ldots, a + (2n-1)d\}$ is the set of weights of vertices of Q_n. Clearly, the sum of weights in the set W is

$$\sum_{v \in V(Q_n)} w(v) = 2na + dn(2n-1).$$

Since the edges of Q_n are labeled by the set of integers $\{1, 2, \ldots, 4n\}$ and since each of these labels is used twice in the computation of the weights of vertices, the sum of all the edge labels used to calculate the weights of vertices is equal to

$$2(1 + 2 + \cdots + |E(Q_n)|) = 4n(1 + 4n).$$

Thus the following equation holds

$$2a + d(2n-1) = 4(1 + 4n). \tag{1}$$

The equation (1) is a linear Diophantine equation to be solved in the unknowns $a, d \in \mathbf{N}$.

3 Necessary conditions

The following theorem gives the necessary conditions for an (a,d)-antimagic labeling of Q_n.

Theorem 1. *If Q_n, $n \geq 3$, is (a,d)-antimagic, then d is even, $0 < d < 8$, and $a = 8n - \frac{d}{2}(2n-1) + 2$.*

Proof: Let Q_n be an (a,d)-antimagic. From the linear Diophantine equation (1) we have

$$d = \frac{4(4n+1) - 2a}{2n-1}, \tag{2}$$

and we can see that d is even.

By putting $a \geq 10$ ($a = 10$ is the minimal value of weight which can be assigned to a vertex of degree four) we get the upper bound on the value d, i.e. $0 < d < 8$. This implies that the equation (1) has exactly the three different solutions $(a,d) = (6n+3, 2)$ or $(a,d) = (4n+4, 4)$ or $(a,d) = (2n+5, 6)$, respectively.

4 (a,d)-antimagic labelings

In the sequel we shall use the functions

$$\lambda(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases} \tag{3}$$

and

$$\delta(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 2 \\ 1 & \text{if } x \equiv 1 \pmod 2 \end{cases} \tag{4}$$

to simplify later notations.

If n is odd , $n \geq 3$, we construct an edge labeling g_1 and if $n \equiv 0 \pmod 4$, $n \geq 4$, we construct an edge labeling g_2 of an antiprism Q_n in the following way.

$$g_1(x_{1,i}x_{1,i+1}) = \left(\frac{n+1}{2}+i\right)\lambda\left(i,\frac{n-1}{2}\right) + \left(i-\frac{n-1}{2}\right)\lambda\left(\frac{n+1}{2},i\right)$$
for $i \in I$,

$$g_1(x_{2,i}x_{2,i+1}) = \begin{cases} n+1+i & \text{if } i \in I-\{n\} \\ n+1 & \text{if } i=n \end{cases},$$

$$g_1(x_{1,i}x_{2,i}) = (4n-2i+1)\lambda\left(i,\frac{n-1}{2}\right) + (5n-2i+1)\lambda\left(\frac{n+1}{2},i\right)$$
for $i \in I$,

$$g_1(x_{1,i}x_{2,i-1}) = (2n+2i-3)\lambda\left(i,\frac{n+1}{2}\right) + (n+2i-1)\lambda\left(\frac{n+3}{2},i\right)$$
for $i \in I-\{1\}$,

$$g_1(x_{1,1}x_{2,n}) = 3n.$$

$$g_2(x_{1,i}x_{1,i+1}) = i \text{ for } i \in I,$$

$$g_2(x_{2,i}x_{2,i+1}) = (n+1+i)\lambda\left(i,\frac{n}{2}-1\right) + [(n+i)\delta(i) + (n+2+i)\delta(i+1)]\lambda\left(\frac{n}{2},i\right) \text{ for } i \in I-\{n\},$$

$$g_2(x_{2,n}x_{2,1}) = n+1,$$

$$g_2(x_{1,i}x_{2,i}) = (2n+2i-1)\lambda\left(i,\frac{n}{2}\right) + (n+2i)\lambda\left(\frac{n}{2}+1,i\right)$$
for $i \in I$,

$$g_2(x_{1,i}x_{2,i-1}) = (4n-2i+3)\lambda\left(i,\frac{n}{2}\right) + (5n-2i+2)\lambda\left(\frac{n}{2}+1,i\right)$$
for $i \in I-\{1\}$,

$$g_2(x_{1,1}x_{2,n}) = 3n+1.$$

Theorem 2. *For $n \geq 3$, $n \not\equiv 2 \pmod 4$, the antiprism Q_n has an $(6n+3,2)$-antimagic labeling.*

Proof: First, we shall show that the labeling g_1 uses each integer $1,2,\ldots,|E(Q_n)|$ and the labeling g_2 is dealt with similarly.
If $i \in I$ then $g_1(x_{1,i}x_{1,i+1})$ is equal successively to $[1,2,3,\ldots,\frac{n+1}{2},\frac{n+3}{2},\ldots,n]$; if $i \in I$ then $g_1(x_{2,i}x_{2,i+1})$ successively attain values of $[n+1,n+2,\ldots,2n]$; if $i \in I$ then $g_1(x_{1,i}x_{2,i})$ successively assume values of $[3n+1,3n+2,\ldots,4n]$ and finally if $i \in I$ then $g_1(x_{1,i}x_{2,i-1})$ is equal successively to $[2n+1,2n+2,\ldots,3n]$.

Let us denote the weights (under a edge labeling g) of vertices $x_{1,i}$ and

$x_{2,i}$ of Q_n by

$$w_g(x_{1,i}) = g(x_{1,i}x_{1,i+1}) + g(x_{1,i-1}x_{1,i}) + g(x_{1,i}x_{2,i}) + g(x_{1,i}x_{2,i-1}) \quad (5)$$

$$w_g(x_{2,i}) = g(x_{2,i}x_{2,i+1}) + g(x_{2,i-1}x_{2,i}) + g(x_{1,i}x_{2,i}) + g(x_{1,i+1}x_{2,i}), \quad \text{for } i \in I. \quad (6)$$

It is not difficult to check that the weights of vertices under the labelings g_1 and g_2 constitute the sets

$W_1 = \{w_{g_1}(x_{1,i})\colon i \in I\} = \{w_{g_2}(x_{1,i})\colon i \in I\} = \{6n+3, 6n+5, \dots, 8n+1\}$
and
$W_2 = \{w_{g_1}(x_{2,i})\colon i \in I\} = \{w_{g_2}(x_{2,i})\colon i \in I\} = \{8n+3, 8n+5, \dots, 10n+1\}$.

We can see that each vertex of Q_n receives exactly one label of weight from $W_1 \cup W_2$ and each number from $W_1 \cup W_2$ is used exactly once as a label of weight and further that the set $W = W_1 \cup W_2 = \{a, a+d, \dots, a+(|V(Q_n)| -1)d\}$, where $a = 6n+3$ and $d = 2$.

This proves that g_1 and g_2 are $(6n+3, 2)$-antimagic labelings.

Now, we define an edge labeling g_3 (if n is odd, $n \geq 3$) and edge labeling g_4 (if $n \equiv 0 \pmod 4$, $n \geq 4$) of an antiprism Q_n as follows, where again the functions $\lambda(x,y)$ and $\delta(x)$ defined in (3), (4) are used:

$$g_3(x_{1,i}x_{1,i+1}) = (n+2i+1)\,\lambda\left(i, \frac{n-1}{2}\right) + (2i-n+1)\,\lambda\left(\frac{n+1}{2}, i\right) \quad \text{for } i \in I,$$

$$g_3(x_{2,i}x_{2,i+1}) = \begin{cases} 2n+2i+2 & \text{if } i \in I - \{n\} \\ 2n+2 & \text{if } i = n \end{cases},$$

$$g_3(x_{1,i}x_{2,i}) = (4n-4i+1)\lambda\left(i, \frac{n-1}{2}\right) + (6n-4i+1)\lambda\left(\frac{n+1}{2}, i\right) \quad \text{for } i \in I,$$

$$g_3(x_{1,i}x_{2,i-1}) = (4i-7)\lambda\left(i, \frac{n+1}{2}\right) + (4i-2n-3)\lambda\left(\frac{n+3}{2}, i\right) \quad \text{for } i \in I - \{1\},$$

$$g_3(x_{1,1}x_{2,n}) = 2n-1.$$

$$
\begin{aligned}
g_4(x_{1,i}x_{1,i+1}) &= [(n+2i+2)\delta(i) + (n+2i-2)\delta(i+1)]\,\lambda\left(i,\frac{n}{2}\right) \\
&\quad + (2i-n)\lambda\left(\frac{n}{2}+1,i\right) \text{ for } i \in I, \\
g_4(x_{2,i}x_{2,i+1}) &= \begin{cases} 2n+2i+2 & \text{if } i \in I-\{n\} \\ 2n+2 & \text{if } i = n \end{cases}, \\
g_4(x_{1,i}x_{2,i}) &= (4n-4i+1)\lambda\left(i,\frac{n}{2}\right) + (6n-4i+3)\lambda\left(\frac{n}{2}+1,i\right) \\
&\quad \text{for } i \in I, \\
g_4(x_{1,i}x_{2,i-1}) &= (4i-7)\lambda\left(i,\frac{n}{2}+1\right) + (4i-2n-5)\lambda\left(\frac{n}{2}+2,i\right) \\
&\quad \text{for } i \in I-\{1\}, \\
g_4(x_{1,1}x_{2,n}) &= 2n-1.
\end{aligned}
$$

Theorem 3. *For $n \geq 3$, $n \not\equiv 2 \pmod 4$, the antiprism Q_n has an $(4n+4,4)$-antimagic labeling.*

Proof: Label the edges of Q_n by g_3 and g_4, respectively. We obtain the labelings with labels from the set $\{1,2,3,\dots,|E(Q_n)|\}$ and this implies that the labelings g_3 and g_4 are the bijections from the edge set $E(Q_n)$ onto the set $\{1,2,3,\dots,4n\}$.

By direct computation, we obtain that the weights of vertices of Q_n under the labelings g_3 and g_4 constitute the sets

$$W_3 = \{w_{g_3}(x_{1,i}) \colon i \in I\} = \{w_{g_4}(x_{1,i}) \colon i \in I\} = \{4n+4i \colon i \in I\}$$

and

$$W_4 = \{w_{g_3}(x_{2,i}) \colon i \in I\} = \{w_{g_4}(x_{2,i}) \colon i \in I\} = \{8n+4i \colon i \in I\}$$

where $w_g(x_{1,i})$ and $w_g(x_{2,i})$ are defined in (5) and (6), respectively.

We see that the set $W_3 \cup W_4 = \{a, a+4, a+8, \dots, a+(2n-1)4\}$, where $a = 4n+4$, is the set of weights of all vertices of Q_n and it can be seen that the induced mapping $f\colon V(Q_n) \to W_3 \cup W_4$ is bijective.

5 Conclusion and open problems

Lemma 4. *Antiprism Q_3 is not $(2n+5,6) = (11,6)$-antimagic.*

Proof: Assume that Q_3 is $(11,6)$-antimagic. Then the set of weights of vertices is $\{11,17,23,29,35,41\}$. The smallest value of weight of vertex ($w(x_1) = a = 11$) can be obtained only under the quadruple of values of adjacent edges $(1,2,3,5)$. The following value of weight of vertex

($w(x_2) = a + d = 17$) can be obtained under the quadruples $(1,2,3,11)$, $(1,2,4,10)$, $(1,2,5,9)$, $(1,2,6,8)$, $(1,3,4,9)$, $(1,3,5,8)$, ... Each of these quadruples contains a pair or a triple of values from the set $\{1,2,3,5\}$, therefore it is impossible to arrange these quadruples on the edges of vertex x_2 and this contradicts the fact that Q_3 is $(11,6)$-antimagic.

We know $(2n+5,6)$-antimagic labelings for Q_4 and Q_7 (Figures 1 and 2). This prompts us to propose the following:

Conjecture 1. *For $n \geq 4$ the antiprism Q_n has an $(2n+5,6)$-antimagic labeling.*

Concluding this paper, let us pose the following two conjectures:

Conjecture 2. *For $n \equiv 2 \pmod 4$, $n \geq 6$, the antiprism Q_n has an $(6n+3,2)$-antimagic labeling.*

Conjecture 3. *For $n \equiv 2 \pmod 4$, $n \geq 6$, the antiprism Q_n is $(4n+4,4)$-antimagic.*

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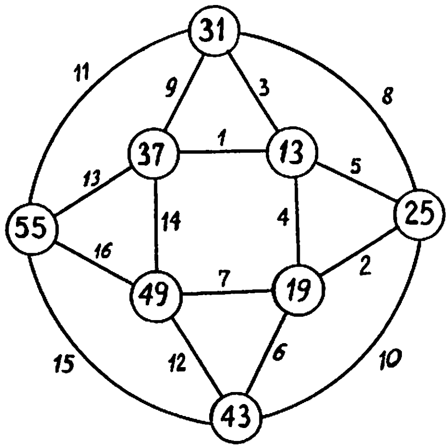

Figure 1. (13, 6)-antimagic labeling of Q_4

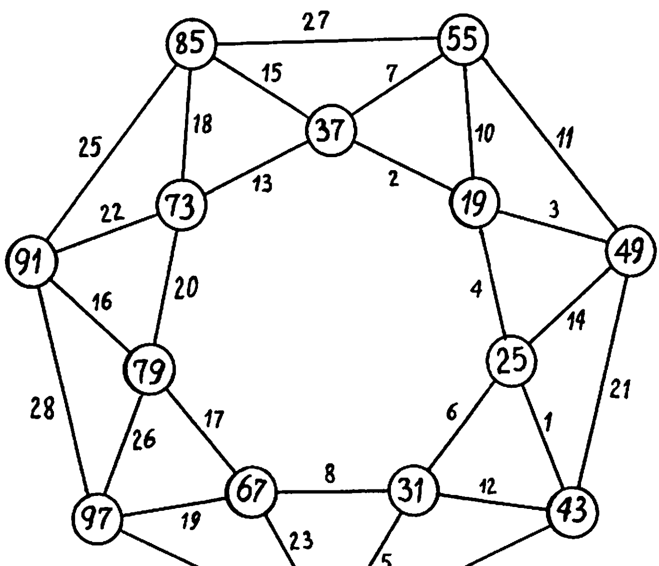

Figure 2. (19, 6)-antimagic labeling of Q_7