On The Total Chromatic Number of Steiner Systems

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Abstract

We give a conjecture for the total chromatic number χ_T of all Steiner systems and show its relationship to the celebrated Erdős, Faber, Lovász conjecture. We show that our conjecture holds for projective planes, resolvable Steiner systems and cyclic Steiner systems by determining their total chromatic number.

1 Introduction

1.1 Hypergraphs

A *hypergraph* H is a pair $(V(H), E(H))$, where $V(H)$ is a finite set of *vertices* and $E(H)$ is a finite family of non-empty subsets of $V(H)$ called *hyperedges* or *blocks*, with $\bigcup_{E\in E(H)} E = V(H)$. H is *linear* if for all distinct $E, E' \in E(H)$, $|E \cap E'| \leq 1$, so for a linear hypergraph there may be no repeated hyperedges of cardinality greater than one. Distinct vertices $v, v' \in V(H)$ are *adjacent* if there is some hyperedge $E \in E(H)$ with $v, v' \in E$. Distinct hyperedges $E, E' \in E(H)$ are *adjacent* if $E \cap E' \neq \emptyset$. Vertex $v \in V(H)$ is *incident* with hyperedge $E \in E(H)$, and vice versa, if $v \in E$.

The *dual* of $H = (\{v_1, v_2, \ldots, v_n\}, [E_1, E_2, \ldots, E_m])$, H^*, is the hypergraph whose vertices $\{e_1, e_2, \ldots, e_m\}$ correspond to the hyperedges of H, and with hyperedges

$$V_i = \{e_j : v_i \in E_j \text{ in } H\} \qquad (i = 1, 2, \ldots, n).$$

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The *rank* of H, rank(H), is the maximum cardinality of a hyperedge in $E(H)$. A hyperedge of rank one is a *loop*. If all hyperedges of a hypergraph H have the same cardinality r then we say that H is r-*uniform*. The *degree* of a vertex $v \in V(H)$, $\deg_H(v)$, is the number of hyperedges containing v. The maximum degree among vertices of H is denoted $\Delta(H)$. If all vertices of a hypergraph H have the same degree Δ then we say that H is Δ-*regular*.

The *neighbour set* of a vertex $v \in V(H)$, $N_H(v)$, is the set of vertices adjacent to v. For $S \subseteq V(H)$, define $N_H(S) = \bigcup_{w \in S} N_H(w)$.

The *2-section* of H, H_2, is the simple graph with vertex set $V(H)$ where distinct $x, y \in V(H)$ are adjacent in H_2 if and only if they are adjacent in H. The *line graph* of H, $L(H)$, is the simple graph with vertex set $E(H)$ where distinct $E, E' \in E(H)$ are adjacent in $L(H)$ if and only if E and E' are adjacent in H. Then $L(H)$ is isomorphic to $(H^*)_2$. The *incidence graph* of H, $I(H)$, is the bipartite graph with vertices $V(H) \cup E(H)$ and bipartition $(V(H), E(H))$ where $v \in V(H)$ is adjacent to $E \in E(H)$ if and only if v is contained in the hyperedge E of H. Then $I(H)$ is isomorphic to $I(H^*)$ and $I(H)$ uniquely defines H up to isomorphism and duality.

A *(strong) vertex colouring* of hypergraph H is a mapping $C : V(H) \to \{1, 2, \ldots, k\}$ such that every pair of adjacent vertices receives different colours. The smallest k for which a vertex colouring exists is the *chromatic number* $\chi(H)$. A *(hyper)edge colouring* of H is a mapping $C : E(H) \to \{1, 2, \ldots, k_e\}$ such that every pair of adjacent hyperedges receives different colours. For each i the set of edges coloured i forms a *matching*. The smallest k_e for which such a colouring exists is the *(hyper)edge chromatic number* $\chi_e(H)$. A *total colouring* of H is a mapping $C : (V(H) \cup E(H)) \to \{1, 2, \ldots, k_T\}$ such that every pair of adjacent vertices, every pair of adjacent hyperedges and every incident vertex and hyperedge receive different colours. The smallest k_T for which such a colouring exists is the *total chromatic number* $\chi_T(H)$. Note that a total colouring of H defines a total colouring of H^*, hence $\chi_T(H) = \chi_T(H^*)$. This "self-duality" is a very useful properties of hypergraph total colourings of hypergraphs. Note also that since a total colouring induces both a vertex colouring and a hyperedge colouring then $\chi_T(H) \geq \max\{\chi^s(H), \chi_e(H)\}$ for all H.

The study of the total chromatic number for hypergraphs and in particular linear hypergraphs, is motivated in part by the total colouring conjecture, posed independently by Behzad [1] and Vizing [18]:

Total Colouring Conjecture (Behzad, Vizing). *Let G be a simple graph. Then*

$$\chi_T(G) \leq \Delta(G) + 2. \square$$

A stronger conjecture for hypergraphs was given in [7].

Total Colouring Conjecture for Hypergraphs. *Let H be a linear hypergraph without loops or vertices of degree one. Then*

$$\chi_T(H) \leq \min\{\Delta(H_2), \Delta(L(H))\} + 2.\Box$$

Evidence for Behzad and Vizing's total colouring conjecture for graphs has been gathered in two principle ways, first by proving the conjecture true for a wide range of classes of graphs and secondly by bounding the total chromatic number for all graphs. Recent surveys of graph total colouring are given in Hind [13] and in Yap's book [19]. In [6], [15] results are proven about total chromatic numbers of specific classes of hypergraphs. Upper bounds on the total chromatic number of all hypergraphs are given in [7]. Results concerning the structure of the total graph for hypergraphs are given in [8].

Total colourings of hypergraphs are related to a graphical model for radio frequency assignment. In [12, 20] we encounter the $L(a_1, a_2)$-labelling paradigm. Given simple graph G an $L(a_1, a_2)$-labelling assigns a number from the interval $[1, q]$ to each vertex such that for adjacent vertices, where there is significant radio interference, the labels must be at least a_1 apart and for vertices at distance two apart, where there are minor radio interference effects, the labels must be at least a_2 apart. The $L(a_1, a_2)$-labelling number is the smallest q which admits such a labelling, corresponding to the most efficient use of the radio spectrum. Then the total colouring of a hypergraph H corresponds to the $L(1, 1)$-labelling number of its incidence graph $I(H)$.

1.2 Steiner Systems

A *Steiner system* $S(2, k, v)$ is a linear k-uniform hypergraph on v vertices such that each pair of vertices is contained in exactly one hyperedge. Hence a Steiner system $S(2, k, v)$ has $\frac{v(v-1)}{k(k-1)}$ hyperedges and is regular of degree $\frac{v-1}{k-1}$. Clearly v and k must be chosen so that these quantities are integers for an $S(2, k, v)$ to be possible.

A Steiner systems for which $v = k$, which therefore have only one hyperedge has $\chi_T(H) = v + 1$. When k=2 we have the complete graphs. The total chromatic number for complete graphs is given in [2]. We will consider only Steiner systems with hyperedge cardinality $k \geq 3$ in the remainder of this paper.

In this paper we find the total chromatic number for three particular classes of Steiner systems: finite projective planes, resolvable Steiner systems and cyclic Steiner systems.

The *finite projective plane* of rank r is a hypergraph having $r^2 - r +$

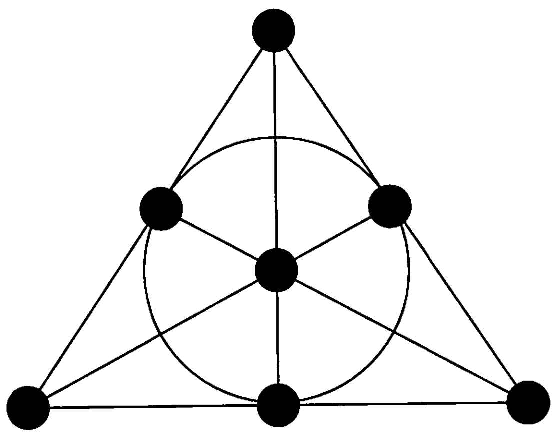

Figure 1: The projective plane of rank 3 or Fano configuration.

1 vertices ("points") and $r^2 - r + 1$ hyperedges ("lines"), satisfying the following axioms:

1. every point belongs to exactly r lines;
2. every line contains exactly r points;
3. two distinct points are in one and only one line;
4. two distinct lines have exactly one point in common;

We show in figure 1.2 a common representation of the "Fano configuration" which is the projective plane of rank 3. If H is a finite projective plane, then from the definition it is clear that H is self-dual, i.e. H and H^* are isomorphic. Projective planes do not exist for any choice of rank r, but it is known that they certainly do exist for $r = p^\alpha + 1$ for prime p (see [3]).

A *resolvable* Steiner system is a Steiner system whose edges have a partition into perfect matchings. In order that a Steiner system $S(2, k, v)$ is resolvable, we must have that k divides v. We have immediately that a resolvable Steiner system has edge chromatic number $\frac{v-1}{k-1}$. An example of a resolvable Steiner system is the *affine plane*. The affine plane of rank r is the hypergraph obtained from the projective plane of rank $r + 1$ by deleting a hyperedge and all vertices contained in the hyperedge. In figure 1.2 we illustrate the affine plane of rank 3 together with a resolution of its hyperedges.

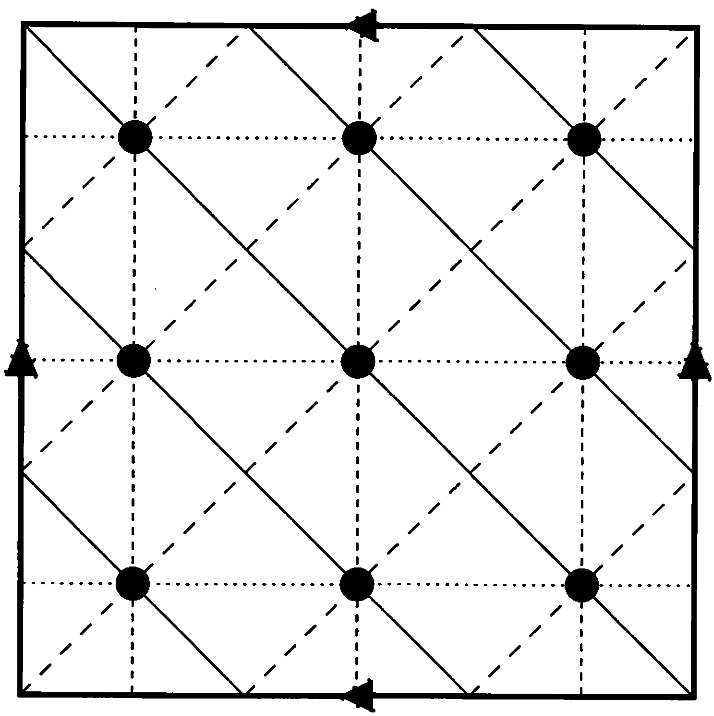

Figure 2: The affine plane of rank 3.

A Steiner system $S(2,k,v)$ is *cyclic* if its vertex set is $\{0,1,\ldots,v-1\}$ and the mapping $i \to i+1 \pmod v$ is an automorphism. Colbourn and Colbourn showed in [4] that the edge chromatic number of a cyclic Steiner system on $S(2,k,v)$ is at most v by constructing an edge colouring. We extend their result to show that a cyclic Steiner system on v vertices has *total* chromatic number at most v.

2 A Colouring Conjecture Related to the Erdős, Faber, Lovász Conjecture

Since every pair of vertices is contained in some hyperedge, the strong vertex chromatic number of a Steiner system on v vertices is v. Thus the total chromatic number of a Steiner system on v vertices is at least v. We present evidence that in fact for all Steiner systems $S(2,k,v)$ there is a total colouring using v colours. So we have

Conjecture 1 *Let H be a Steiner system $S(2,k,v)$, with $3 \le k < v$, then $\chi_T(H) = v$.*□

In 1972, Erdős, Faber and Lovász presented a conjecture concerning the vertex chromatic number of an apparently simple class of graphs, consisting of n copies of the complete graph K_n where each pair of complete graphs has at most one vertex in common. They conjectured that the members of this class may be vertex coloured using n colours, which is clearly the minimum possible. Prior to his death Erdős offered \$500 for resolution of the conjecture [9]. In [14], Hindman observed that we may use hypergraph duality to rewrite this conjecture as:

Erdős, Faber, Lovász conjecture (hypergraph edge colouring form). *Let H be a linear hypergraph on n vertices with $\Delta(H) \le n$. Then $\chi_e(H) \le n$.*□

Here $\Delta(H) \le n$ is simply a technical condition to restrict the number of loops incident with each vertex. These conjectures are related by the following lemma

Lemma 1 *Let H be a linear hypergraph on n vertices with $\Delta(H) \le n$. Then there exists a linear hypergraph H' on $n+1$ vertices with $\Delta(H') \le n+1$ such that $\chi_T(H) \le \chi_e(H')$.*

Proof: H' is constructed from H by adding a new vertex v to H and by adding a 2-edge from v to every vertex of H. Then if we have a hyperedge colouring of H', we may obtain a total colouring of H as follows: the colouring of hyperedges of H is simply the colouring of H' restricted to the edges of H, then we may colour each vertex w of H with the colour of $\{v, w\}$ in the hyperedge colouring of H'.□

Hence if the Erdős, Faber, Lovász conjecture were to be proven true then at most $v+1$ colours would suffice to total colour a linear hypergraph, and more particularly a Steiner system, on v vertices. If conjecture 1 were to be proven true then this would prove the Erdős, Faber, Lovász conjecture for an important class of hypergraphs. Erdős expressed interest in this as a possible angle of attack [9]. This might use strong structural results for hypergraph total colouring such as those in [8]. We will show in this paper that conjecture 1 holds for projective planes, resolvable Steiner systems and cyclic Steiner systems.

Let H be a Steiner triple system $S(2,3,v)$. In [11] Gionfriddo and Tuza show that for the *hereditary closure* $\hat{H}$ of a cyclic, resolvable Steiner triple system, we have $\frac{3v-1}{2} \leq \chi_T(\hat{H}) \leq \frac{3v-1}{2}+3$. Recall that a hypergraph H has the edge colouring property if $\chi_e(H) = \Delta(H)$, i.e. if H has a resolution. In the same paper it is shown that for H a resolvable Steiner system $S(2,k,v)$, its hereditary closure $\hat{H}$ also has the edge colouring property. A *nearly resolvable* Steiner system $H = (\mathcal{V}, \mathcal{E})$ is a Steiner system such that for some vertex $x \in \mathcal{V}$, $\mathcal{E}_x = \{E \in \mathcal{E} : x \notin E\}$ has a resolution into parallel classes. In [10] Gionfriddo and Milici show that the hereditary closure of a nearly resolvable Steiner system has the edge colouring property.

In this paper we will show that for three particular classes of Steiner systems conjecture 1 holds: finite projective planes, resolvable Steiner systems and cyclic Steiner systems.

3 Preliminaries

The results of this paper arise from the construction of the following bipartite graph:

Definition 1 *Let $H = (\mathcal{V}, \mathcal{E})$ be an edge coloured hypergraph with edge colour classes $M_1, M_2, \ldots, M_\gamma$ and let $\mathcal{M} = \{M_1, M_2, \ldots, M_\gamma\}$. Define the* colour class non-incidence *graph of edge-coloured H to be the bipartite graph B with vertex set $\mathcal{M} \cup \mathcal{V}$ where $(M_i, v_j) \in E(B) \Leftrightarrow v_j$ is* not *in any of the hyperedges of M_i.*□

Then we have the following lemma:

Lemma 2 *Let $H = (\mathcal{V}, \mathcal{E})$ be a Steiner system $S(2,k,v)$. Suppose that H has a hyperedge colouring with at most v colours and colour classes $\mathcal{M} = \{M_1, M_2, \ldots, M_\gamma\}$ such that there is a perfect matching from $\mathcal{M}$ into $\mathcal{V}$ in the corresponding colour class non-incidence graph. Then $\chi_T(H) = v$.*

Proof: First colour all vertices with distinct colours. Then reassign each hyperedge colour class with the colour of the vertex to which it is matched.□

Using Hall's theorem for bipartite matching together with the above lemma gives us:

Corollary 1 *Let $H = (\mathcal{V}, \mathcal{E})$ be a Steiner system $S(2,k,v)$. Suppose that there is an edge colouring of H with colour classes $\mathcal{M} = \{M_1, M_2, \ldots, M_\gamma\}$ with $\gamma \leq v$. Suppose further that for each $j = 0, 1, 2, \ldots \frac{v-1}{k-1} - 1$,*

$$|\{i : k\,|M_i| \geq v - j\}| \leq j.$$

Then $\chi_T(H) = v$.

Proof: Consider the colour class non-incidence graph B. For each $j = 0, 1, 2, \ldots, \frac{v-1}{k-1} - 1$ the hypotheses ensure that we have at most j vertices of $\mathcal{M}$ with degree less than or equal to j in B. Consider arbitrary $S \subseteq \mathcal{M}$ with $|S| = j$ for $0 \leq j \leq \frac{v-1}{k-1} - 1$. Let $j' = \max\{M_i \in S : \deg_B(M_i)\}$. Then the hypotheses ensure that $j \leq j'$ and thus $|N_B(S)| \geq |S|$.

If we have a subset $S \subseteq \mathcal{M}$ with $|S| = \frac{v-1}{k-1}$ then it must contain some M_i of degree at least $\frac{v-1}{k-1}$ so $|N_B(S)| \geq \frac{v-1}{k-1} \geq |S|$.

Since H is a Steiner system for each vertex $w \in \mathcal{V}$, w has degree $\frac{v-1}{k-1}$ in H and degree $\gamma - \frac{v-1}{k-1}$ in B. Thus any $S \subseteq \mathcal{M}$ with $|S| > \frac{v-1}{k-1}$ will have $|N_B(S)| = v \geq |S|$.

Hence Hall's condition is satisfied and $\chi_T(H) = v$ by lemma 2.□

We may state the above corollary more generally for all hypergraphs, but we will not require this generality in the following. We note a useful simplification of the above corollary. Let H be a Steiner system $S(2,k,v)$.If we have a hyperedge colouring of H with at most v colour classes where the union of the hyperedges of each colour class contains at most $v - \frac{v-1}{k-1}$ vertices then $\chi_T(H) = v$.

In [20], Yeh shows that if G is the bipartite incidence graph of a finite projective plane H on n vertices, then there is an $L(2,1)$ labelling of G using n labels, giving a total colouring using n colours. Here we use the above lemma to provide a simpler proof.

Theorem 1 *Let H be a finite projective plane of rank $k \geq 3$, which thus has $v = k^2 - k + 1 = (k-1)^2 + k$. Then $\chi_T(H) = v$.*

Proof: Since H is intersecting, in any proper hyperedge colouring, each hyperedge colour class will contain only a single edge. So the union of the hyperedges of each colour class contains k vertices. Now

$$v - \frac{v-1}{k-1} = (k-1)^2 > k \text{ for } k \geq 3$$

and so by corollary 1 we have that $\chi_T(H) = v$.□

4 χ_T for Resolvable Steiner Systems

To prove the result for resolvable Steiner systems we make use of the following result due to Tits ([16]). Its proof is taken from [17].

Lemma 3 *Let $H = (\mathcal{V}, \mathcal{E})$ be a Steiner system $S(2,k,v)$, for $3 \leq k < v$, then*

$$v \geq 3(k-1)$$

Proof: Choose a set S of three vertices which are not contained in any hyperedge of H. For each set $T \subsetneq S$ of cardinality 2, there is exactly one hyperedge E_T containing T, and any vertex not in S is incident with at most one such edge, since any two such hyperedges already have a vertex in common in S. Hence the union of all three such hyperedges E_T contains $3 + 3(k-2)$ distinct vertices and we have the result.□

Theorem 2 *Let $H = (\mathcal{V}, \mathcal{E})$ be a resolvable Steiner system $S(2,k,v)$ with $k \geq 3$ and $v > k$, then $\chi_T(H) = v$.*

Proof: H has $\frac{v(v-1)}{k(k-1)}$ edges, which may be partitioned into $\frac{v-1}{k-1}$ perfect matchings, each with $\frac{v}{k}$ hyperedges. Split each matching into two almost equal halves, containing $\lfloor \frac{v}{2k} \rfloor$ hyperedges and $\lceil \frac{v}{2k} \rceil$ hyperedges. These will be our hyperedge colour classes. Let $\mathcal{M} = \{M_1, M_2, \ldots, M_{2\frac{v-1}{k-1}}\}$ be the set of these "half" matchings, where

$$\begin{array}{ll} |M_{2i-1}| = \lfloor \frac{v}{2k} \rfloor & i = 1,2,\ldots,\frac{v-1}{k-1} \\ |M_{2i}| = \lceil \frac{v}{2k} \rceil & i = 1,2,\ldots,\frac{v-1}{k-1} \\ M_{2i-1} \cup M_{2i} \text{ is a perfect matching} & i = 1,2,\ldots,\frac{v-1}{k-1} \end{array}$$

We will show that the set $\mathcal{M}$ of hyperedge colour classes can be constructed in such a way as to satisfy the requirements of corollary 1 or lemma 2.

First note that for $i = 1,2,\ldots,2\frac{v-1}{k-1}$

$$\left| \bigcup_{E \in M_i} E \right| \leq \frac{v+k}{2}.$$

So to satisfy the requirements of corollary 1 it suffices to show that

$$\frac{v+k}{2} \leq v - \frac{v-1}{k-1}$$

or equivalently that

$$v\frac{k-3}{2} - \frac{k(k-1)}{2} + 1 \geq 0. \quad (1)$$

We consider three cases:

1. **$k \geq 5$.** Using lemma 3 we have that

$$v\frac{k-3}{2} - \frac{k(k-1)}{2} + 1 \geq 3(k-1)\frac{k-3}{2} - \frac{k(k-1)}{2} + 1$$

but

$$3(k-1)\frac{k-3}{2} - \frac{k(k-1)}{2} + 1 = k^2 - \frac{11}{2}k + \frac{11}{2} \geq 0 \text{ for } k \geq 5.$$

2. **$k = 4$.** Putting $k = 4$ into inequality (1) yields the requirement that $v \geq 10$ for the conditions of corollary 1 to be satisfied. Noting that by lemma 3 we must have $v \geq 3(k-1) = 9$ and that there does not exist an $S(2,4,9)$ completes the proof in this case.

3. **$k = 3$.** Here we must have that 2 divides $v-1$ and that 3 divides v, hence $v \equiv 3 (\text{mod } 6)$. Let B be the colour class non-incidence graph associated with the colouring as given above. Then

$$\deg_B(M_i) = \begin{cases} \frac{v+3}{2} & i = 1,3,5,\ldots,2\frac{v-1}{k-1}-1 \\ \frac{v-3}{2} & i = 2,4,6,\ldots,2\frac{v-1}{k-1} \end{cases}$$

Since for $w \in \mathcal{V}$, $\deg_B(w) = \frac{v-1}{2}$, if we consider a set $S \subseteq \mathcal{M}$ with more than $\frac{v-1}{2}$ vertices, we have that $|N_B(S)| = v > |S|$ as in corollary 1. Any subset S' of $\mathcal{M}$ with at most $\frac{v-3}{2}$ vertices will satisfy $|N_B(S')| \geq |S'|$. Any subset $S'' \subseteq \mathcal{M}$ of cardinality $\frac{v-1}{2}$ which contains some vertex $M_i (i = 1,3,5,\ldots,2\frac{v-1}{k-1}-1)$ will satisfy $|N_B(S)| \geq |S|$. Suppose that the set $S = \{M_2, M_4, M_6, \ldots, M_{2\frac{v-1}{k-1}}\}$ has $|N_B(S)| = \frac{v-3}{2}$. Then we change slightly our set of "half matchings". Let $E \in M_1$. We will get the set of "half matchings" $\{M_1', M_2', \ldots, M_{2\frac{v-1}{k-1}}'\}$,

where

$$\begin{aligned} M_1' &= M_2 \cup \{E\} \\ M_2' &= M_1 - \{E\} \\ M_i' &= M_i \qquad \text{for } i = 3,4,\ldots,2\tfrac{v-1}{k-1} \end{aligned}$$

In the corresponding colour class non-incidence graph B' we will then have $|N_{B'}(S'')| = \frac{v+3}{2} \geq |S''|$. Hence Hall's condition is satisfied and there is a perfect matching from $\mathcal{M}$ to $\mathcal{V}$ in B (or B'). Using lemma 2 completes the proof.□

5 χ_T for Cyclic Steiner Systems

In order to prove the result for cyclic Steiner systems we will require a few more preliminaries concerning them. See [4],[5] for further details. Let H be a cyclic Steiner system $S(2,k,v)$. Then if the vertex set of H is $\{0,1,\ldots,v-1\}$ the mapping $i \to i+1$ is an automorphism. The automorphism partitions the blocks of the Steiner system into *orbits*. When $v \equiv 1 \pmod{k(k-1)}$ each orbit contains exactly v blocks. When $v \equiv k \pmod{k(k-1)}$ each orbit except one contains v blocks. The exception, the short orbit, contains $\frac{v}{k}$ blocks. All of these preliminaries are taken from [5].

The study of cyclic Steiner systems and difference families are inextricably linked. A $(v,k,1)$ *difference set* $D = \{d_1, d_2, \ldots, d_k\}$ is a set of k residues modulo v such that for any residue $x \not\equiv 0 \pmod{v}$ the congruence $d_i - d_j \equiv x \pmod{v}$ has exactly one solution pair (d_i, d_j) with $d_i, d_j \in D$. Every $(v,k,1)$ difference set generates a cyclic Steiner system $S(2,k,v)$, whose blocks are $B(i) = \{d_1 + i, d_2 + i, \ldots, d_k + i\} \pmod{v}$, $i = 0, 1, \ldots v-1$. The difference set is then referred to as the *starter block* of the Steiner system. A $(v,k,1)$ *difference family* is a collection of such sets $D_1, D_2, \ldots, D_n$ each of cardinality k such that each residue $x \not\equiv 0 \pmod{v}$ has exactly one solution pair (d_i, d_j) with $d_i, d_j \in D_m$ from some m. Each $(v,k,1)$ difference family generates a cyclic Steiner system in the same manner as before. For example the difference family $(0,1,4), (0,2,7)$ generates the cyclic Steiner system $S(2,3,13)$. Using this definition, an $S(2,3,15)$ design cannot be represented as a difference family. However, this Steiner system may be generated by two starter blocks modulo 15, together with the *extra starter block* $(0,5,10)$, which gives rise to an orbit of the automorphism group containing 3 blocks.

Consider a long orbit of a cyclic Steiner system $S(2,k,v)$, generated by the starter block $D = (d_1, d_2, \ldots, d_k)$. We require two properties of the block adjacency graph G_D for this orbit. First note that this graph is $k(k-1)$-regular. Then if I is an independent set in this G_D we have

$$|\{vw \in E(G_D) : v \in I, w \notin I\}| = k(k-1)\,|I| \leq |E(G_D)| = \frac{vk(k-1)}{2}$$

so we must have $|I| \leq \frac{v}{2}$. Second note that for G_D to consist of $(k(k-1)+1)$-cliques, we must have $v = l(k(k-1)+1)$ for some integer l and then $D = (la_1, la_2, \ldots, la_k)$ for $(a_1, a_2, \ldots, a_k)$ a difference set modulo $k(k-1)+1$. Note that l must occur in the difference set D in this case, so this may happen for at most one starter block of the difference family for given $S(2,k,v)$. See [5] for further results concerning cyclic Steiner systems.

We are now ready to prove the main result.

Theorem 3 *Let $H = (\mathcal{V}, \mathcal{E})$ be a cyclic Steiner system $S(2,k,v)$ with $k \geq 3$ and $v > k$, then $\chi_T(H) = v$.*

Proof: We will construct a hyperedge colouring which satisfies the conditions of corollary 1. Consider the two cases:

1. $v \equiv 1 \pmod{k(k-1)}$. For each of the orbits of blocks construct the block intersection graph. This graph is regular of degree $k(k-1)$. Using Brooks' theorem this graph may be coloured using at most $k(k-1)$ colours unless it is composed of $k(k-1)+1$-cliques. At most one orbit may yield such a graph.

   We have $\frac{v-1}{k(k-1)}$ orbits and each may be coloured using $k(k-1)$ colours except one which may require $k(k-1)+1$ colours. Hence v colours will suffice to colour the hyperedges. In the above discussion we have shown that the hyperedges of each colour class contain at most $\frac{v}{2}$ vertices and we have
$$v - \frac{v-1}{k-1} > \frac{v}{2}$$
   for $v \geq 3$. Hence by corollary 1, $\chi_T(H) = v$.

2. $v \equiv k \pmod{k(k-1)}$. Here we have $\frac{v-k}{k(k-1)}$ full orbits and one short orbit. The full orbits may be coloured as above. Hence the hyperedges of each full orbit colour class will contain less than $v - \frac{v-1}{k-1}$ vertices. The short orbit may be coloured using two colours. One colour will colour a single edge of the short orbit and the other colours the remainder. Hence the hyperedges of all colour classes contain at most $v - \frac{v-1}{k-1}$ vertices except one whose hyperedges contain $v-k$ vertices. Hence by corollary 1, $\chi_T(H) = v$.□

6 Conclusion and Open Problems

The Erdős, Faber and Lovász conjecture remains one of the outstanding unsolved problems of combinatorial mathematics. In this paper we have presented a conjecture relating the Erdős, Faber and Lovász conjecture to one for hypergraph total colouring, and provided evidence for our conjecture by determining the total chromatic number for projective planes, resolvable Steiner systems and cyclic Steiner systems. It would be interesting to prove the truth of conjecture 1 for a wider range of Steiner systems, or to find a counterexample. It is our hope that by consideration of total colouring of hypergraphs we may open up a new and interesting angle of attack for the Erdős, Faber Lovász conjecture.

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References

[1] M. Behzad. *Graphs and their Chromatic Numbers.* PhD thesis, Michigan State University, 1965.

[2] M. Behzad, G. Chartrand, and J. K. Cooper. The colour numbers of the complete graphs. *Journal of the London Mathematical Society*, 44:226–228, 1967.

[3] Claude Berge. *Hypergraphs: Combinatorics of Finite Sets.* North-Holland, Amsterdam, 1989.

[4] C. J. Colbourn and M. J. Colbourn. The chromatic index of cyclic Steiner 2-designs. *International Journal of Mathematics and Mathematical Sciences*, 5:823–825, 1982.

[5] M. J. Colbourn and R. A. Mathon. On cyclic Steiner 2-designs. *Annals of Discrete Mathematics*, 7:215–253, 1980.

[6] P. Cowling. Strong total chromatic numbers of complete hypergraphs. *Discrete Mathematics*, 138:207–212, 1995.

[7] P. Cowling. Total colouring of hypergraphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, XIX:151–160, 1995.

[8] P. Cowling. The total graph of a hypergraph. *Discrete Mathematics*, 167/168:215–236, 1997.

[9] P. Erdős. Personal communication. 1993.

[10] M. Gionfriddo and S. Milici. A result concerning two conjectures of Berge and Chvátal. *Discrete Mathematics*, 124:79–86, 1994.

[11] M. Gionfriddo and Z. Tuza. On conjectures of Berge and Chvátal. *Discrete Mathematics*, 124:79–86, 1994.

[12] J. R. Griggs and R. Yeh. Labelling graphs with a condition at distance 2. *Siam Journal of Discrete Mathematics*, 5(4):586–595, 1992.

[13] H. R. Hind. Recent developments in total colouring. *Discrete Mathematics*, 125:211–218, 1994.

[14] N. Hindman. On a conjecture of Erdős, Faber and Lovász about n-colourings. *Canadian Jounal of Mathematics*, 33:563–570, 1981.

[15] J. C. Meyer. Nombre chromatique total d'un hypergraphe. *Journal of Combinatorial Theory, Series B*, 24:44-50, 1978.

[16] J. Tits. Sur les systèmes de Steiner associés aux trois 'grands' groupes de mathieu. *Rend. Math. e Appl.*, 23(5):166–184, 1964.

[17] J.H. van Lint and R.M. Wilson. *A Course in Combinatorics*. Cambridge University Press, 1992.

[18] V. G. Vizing. Some unsolved problems in graph theory. *Uspekhi Mat. Nauk*, 23:117–134, 1968.

[19] H. P. Yap. *Total colourings of graphs*. Lecture Notes in Mathematics 1623 (Springer-Verlag), 1996.

[20] R. Yeh. *Labelling Graphs with a Condition at Distance 2*. PhD thesis, University of South Carolina, 1990.