Restrained domination in graphs with minimum degree two

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ABSTRACT. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set if every vertex not in S is adjacent to a vertex in S. Furthermore, a set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set, while the restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. We show that if a connected graph G of order n has minimum degree at least 2 and is not one of eight exceptional graphs, then $\gamma_r(G) \leq (n-1)/2$. We show that if G is a graph of order n with $\delta = \delta(G) \geq 2$, then $\gamma_r(G) \leq n(1 + (\frac{1}{\delta})^{\frac{\delta}{\delta-1}} - (\frac{1}{\delta})^{\frac{1}{\delta-1}})$.

1 Introduction

In this paper, we follow the notation of [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the open neighborhood of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S by $N[S] = N(S) \cup S$. The subgraph of G induced by the vertices in S is denoted by $\langle S \rangle$. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$).

A set $S \subseteq V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. (That is, $N[S] = V$.) The domination number of G, denoted

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by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [2] includes a chapter on domination. For a more thorough study of domination in graphs, see Haynes, Hedetniemi and Slater [5, 6].

In this paper we study a variation on the domination theme which is called restrained domination. A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. Clearly, $\gamma_r(G) \geq \gamma(G)$. This concept of restrained domination in graphs was introduced and studied by Domke et al. [3, 4].

McCuaig and Shepherd [7] have shown that if a connected graph G of order n has minimum degree at least 2 and is not one of seven exceptional graphs, then $\gamma(G) \leq 2n/5$. Alon [1] showed that if G is a graph of order n with $\delta = \delta(G) \geq 2$, then $\gamma(G) \leq n\,[1 + \ln(\delta + 1)]/(\delta + 1)$.

In this paper we investigate upper bounds on the restrained domination number of a connected graph. We show that if a connected graph G has minimum degree at least 2 and is not one of eight exceptional graphs, then $\gamma_r(G) \leq (n-1)/2$. Furthermore, we show that if G is a graph of order n with $\delta = \delta(G) \geq 2$, then $\gamma_r(G) \leq n\,[1 + (\frac{1}{\delta})^{\frac{\delta}{\delta-1}} - (\frac{1}{\delta})^{\frac{1}{\delta-1}}]$.

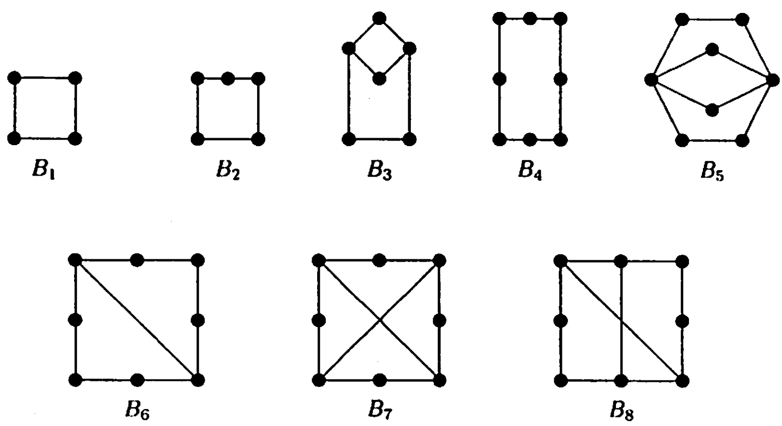

Figure 1. The collection of $\mathcal{B}$ of grpahs

2 Small values of n

In this section, we examine the restrained domination number, $\gamma_r(G)$, of connected graphs G of order n, where $3 \leq n \leq 8$, with minimum degree $\delta(G) \geq 2$. Let $\mathcal{B}$ be the collection of graphs shown in Figure 1. We shall prove:

Theorem 1 *If $G = (V, E)$ is a connected graph of order $n \leq 8$ with $\gamma_r(G) > (n-1)/2$ and $\delta(G) \geq 2$, then $G \in \mathcal{B}$.*

Let G be a connected graph of order n with $\delta(G) \geq 2$. If $n = 3$, then $G \cong C_3$ and $\gamma_r(G) = 1 = (n-1)/2$. If $n = 4$, then either $\gamma_r(G) = 1$ or $G \cong B_1$ and $\gamma_r(G) = 2 = n/2$. Hence B_1 is the only graph on three or four vertices that satisfies the hypothesis of Theorem 1.

The following result will prove to be useful.

Lemma 2 *Let $G = (V, E)$ be a graph of order $n \geq 5$ with $\delta(G) \geq 2$. If $\Delta(G) = n-2$, then $\gamma_r(G) \leq (n-1)/2$.*

Proof. Since $n \geq 5$, we have $\Delta(G) \geq 3$. Let $N(v_1) = \{v_2, \ldots, v_{n-1}\}$ and $V - N[v_1] = \{v_n\}$. Since $\delta(G) \geq 2$, we may assume that $N(v_n) = \{v_2, v_3, \ldots, v_k\}$ where $k \geq 3$. If $\{v_1, v_2\}$ or $\{v_1, v_3\}$ is a restrained dominating set of G, then $\gamma_r(G) \leq 2 \leq (n-1)/2$. Suppose, then, that neither $\{v_1, v_2\}$ nor $\{v_1, v_3\}$ is a restrained dominating set of G. Let $i \in \{2, 3\}$, and let S_i be the set of isolated vertices in $G - \{v_1, v_i\}$. Since $\{v_1, v_i\}$ is not a restrained dominating set of G, $S_i \neq \emptyset$. If $|S_i| \leq (n-1)/2 - 2$, then $\{v_1, v_i\} \cup S_i$ is a restrained dominating set of G, whence $\gamma_r(G) \leq (n-1)/2$. Suppose, then, that $|S_i| \geq n/2 - 2$. Every vertex in S_i is adjacent only to v_1 and v_i, so $S_2 \cap S_3 = \emptyset$. Furthermore, $v_i \notin S_2 \cup S_3$. Thus $V - (S_2 \cup S_3 \cup \{v_2, v_3\})$ is a restrained dominating set of G, whence $\gamma_r(G) \leq n - |S_2| - |S_3| - 2 \leq n - (n/2 - 2) - (n/2 - 2) - 2 = 2 \leq (n-1)/2$. □

Lemma 3 *If $G = (V, E)$ is a connected graph of order 5 with $\gamma_r(G) \geq 3$ and $\delta(G) \geq 2$, then $G \cong B_2$.*

Proof. If $\Delta(G) = 4$, then $\gamma_r(G) = 1$, a contradiction. If $\Delta(G) = 3$, then, by Lemma 2, $\gamma_r(G) \leq 2$, a contradiction. Hence $\Delta(G) = 2$, i.e., $G \cong B_2$. □

Lemma 4 *Let $G = (V, E)$ be a connected graph of order 6 with $\delta(G) \geq 2$. If $\Delta(G) \neq 3$, then $\gamma_r(G) \leq 2$.*

Proof. If $\Delta(G) = 2$, then $G \cong C_6$. If $\Delta(G) = 4$, then, by Lemma 2, $\gamma_r(G) \leq 2$. If $\Delta(G) = 5$, then $\gamma_r(G) = 1$. □

Lemma 5 *If $G = (V, E)$ is a connected graph of order 6 with $\gamma_r(G) \geq 3$ and $\delta(G) \geq 2$, then $G \cong B_3$.*

Proof. By Lemma 4, we know that $\Delta(G) = 3$. Let v_1 be a vertex of degree 3, and let $N(v_1) = \{v_2, v_3, v_4\}$ and $V - N[v_1] = \{v_5, v_6\}$. We show that no vertex from $\{v_2, v_3, v_4\}$ is adjacent to both v_5 and v_6. If this is not the case, then we may assume that v_2v_5 and v_2v_6 are edges. Since $\{v_1, v_2\}$ is not a restrained dominating set, at least one vertex is isolated in $G - \{v_1, v_2\}$. Since $\delta(G) \geq 2$ and $deg\, v_1 = deg\, v_2 = 3 = \Delta(G)$, each $v_i \in V(G) - \{v_1, v_2\}$ must have a neighbor in $G - \{v_1, v_2\}$, a contradiction. Hence no vertex from $\{v_2, v_3, v_4\}$ is adjacent to both v_5 and v_6.

Since $\delta(G) \geq 2$, we may assume that v_2v_5 and v_4v_6 are edges. Since v_3 is adjacent to at most one of v_5 and v_6, and since v_5 and v_6 have degree at least 2, v_5v_6 must be an edge. Now if v_3 is adjacent to v_2 or v_4, say v_2v_3 is an edge, then $\{v_1, v_5\}$ is a restrained dominating set, contradicting the fact that $\gamma_t(G) = 3$. Thus, v_3 is adjacent to neither v_2 not v_4. We may assume that v_3v_5 is an edge and, hence, v_3v_6 is not an edge. Hence $G \cong B_3$. □

Lemma 6 *If $G = (V, E)$ is a connected graph of order 7 with $\delta(G) \geq 2$, then $\gamma_r(G) \leq 3$.*

Proof. If $\Delta(G) = 6$, then $\gamma_r(G) = 1$. If $\Delta(G) = 5$, then, by Lemma 2, $\gamma_r(G) \leq 3$.

Suppose that $\Delta(G) = 4$. Let v_1 be a vertex of degree 4, and let $N(v_1) = \{v_2, v_3, v_4, v_5\}$ and $V - N[v_1] = \{v_6, v_7\}$. Suppose some vertex from $\{v_2, v_3, v_4, v_5\}$, say v_2, is adjacent to both v_6 and v_7. If $\{v_1, v_2\}$ is a restrained dominating set, then $\gamma_r(G) = 2$. Otherwise, if $\{v_1, v_2\}$ is not a restrained dominating set, then at least one vertex is isolated in $G - \{v_1, v_2\}$. We may assume v_3 is isolated in $G - \{v_1, v_2\}$. Thus, v_3 is adjacent only to v_1 and v_2. Hence v_2 is adjacent to only v_1, v_3, v_6, v_7. It follows that $G - \{v_1, v_2, v_3\}$ has no isolated vertex. Thus, $\{v_1, v_2, v_3\}$ is a restrained dominating set, so $\gamma_r(G) \leq 3$. Suppose, then, that no vertex from $\{v_2, v_3, v_4, v_5\}$ is adjacent to both v_6 and v_7. We may assume that v_2v_6 and v_5v_7 are edges. If v_6v_7 is not an edge, then we may assume that v_3v_6 and v_4v_7 are edges. But then $\{v_1, v_3, v_4\}$ is a restrained dominating set, so $\gamma_r(G) \leq 3$. On the other hand, if v_6v_7 is an edge, then either there is an edge joining at least one of v_2 and v_5 to one of v_3 and v_4 or not. If there is an edge joining $\{v_2, v_5\}$ and $\{v_3, v_4\}$, say v_2v_3 is an edge, then $\{v_3, v_4, v_7\}$ is a restrained dominating set, so $\gamma_r(G) \leq 3$. Suppose, then, that there is no edge joining v_2 or v_5 to v_3 or v_4. If v_3v_4 is an edge, then $\{v_1, v_2, v_5\}$ is a restrained dominating set, so $\gamma_r(G) \leq 3$. Hence we may assume that v_3v_4 is not an edge. Assume, without loss of generality, that v_3v_6 is an edge. If $v_4v_6 \in E(G)$ or $v_4v_7 \in E(G)$, then $\{v_3, v_6, v_7\}$ is a restrained dominating set of G. In both cases, $\gamma_r(G) \leq 3$. Hence if $\Delta(G) = 4$, then $\gamma_r(G) \leq 3$.

Suppose $\Delta(G) = 3$. Let v_1 be a vertex of degree 3, and let $N(v_1) = \{v_2, v_3, v_4\}$ and $V - N[v_1] = \{v_5, v_6, v_7\}$. Suppose some vertex from $\{v_2, v_3, v_4\}$

is adjacent to two of the vertices v_5, v_6 and v_7. We may assume v_2v_5 and v_2v_6 are edges. Thus v_2 is adjacent only to v_1, v_5, v_6. If $\{v_1, v_2, v_7\}$ is a restrained dominating set, then $\gamma_r(G) \leq 3$. Otherwise, if $\{v_1, v_2, v_7\}$ is not a restrained dominating set, then $G - \{v_1, v_2, v_7\}$ contains an isolated vertex. If v_i is isolated in $G - \{v_1, v_2, v_7\}$, then v_i is adjacent to v_7 and $\{v_1, v_2, v_i\}$ is a restrained dominating set, and so $\gamma_r(G) \leq 3$. Suppose, then, that each of v_2, v_3, and v_4 is adjacent to at most one of v_5, v_6 and v_7. Since $\delta(G) \geq 2$, it follows that there are at least two edges in the subgraph induced by v_5, v_6 and v_7. We may assume that v_5v_6 and v_6v_7 are edges. If $\{v_1, v_6\}$ is a restrained dominating set, then $\gamma_r(G) = 2$. Otherwise, if $\{v_1, v_6\}$ is not a restrained dominating set, then one of v_2, v_3, v_4, say v_3, is isolated in $G - \{v_1, v_6\}$. Thus v_3 is adjacent to only v_1 and v_6, and v_6 is adjacent only to v_3, v_5 and v_7. It follows then that $\{v_1, v_3, v_6\}$ is a restrained dominating set, so $\gamma_r(G) \leq 3$. Hence if $\Delta(G) = 3$, then $\gamma_r(G) \leq 3$. Finally, if $\Delta(G) = 2$, then $G \cong C_7$ and $\gamma_r(G) = 3$. □

Lemma 7 *If $G = (V, E)$ is a connected graph of order 8 with $\delta(G) \geq 2$ and with $\Delta(G) \geq 5$, then $\gamma_r(G) \leq 3$.*

Proof. If $\Delta(G) = 7$, then $\gamma_r(G) = 1$. If $\Delta(G) = 6$, then, by Lemma 2, $\gamma_r(G) \leq 3$. Hence we may assume that $\Delta(G) = 5$. Let v_1 be a vertex of degree 5, and let $N(v_1) = \{v_2, v_3, v_4, v_5, v_6\}$ and $V - N[v_1] = \{v_7, v_8\}$.

Suppose firstly that some vertex from $\{v_2, v_3, v_4, v_5, v_6\}$, say v_2, is adjacent to both v_7 and v_8. If $\{v_1, v_2\}$ is a restrained dominating set, then $\gamma_r(G) = 2$. Otherwise, if $\{v_1, v_2\}$ is not a restrained dominating set, then at least one vertex is isolated in $G - \{v_1, v_2\}$. We may assume v_3 is isolated in $G - \{v_1, v_2\}$. Thus, v_3 is adjacent only to v_1 and v_2. Now if $\{v_1, v_2, v_3\}$ is a restrained dominating set, then $\gamma_r(G) \leq 3$. Otherwise, if $\{v_1, v_2, v_3\}$ is not a restrained dominating set, then at least one vertex is isolated in $G - \{v_1, v_2, v_3\}$. We may assume v_4 is isolated in $G - \{v_1, v_2, v_3\}$, so v_4 is adjacent only to v_1 and v_2. Hence v_2 has degree 5 and is adjacent to only v_1, v_3, v_4, v_7, v_8. It follows that $G - \{v_1, v_2, v_3, v_4\}$ has no isolated vertex. If v_5v_6 is an edge, then $\{v_1, v_7, v_8\}$ is a restrained dominating set and $\gamma_r(G) \leq 3$. On the other hand, if v_5v_6 is not an edge, then without loss of generality, $v_5v_7 \in E(G)$ and $v_6v_8 \in E(G)$ and $\{v_1, v_5, v_6\}$ is a restrained dominating set implying that $\gamma_r(G) \leq 3$.

Suppose next that no vertex from $\{v_2, v_3, v_4, v_5, v_6\}$ is adjacent to both v_7 and v_8. We may assume that v_2v_7 and v_6v_8 are edges. If v_7v_8 is not an edge, then we may assume that v_3v_7 and v_5v_8 are edges. But then at least one of $\{v_1, v_3, v_5\}$ and $\{v_1, v_2, v_6\}$ is a restrained dominating set, so $\gamma_r(G) \leq 3$. On the other hand, if v_7v_8 is an edge, then either $\{v_1, v_2, v_6\}$ is a restrained dominating set, in which case $\gamma_r(G) \leq 3$, or not. If $\{v_1, v_2, v_6\}$ is not a restrained dominating set, then since v_7 and v_8 are not isolated in $G - \{v_1, v_2, v_6\}$, we may assume that v_5 is isolated in $G - \{v_1, v_2, v_6\}$.

Further, we may assume that v_5v_6 is an edge. If $\{v_1, v_2, v_7\}$ is a restrained dominating set, then $\gamma_r(G) \leq 3$. If $\{v_1, v_2, v_7\}$ is not a restrained dominating set, then we may assume that v_3 is isolated in $G - \{v_1, v_2, v_7\}$. In particular, v_3 is adjacent to at least one of v_2 and v_7. But then $\{v_1, v_4, v_8\}$ is a restrained dominating set and $\gamma_r(G) \leq 3$. Hence if $\Delta(G) = 5$, then $\gamma_r(G) \leq 3$. □

Lemma 8 *Let $G = (V, E)$ be a connected graph of order 8 with $\delta(G) \geq 2$ and $\Delta(G) = 4$. If $\gamma_r(G) \geq 4$, then $G \cong B_5$.*

Proof. Let v_1 be a vertex of degree 4, and let $N(v_1) = \{v_2, v_3, v_4, v_5\}$ and $V - N[v_1] = \{v_6, v_7, v_8\}$. We show firstly that every vertex of $\{v_2, v_3, v_4, v_5\}$ is adjacent to at most one vertex from $\{v_6, v_7, v_8\}$. If this is not the case, then we may assume that v_2v_6 and v_2v_7 are edges. If v_2v_8 is an edge, then, since $\Delta(G) = 4$, v_2 is adjacent to only v_1, v_6, v_7 and v_8. But then $\{v_1, v_2\}$ is a restrained dominating set, contradicting the fact that $\gamma_r(G) \geq 4$. Hence v_2v_8 is not an edge.

Suppose now that v_6v_8 and v_7v_8 are both edges. Since $\{v_1, v_8\}$ is not a restrained dominating set, we may assume that v_5 is adjacent to only v_1 and v_8. Now since $\{v_1, v_5, v_8\}$ is not a restrained dominating set, we may assume that v_4 is isolated in $G - \{v_1, v_5, v_8\}$. Thus, v_4 is adjacent to only v_1 and v_8. But then $\{v_3, v_6, v_8\}$ is a restrained dominating set, contradicting the fact that $\gamma_r(G) \geq 4$. Hence at most one of v_6v_8 and v_7v_8 is an edge.

Suppose that either v_6v_8 or v_7v_8 is an edge, say v_7v_8. Suppose v_8 is adjacent to at least two of v_3, v_4, v_5, say to v_4 and v_5. Then $\{v_3, v_6, v_8\}$ is a restrained dominating set, a contradiction. Hence v_8 is adjacent to exactly one of v_3, v_4, v_5, say to v_5 (so v_8 is adjacent to only v_5 and v_7). If $\{v_2, v_5, v_6\}$ is a restrained dominating set, then $\gamma_r(G) \leq 3$, a contradiction. Hence v_3 or v_4, say v_3, is not dominated by $\{v_2, v_5, v_6\}$. Hence v_3 is adjacent to v_1 and to at least one of v_4 and v_7, but to no other vertex. If $\{v_4, v_6, v_7\}$ is a restrained dominating set, then $\gamma_r(G) \leq 3$, a contradiction. Hence v_5 must be adjacent to v_1, v_8 and possibly v_2. If v_3v_4 is an edge, then $\{v_1, v_2, v_5\}$ is a restrained dominating set of G, a contradiction. Hence, v_3v_4 is not an edge of G and v_3v_7 is therefore an edge of G. If v_4 is adjacent to either v_6 or v_7, then $\{v_1, v_2, v_5\}$ is a restrained dominating set of G, a contradiction. It follows that v_4 is adjacent to v_1 and v_2 only. Since $\delta(G) \geq 2$, v_6 is adjacent to v_7. But then $\{v_4, v_7, v_8\}$ is a restrained dominating set of G, a contradiction.

We may assume that v_4v_8 and v_5v_8 are edges. Since $\{v_2, v_8\}$ is not a restrained dominating set, v_3v_8 cannot be an edge, so v_8 is adjacent to only v_4 and v_5. Further, since $\{v_2, v_3, v_8\}$ is not a restrained dominating set, one of v_6 or v_7, say v_7, is isolated in $G - \{v_2, v_3, v_8\}$. Thus v_7 is adjacent to only v_2 and v_3. But then $\{v_3, v_6, v_8\}$ is a restrained dominating set, contradicting the fact that $\gamma_r(G) \geq 4$.

Hence every vertex of $\{v_2, v_3, v_4, v_5\}$ is adjacent to at most one vertex from v_6, v_7 or v_8. Since $\delta(G) \geq 2$, it follows that there is at least one edge in the subgraph induced by v_6, v_7 and v_8. First consider the case when the subgraph induced by v_6, v_7 and v_8 has exactly one edge. Without loss of generality, assume that v_7v_8 is the only edge of the subgraph induced by v_6, v_7 and v_8. Since $\delta(G) \geq 2$, we may assume that v_5v_8, v_4v_7, v_2v_6 and v_3v_6 are edges of G. But then $\{v_4, v_5, v_6\}$ is a restrained dominating set of G, which is a contradiction. Now consider the case when the subgraph induced by v_6, v_7 and v_8 has at least two edges, v_6v_7 and v_6v_8, say. Since $\{v_1, v_6\}$ is not a restrained dominating set, one of v_2, v_3, v_4, v_5, say v_2, is isolated in $G - \{v_1, v_6\}$. Thus v_2 is adjacent to only v_1 and v_6. Furthermore, since $\{v_1, v_2, v_6\}$ is not a restrained dominating set, one of v_3, v_4, v_5, say v_3, is isolated in $G - \{v_1, v_2, v_6\}$. Thus v_3 is adjacent to only v_1 and v_6, and v_6 is adjacent only to v_2, v_3, v_7 and v_8. Now since $\{v_1, v_7, v_8\}$ is not a restrained dominating set, v_4v_5 cannot be an edge of G. We may assume that v_4v_7 is an edge. Since $\{v_1, v_5, v_8\}$ is not a restrained dominating set, v_5v_7 cannot be an edge. Thus v_5 must be adjacent to only v_1 and v_8. Furthermore, since $\{v_1, v_4, v_7\}$ is not a restrained dominating set, v_8 cannot be adjacent to v_4 or v_7. Thus, $G \cong B_5$. □

Lemma 9 *Let $G = (V, E)$ be a connected graph of order 8 with $\delta(G) \geq 2$ and with $\Delta(G) = 3$. If $\gamma_r(G) \geq 4$, then $G \in \{B_6, B_7, B_8\}$.*

Proof. Let v be a vertex of degree 3. First we show that v is within distance two of u for all $u \in V(G)$. Suppose that $d(u, v) = 3$ for some $u \in V(G)$. If u has degree 3, then $\{u, v\}$ is a restrained dominating set and $\gamma_r(G) = 2$, contradicting the fact that $\gamma_r(G) \geq 4$. Hence u has degree 2. Let $V - N[\{u, v\}] = \{w\}$, $N(v) = \{a, b, c\}$ and $N(u) = \{x, y\}$. Suppose that w is adjacent to x or y, say to y. If xy is an edge, then y is adjacent to only u, w, x, whence $\{v, y\}$ is a restrained dominating set, a contradiction. Hence xy is not an edge. Since $\{u, v, y\}$ is not a restrained dominating set, there must be an isolated vertex in $G - \{u, v, y\}$. Since x and w are not isolated in $G - \{u, v, y\}$, we may assume that c is adjacent to only v and y. Now since $\{v, x, w\}$ is not a restrained dominating set, a and b must be nonadjacent vertices. But then each of a and b must be adjacent to at least one of x and w. Thus, $\{c, x, w\}$ is a restrained dominating set, a contradiction. Hence wx and wy cannot be edges of G. Thus, we may assume that aw and bw are edges. Since $\{c, u, w\}$ is not a restrained dominating set, there must be an isolated vertex in $G - \{c, u, w\}$. Since x and y are the only possible isolated vertices in $G - \{c, u, w\}$, we may assume that y is adjacent to u, to at least one of c and w, and to no other vertex. But then $\{c, w, x\}$ is a restrained dominating set, a contradiction. Hence v is within distance 2 from every vertex of G.

Let $v = v_1$ and let $N(v_1) = \{v_2, v_3, v_4\}$. Furthermore, let $V - N[v_1] = \{v_5, v_6, v_7, v_8\}$. Since all vertices are at distance at most 2 from v_1, at least one vertex in $N(v_1)$ is adjacent to two vertices in $V - N[v_1]$. We may assume that v_2 is adjacent to v_5 and v_6. If v_3 is adjacent to both v_7 and v_8, then $\{v_1, v_2, v_3\}$ is a restrained dominating set and $\gamma_r(G) \leq 3$, a contradiction. Similarly, if v_4 is adjacent to both v_7 and v_8, then $\gamma_r(G) \leq 3$, a contradiction. Hence we may assume that v_3v_7 and v_4v_8 are edges, while v_3v_8 and v_4v_7 are not edges. Now since $\{v_2, v_7, v_8\}$ is not a restrained dominating set, $G - \{v_2, v_7, v_8\}$ contains an isolated vertex which is necessarily v_5 or v_6. We may assume that v_6 is isolated in $G - \{v_2, v_7, v_8\}$. Furthermore, we may assume that v_6v_7 is an edge. If v_6v_8 is an edge, then $\{v_1, v_2, v_6\}$ is a restrained dominating set, a contradiction. So we may assume v_6 is adjacent only to v_2 and v_7. If v_4v_5 is an edge, then $\{v_1, v_4, v_6\}$ is a restrained dominating set, a contradiction. Hence v_4v_5 is not an edge.

Suppose v_5v_7 is an edge. Then v_7 is adjacent only to v_3, v_5, v_6. But since $\delta(G) \geq 2$, v_5v_8 must be an edge, so v_8 is adjacent only to v_4 and v_5. But then $\{v_1, v_2, v_5\}$ is a restrained dominating set, a contradiction. Hence v_5v_7 is not an edge.

Suppose v_3v_5 is an edge. Then v_3 is adjacent only to v_1, v_5 and v_7. Since $\{v_2, v_5, v_8\}$ is not a restrained dominating set, v_7v_8 cannot be an edge. Thus, v_8 is adjacent only to v_4 and v_5. However, v_5 now has degree 3, so there are no further edges in G. Thus $G \cong B_8$.

Suppose v_3v_5 is not an edge. Then v_5v_8 must be an edge, for otherwise v_5 is adjacent only to v_2. Since $\{v_1, v_7, v_8\}$ is not a restrained dominating set, v_3v_4 cannot be an edge. Thus, v_3 is adjacent only to v_1 and v_7, while v_4 is adjacent only to v_1 and v_8. Now either v_7v_8 is not an edge, in which case $G \cong B_6$, or v_7v_8 is an edge, in which case $G \cong B_7$. This completes the proof of the lemma. □

The following result is immediate.

Lemma 10 *If G is a connected 2-regular graph of order 8, then $G \cong B_4$ and $\gamma_r(G) = 4$.*

Theorem 1 is an immediate consequence of Lemmas 3 to 10. Let $\mathcal{B}^* = \{B_1, B_2, \ldots, B_5\}$. We will refer to a graph in the collection $\mathcal{B}^*$ as a *bad graph.* We close this section by making two observations about bad graphs which we will use in proving our main result of the next section.

Observations

(1) For any bad graph $G = (V, E)$ and $v \in V$, there is a minimum restrained dominating set of G that contains v.

(2) For any bad graph $G = (V, E)$ and $v \in V$, there is a set $D \subset V$ satisfying

(*i*) D dominates $V - \{v\}$,
(*ii*) $\langle V - D \rangle$ contains no isolated vertex, and
(*iii*) $|D| = \gamma_r(G) - 1$.

3 The upper bound

In this section we shall prove:

Theorem 11 *Let G be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$. If $G \notin \mathcal{B}$, then*

$$\gamma_r(G) \leq (n-1)/2.$$

Proof. We have shown (see Theorem 1) the statement to be true for $n \leq 8$. Assume, to the contrary, that the theorem is false. Among all counterexamples, let $G = (V, E)$ be one of minimum size. Then G is a connected graph of order $n \geq 9$ with $\delta(G) \geq 2$ satisfying $\gamma_r(G) \geq n/2$. If $G \cong C_n$, then $\gamma_r(G) \leq (n-1)/2$, a contradiction. Hence G is not a cycle, so G contains at least one vertex of degree 3.

Let $S = \{v \in V \mid \deg v \geq 3\}$. For each $v \in S$, we define the *2-graph of v* to be the connected component of $G - (S - \{v\})$ that contains v. So each vertex of the 2-graph of v has degree 2 in G, except for v. Furthermore, the 2-graph of v consists of edge-disjoint cycles through v, which we call *2-graph cycles*, and paths emanating from v, which we call *2-graph paths*.

Claim 12 *The set S is independent.*

Proof. Assume $e = uv$ is an edge, where $u, v \in S$. Since $\gamma_r(G - e) \geq \gamma_r(G) > (n-1)/2$, and since $\delta(G - e) \geq 2$, the minimality of G implies that e must be a bridge since otherwise $G - e$ would be a connected graph which would be a smaller counterexample. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of $G - e$ where $u \in V_1$. For $i = 1, 2$, let $|V_i| = n_i$. Each G_i satisfies $\delta(G_i) \geq 2$ and is connected. Hence, by the minimality of G, for each $i = 1, 2$ either G_i is in $\mathcal{B}^*$ (since B_4 is a spanning subgraph of the graphs B_6, B_7 and B_8) or $\gamma_r(G_i) \leq (n_i - 1)/2$ for $i = 1, 2$.

If $\gamma_r(G_i) \leq (n_i - 1)/2$ for $i = 1, 2$, then, letting D_i denote a minimum restrained dominating set of G_i ($i = 1, 2$), $D_1 \cup D_2$ is a restrained dominating set of G of cardinality $|D_1| + |D_2| \leq (n_1 - 1)/2 + (n_2 - 1)/2 = (n-2)/2$, a contradiction. Hence G_1 or G_2, say G_1, must belong to $\mathcal{B}^*$. Suppose $\gamma_r(G_2) \leq (n_2 - 1)/2$. If $G_1 \cong B_2$, then $\gamma_r(G) \leq 2 + \gamma_r(G_2) \leq 2 + (n_2 - 1)/2 = n/2 - 1$, a contradiction. Hence $G_1 \in \{B_1, B_3, B_4, B_5\}$. But then $\gamma_r(G) \leq \gamma_r(G_1) + \gamma_r(G_2) \leq n_1/2 + (n_2 - 1)/2 = (n-1)/2$, a contradiction. Hence both G_1 and G_2 belong to $\mathcal{B}^*$.

Assume, firstly, that $G_1 \cong B_2$. If $G_2 \cong B_2$, then $\gamma_r(G) = 4 < (n-1)/2$, a contradiction. Hence $G_2 \in \{B_1, B_3, B_4, B_5\}$. It now follows from Observation (2) that $\gamma_r(G) \le (\gamma_r(G_1)-1)+(\gamma_r(G_2)-1)+1 = 2+(n_2/2-1)+1 = (n-1)/2$, a contradiction. Hence $G_1 \ncong B_2$. Similarly, $G_2 \ncong B_2$. Thus $G_1, G_2 \in \{B_1, B_3, B_4, B_5\}$. It follows from Observation (2) that $\gamma_r(G) \le (\gamma_r(G_1)-1)+(\gamma_r(G_2)-1)+1 \le (n_1/2-1)+(n_2/2-1)+1 = n/2-1$, a contradiction. We deduce, therefore, that S must be independent. □

Claim 13 $|S| \ge 2$

Proof. If $|S| = 1$, then G consists of (at least two) edge-disjoint cycles passing through a common vertex v. If each of these cycles is a 5-cycle, then $\gamma_r(G) = (n-1)/2$; otherwise, if one or more of these cycles is not a 5-cycle, then $\gamma_r(G) < (n-1)/2$. Both cases produce a contradiction. Hence $|S| \ge 2$. □

Claim 14 *All 2-graph paths have length* 1.

Proof. By Claim 13, $|S| \ge 2$. Let $P\colon u, v_1, \ldots, v_k, v$ be a longest path joining two vertices u and v of S, every internal vertex of which belongs to $V-S$. By Claim 12, we know that u and v are not adjacent, so $k \ge 1$. We show that $k = 1$. If this is not the case, then $k \ge 2$. We now consider the graph $G' = G - \{v_1, v_2, \ldots, v_k\}$ of order $n' = n-k$; that is, G' is the graph obtained from G by removing the k internal vertices of the path P. Then $\delta(G') \ge 2$.

Assume G' is connected. Then, by the minimality of G, $G' \in \mathcal{B}^*$ or $\gamma_r(G') \le (n'-1)/2 = (n-k-1)/2$. If $\gamma_r(G') \le (n-k-1)/2$, then $\gamma_r(G) \le k/2 + (n-k-1)/2 = (n-1)/2$, a contradiction. On the other hand, if $G' \in \mathcal{B}^*$, then it follows from Observation (2) that $\gamma_r(G) \le k/2 + (n'-1)/2 = k/2 + (n-k-1)/2 = (n-1)/2$, a contradiction. Hence G' is disconnected.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two components of G'. For $i = 1, 2$, let $|V_i| = n_i$. So $n' = n_1 + n_2$. Each G_i satisfies $\delta(G_i) \ge 2$ and is connected. Hence, by the minimality of G, $G_i \in \mathcal{B}^*$ or $\gamma_r(G_i) \le (n_i-1)/2$ for $i = 1, 2$. If $\gamma_r(G_i) \le (n_i-1)/2$ for $i = 1, 2$, then $\gamma_r(G) \le k/2+\gamma_r(G_1)+\gamma_r(G_2) \le k/2 + (n_1-1)/2 + (n_2-1)/2 = n/2-1$, a contradiction. Hence G_1 or G_2, say G_1, must belong to $\mathcal{B}^*$. Suppose $G_1 \cong B_2$. If $G_2 \cong B_2$, then $\gamma_r(G) \le 4 + k/2 = n/2 - 1$, a contradiction. If $G_2 \ncong B_2$, then $\gamma_r(G_2) \le n_2/2 = (n-k-5)/2$ whence $\gamma_r(G) \le 2 + k/2 + (n-k-5)/2 = (n-1)/2$, a contradiction. Hence $G_1 \ncong B_2$ (and $G_2 \ncong B_2$). If $\gamma_r(G_2) \le (n_2-1)/2$, then it follows from Observation (1) that $\gamma_r(G) \le \gamma_r(G_1)+k/2+\gamma_r(G_2) \le n_1/2+k/2+(n_2-1)/2 = k/2+(n-k-1)/2 = (n-1)/2$, a contradiction. Hence $G_2 \in \mathcal{B}^* - B_2$. But then it is easy to check using Observations (1) and (2) that $\gamma_r(G) \leq (n-2)/2$, a contradiction. We deduce, therefore, that $k = 1$. This completes the proof of the claim. □

Claim 15 *There are no 2-graph cycles.*

Proof. Let $v \in S$. Assume that C: $v, v_1, v_2, \ldots, v_k, v$ is a 2-graph cycle of v of length $k+1$, where $k \geq 2$. Let $H = G - (V(C) - \{v\})$.

We show firstly that $\delta(H) \geq 2$. If this is not the case, then v must be adjacent in G to v_1, v_k and exactly one other vertex, u say. By Claim 12, $u \notin S$. By Claim 14, u is adjacent to v and to one other vertex, w say, of S. Hence letting $H^* = G - V(C) - \{u\}$, we note that H^* is connected graph of order $n^* = n - k - 2$ with $\delta(H^*) \geq 2$. Hence, by the minimality of G, $H^* \in \mathcal{B}^*$ or $\gamma_r(H^*) \leq (n^*-1)/2$. If $H^* \in \mathcal{B}^*$, then either $H^* \cong B_2$, in which case $\gamma_r(G) \leq 3 + k/2 = (n-1)/2$, or $H^* \in \{B_1, B_3, B_4, B_5\}$, in which case it is easy to check (using Observation 1) that $\gamma_r(G) \leq n^*/2 + (k+1)/2 = (n-1)/2$. On the other hand, if $\gamma_r(H^*) \leq (n^*-1)/2 = (n-k-3)/2$, then consider a minimum restrained dominating set D^* of H^*. If $w \in D^*$, then $\gamma_r(G) \leq (k+1)/2 + (n-k-3)/2 = (n-2)/2$, while if $w \notin D^*$, then $\gamma_r(G) \leq (k+2)/2 + (n-k-3)/2 = (n-1)/2$. All the above cases produce a contradiction. Hence $\delta(H) \geq 2$.

Since H is a connected graph of order $n' = n - k$ with $\delta(H) \geq 2$, the minimality of G implies that $H \in \mathcal{B}^*$ or $\gamma_r(H) \leq (n'-1)/2 = (n-k-1)/2$. If $\gamma_r(H) \leq (n-k-1)/2$, then $\gamma_r(G) \leq k/2 + (n-k-1)/2 = (n-1)/2$, a contradiction. On the other hand, if $H \in \mathcal{B}^*$, then either $H^* \cong B_2$, in which case $\gamma_r(G) \leq 2 + k/2 = (n-1)/2$, or $H^* \in \{B_1, B_3, B_4, B_5\}$, in which case it is easy to check (using Observations 1 and 2) that $\gamma_r(G) \leq (k+1)/2 + (n'-2)/2 = (n-1)/2$, once again producing a contradiction. We deduce, therefore, that there is no 2-graph cycle of v. Since v is an arbitrary vertex of S, the claim follows. □

By Claims 12 to 15 it follows that G is a bipartite graph with partite sets S and $V - S$, where $|S| \geq 2$. In particular, any subgraph in $\mathcal{B}^*$ is B_1 or B_4. By definition, each vertex of S has degree at least 3 while each vertex of $V - S$ has degree 2. If $|S| = 2$, then G is a complete bipartite graph and the set consisting of one vertex from S and one vertex from $V - S$ is a restrained dominating set of G of cardinality at most $(n-1)/2$, a contradiction. Hence $|S| \geq 3$.

Among all the vertices of S, let v have smallest degree, m say, in G. Let $N(v) = \{v_1, v_2, \ldots, v_m\}$. We now consider the graph $H = G - N[v]$. It follows from our choice of v and since $|S| \geq 3$, that H contains no isolated vertex.

Claim 16 *If* $\delta(H) = 1$, *then* $\gamma_r(G) \leq (n-2)/2$.

Proof. Since $\delta(H) = 1$, there must be a vertex w of S of degree 1 in H. Let x be the neighbor of w in H, and let $N(x) = \{u, w\}$. Necessarily, $u \in S$. We now consider the graph $G' = G - (N[v] \cup \{w, x\})$ of order $n' = n - m - 3$.

If $N(v) \subset N(w)$, then $N(v) = N(w) - \{x\}$ and G' is a connected graph with $\delta(G') \geq 2$. Hence, by the minimality of G, $G' \in \mathcal{B}^*$ or $\gamma_r(G') \leq (n' - 1)/2 = (n - m - 4)/2$. If $\gamma_r(G') \leq (n - m - 4)/2$, then by adding the vertices v and x to a minimum restrained dominating set D' of G' we produce a restrained dominating set of G of cardinality $|D'| + 2 \leq (n - m - 4)/2 + 2 = (n - m)/2 \leq (n - 3)/2$. On the other hand, if $G' \in \mathcal{B}^*$, then it is straightforward to check using Observation (2) and the fact that $\{v, x\}$ is a restrained dominating set of $\langle N[v] \cup \{w, x\}\rangle$ that $\gamma_r(G) \leq (n - 3)/2$.

If $N(v) \not\subset N(w)$, then it follows from our choice of v that v and w have $m - 1$ vertices in common. Let $\{v_m\} = N(v) - N(w)$. Further, let $N(v_m) = \{v, z\}$. We now consider two possibilities depending on whether $u = z$ or $u \neq z$.

Suppose that $u = z$. Then G' is connected. If $\delta(G') \geq 2$, then the minimality of G implies that $G' \in \mathcal{B}^*$ or $\gamma_r(G') \leq (n'-1)/2 = (n-m-4)/2$. If $\gamma_r(G') \leq (n-m-4)/2$, then let D' be a minimum restrained dominating set of G'. If $u \in D'$, consider the following two cases. If $m = 3$, then v_1 and v_2 can be added to D' to produce a restrained dominating set of G of cardinality $|D'|+2 \leq (n-3)/2$. If $m \geq 4$, then $D' \cup \{v, v_1, v_m\}$ is a restrained dominating set of G of cardinality $|D'| + 3 \leq (n - m - 4)/2 + 3 = (n - m + 2)/2 \leq (n-2)/2$. If $u \notin D'$, then adding v and x to D' produces a restrained dominating set of G of cardinality $|D'| + 2 \leq (n - 3)/2$. On the other hand, if $G' \in \mathcal{B}^*$, then it is straightforward to check using Observation (2) that $\gamma_r(G) \leq (n - 3)/2$. So we may assume that $\delta(G') = 1$. Thus u has degree 3 in G. Let y be the neighbor of u different from x and v_m. By our choice of v, it follows that $m = 3$, so both v and w have degree 3. We now consider the graph $G^* = G - N[v] - N[x] - \{y\}$. Then G^* is a connected graph of order $n^* = n - 8$ with $\delta(G^*) \geq 2$. The minimality of G implies that $G^* \in \mathcal{B}^*$ or $\gamma_r(G^*) \leq (n^* - 1)/2 = (n - 9)/2$. If $\gamma_r(G^*) \leq (n - 9)/2$, then adding the vertices $\{w, v_3, y\}$ to a minimum restrained dominating set of G^* produces a restrained dominating set of G of cardinality at most $(n - 3)/2$. If $G^* \in \mathcal{B}^*$, then (using Observation 2) it is straightforward to check that $\gamma_r(G) \leq (n - 3)/2$. Hence if $u = z$, then $\gamma_r(G) \leq (n - 2)/2$.

Suppose, next, that $v \neq z$. Since $u, z \in S$, we know that u and z are nonadjacent. Suppose G' is connected. Then, since $\delta(G') \geq 2$, the minimality of G implies that $\gamma_r(G') \leq (n' - 1)/2 = (n - m - 4)/2$ or $G' \in \mathcal{B}^*$. If $G' \in \mathcal{B}^*$, then (using Observation 2) it is straightforward to check that $\gamma_r(G) \leq (n - 2)/2$. If $\gamma_r(G') \leq (n - m - 4)/2$, then consider a minimum restrained dominating set D' of G'. If $u \in D'$ and $z \notin D'$, then $D' \cup \{v, v_1\}$ is a restrained dominating set of G', whence $\gamma_r(G) \leq$

$2+(n-m-4)/2=(n-m)/2 \le (n-3)/2$. If $u \notin D'$ and $z \in D'$, then $D' \cup \{w, v_1\}$ is a restrained dominating set of G', whence $\gamma_r(G) \le (n-3)/2$. If $u, z \notin D'$, then $D' \cup \{v, x\}$ is a restrained dominating set of G', whence $\gamma_r(G) \le (n-3)/2$. Suppose $u, z \in D'$. If $m=3$, then the two common neighbors, v_1 and v_2, of v and w can be added to D' to produce a restrained dominating set of G of cardinality $|D'|+2 \le (n-3)/2$. If $m \ge 4$, then $D' \cup \{v, v_1, v_m\}$ is a restrained dominating set of G of cardinality $|D'|+3 \le (n-m-4)/2+3=(n-m+2)/2 \le (n-2)/2$. Hence if G' is connected, then $\gamma_r(G) \le (n-2)/2$.

Suppose then that G' is disconnected. Then G' consists of two components, namely a component F_1 containing u and a component F_2 containing z. For $i=1,2$, let F_i have order n_i, so $n'=n_1+n_2$. Now F_i is a connected graph of order n_i with $\delta(F_i) \ge 2$. Hence, by the minimality of G, $\gamma_r(F_i) \le (n_i-1)/2$ or $F_i \in \mathcal{B}^*$. Suppose that $\gamma_r(F_i) \le (n_i-1)/2$ for $i=1,2$. For $i=1,2$, let D_i be a minimum restrained dominating set of F_i. Then $\{v, v_m, x\} \cup D_1 \cup D_2$ is a restrained dominating set of G of cardinality $3+|D_1|+|D_2| \le 3+(n_1-1)/2+(n_2-1)/2=(n-m+1)/2 \le (n-2)/2$. Suppose, then, that F_1 or F_2 belongs to $\mathcal{B}^*$. Then F_1 or $F_2 \in \{B_1, B_4\}$. Since there is only one edge joining F_i, $i=1,2$, to a vertex not in F_i, and B_1 and B_4 both have more than one vertex of degree at least three, $F_i \notin \{B_1, B_4\}$, which is a contradiction. This completes the proof of the claim. □

By Claim 16, $\delta(H) \ge 2$ for otherwise we have a contradiction. Since G is bipartite, so too is H. Let H' be a component of H (possibly, $H=H'$), and suppose H' has order n'. By the minimality of G, $H' \in \mathcal{B}^*$ or $\gamma_r(H') \le (n'-1)/2$.

Claim 17 *If $H' \in \mathcal{B}^*$, then $H' \cong B_1$.*

Proof. Since H' is bipartite, $H' \cong B_1$ or $H' \cong B_4$. So we need only show that $H' \ncong B_4$. Assume, to the contrary, that H' is an 8-cycle, say $u_1, u_2, \ldots, u_8, u_1$. We may assume that $S \cap V(H')=\{u_1, u_3, u_5, u_7\}$. Then every vertex in $N(v)$ is adjacent to at most one vertex of $S \cap V(H')$. Furthermore, each vertex of $S \cap V(H')$ is adjacent to at least one vertex of $N(v)$ since all vertices of S have degree at least 3 in G. By the Pigeonhole Principle, we may assume that u_1 is adjacent to at most $m/4$ vertices of $N(v)$. By our choice of v, it follows that $m=\deg v \le \deg u_1 \le m/4+2$, or, equivalently, $m \le 8/3$ which contradicts the fact that $m \ge 3$. Hence $H' \cong B_1$. □

Claim 18 *If H is disconnected and $H' \cong B_1$, then $\gamma_r(G) \le (n-3)/2$.*

Proof. Let w and y be the two nonadjacent vertices of the 4-cycle H' that belong to S. Let T denote the set of neighbors of v that are adjacent to either w or y. Since w and y have degree at least 3 in G, each of w and y is adjacent to at least one vertex of T. Since H is disconnected, we know that $|T| \leq m-1$. By the Pigeonhole Principle, we may assume that w is adjacent to at most $|T|/2 \leq (m-1)/2$ vertices of T. By our choice of v, it follows that $m = \deg v \leq \deg w \leq (m-1)/2+2$, or, equivalently, $m \leq 3$. Consequently, $m=3$, $|T|=2$ and H has exactly two components. We may assume that $T=\{v_1,v_2\}$ and that v_1w and v_2y are edges.

Let H'' denote the component of H different from H'. Since there is only one edge joining H'' to a vertex not in H'', $H'' \ncong B_1$. Let x be the vertex of H'' adjacent to v_3. The graph H'' is a connected graph of order $n''=n-8$ with $\delta(H'') \geq 2$. Since $H'' \notin \mathcal{B}^*$, the minimality of G implies that $\gamma_r(H'') \leq (n''-1)/2 = (n-9)/2$. Let D'' be a minimum restrained dominating set of H''. Either $x \in D''$, in which case $D'' \cup \{v_1, y\}$ is a restrained dominating set of G, or $x \notin D''$, in which case $D'' \cup \{v_1, v_3, y\}$ is a restrained dominating set of G. In any event, $\gamma_r(G) \leq |D''|+3 \leq (n-9)/2+3 = (n-3)/2$. □

Claim 19 *If H is connected and $H \cong B_1$, then* $\gamma_r(G) \leq (n-3)/2$.

Proof. Let w and y be the two nonadjacent vertices of the 4-cycle H that belong to S. Then every vertex in $N(v)$ is adjacent to either w or y. Furthermore, each of w and y is adjacent to at least one vertex of $N(v)$. By the Pigeonhole Principle, we may assume that w is adjacent to at most $m/2$ vertices in $N(v)$. Let $N_w = N(w) \cap N(v)$. Then $N_w \cup \{y\}$ is a restrained dominating set of G of cardinality $|N_w|+1 \leq m/2+1 = (n-5)/2+1 = (n-3)/2$. □

Let $H_1, \ldots, H_\ell$, $\ell \geq 1$, denote the components of H. For $i=1,\ldots,\ell$, let H_i have order n_i, so $n_1+\cdots+n_\ell = n-m-1$. If $H_i \in \mathcal{B}^*$ for some i, $1 \leq i \leq \ell$, then by Claims 17, 18 and 19, $\gamma_r(G) \leq (n-3)/2$, a contradiction. Hence, by the minimality of G, $\gamma_r(H_i) \leq (n_i-1)/2$ for all $i=1,\ldots,\ell$. For $i=1,\ldots,\ell$, let D_i be a minimum restrained dominating set of H_i, and let $D = \cup_{i=1}^{\ell} D_i$. Then $|D| = \sum_{i=1}^{\ell}|D_i| \leq \sum_{i=1}^{\ell}(n_i-1)/2 = (n-m-1-\ell)/2 \leq (n-m-2)/2$. Let M_1 denote those vertices in $N(v)$ that are adjacent to a vertex of D in G, and let $M_2 = N(v)-M_1$. If $M_2 = \emptyset$, then $D \cup \{v_1\}$ is a restrained dominating set of G of cardinality $|D|+1 \leq (n-m-2)/2+1 = (n-m)/2 \leq (n-3)/2$, a contradiction. Hence $M_2 \neq \emptyset$. For $i=1,2$, let $|M_i| = m_i$, so $m_1+m_2=m$. If $m_1 \geq m/2$, then $D \cup M_2$ is a restrained dominating set of G of cardinality $|D|+m_2 \leq (n-m-2)/2+m/2 = (n-2)/2$. On the other hand, if $m_1 \leq (m-1)/2$, then $D \cup M_1 \cup \{v\}$ is a restrained dominating set of G of cardinality $|D|+m_1+1 \leq (n-m-2)/2+(m+1)/2 = (n-1)/2$. Both possibilities produce a contradiction. This completes the proof of the theorem. □

That there exists a family of connected graphs G of order n with $\delta(G) \geq 2$ satisfying $\gamma_r(G) = (n-1)/2$ may be seen as follows. For $k \geq 2$, let G_k

consist of k edge-disjoint 5-cycles that all pass through a common vertex v. Then G_k is a connected graph of order $n = 4k+1$ with $\delta(G_k) \geq 2$ satisfying $\gamma_r(G_k) = (n-1)/2$. The graph G_3 is shown in Figure 2.

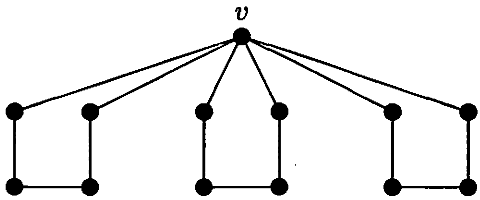

Figure 2. The graph G_3

4 A more general upper bound for $\gamma_r(G)$

We now establish a more general upper bound for $\gamma_r(G)$ of a graph G involving the minimum degree $\delta = \delta(G)$ and the order n of G. Our proof is probabilistic.

Theorem 20 *Let $G = (V, E)$ be a graph of order n and minimum degree $\delta \geq 2$. Then*

$$\gamma_r(G) \leq n(1 + (\frac{1}{\delta})^{\frac{\delta}{\delta-1}} - (\frac{1}{\delta})^{\frac{1}{\delta-1}})$$

Proof. Let $\pi = 1 - (\frac{1}{\delta})^{\frac{1}{\delta-1}}$. Since $\delta \geq 2$, we have $\delta \leq 2^{\delta-1}$, so that $\frac{1}{\delta} \geq \frac{1}{2^{\delta-1}}$, i.e. $1 - (\frac{1}{\delta})^{\frac{1}{\delta-1}} \leq \frac{1}{2}$. Hence, $\pi \leq \frac{1}{2}$. But then $2\pi \leq 1$, so that $\pi \leq 1 - \pi$. Construct a restrained dominating set for G as follows. Take each vertex independently with probability π. Call the resulting set of vertices A. The expected value of $|A|$ is $n\pi$. Let $B = \{x \in N(A) - A |$ there exists $y \in N(A) - A$ such that $xy \in E(G)\}$, let $C = N(A) - A - B$ and let $D = V - N[A]$. Then $S = A \cup C \cup D$ is a restrained dominating set. A vertex is in C if and only if there exists $\ell \geq 1$ such that ℓ of its neighbors are in A and the remaining $\deg(v) - \ell$ of its neighbors are in D. So, $P(v \in C) = (1-\pi)(1-\pi)^{\deg(v)-\ell}\pi^{\ell} = (1-\pi)^{\deg(v)-\ell+1}\pi^{\ell-1}\pi \leq (1-\pi)^{\deg(v)-\ell+1}(1-\pi)^{\ell-1}\pi = (1-\pi)^{\deg(v)}\pi \leq (1-\pi)^{\delta}\pi$. This means that the expected value of $|C|$ is at most $n(1-\pi)^{\delta}\pi$. Also, a vertex v is in D if and only if neither it nor any of its neighbors is in A. So, $P(v \in D) = (1-\pi)^{1+\deg(v)} \leq (1-\pi)^{1+\delta}$. Hence, the expected value of $|D|$ is at most $n(1-\pi)^{1+\delta}$. Therefore, $E(|S|) \leq n((1-\pi)^{\delta}\pi + (1-\pi)^{\delta+1} + \pi) = n((1-\pi)^{\delta}(\pi + (1-\pi)) + \pi) = n((1-\pi)^{\delta} + \pi) = n((\frac{1}{\delta})^{\frac{\delta}{\delta-1}} + 1 - (\frac{1}{\delta})^{\frac{1}{\delta-1}})$. □

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