# All admissible 3-$(v, 4, \lambda)$ directed designs exist

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**Abstract**

In a $t$-$(v, k, \lambda)$ *directed design* the blocks are ordered $k$-tuples and every ordered $t$-tuple of distinct points occurs in exactly $\lambda$ blocks (as a subsequence). We show that a simple 3-$(v, 4, 2)$ directed design exists for all $v$. This completes the proof that the necessary condition $\lambda v \equiv 0 \pmod 2$ for the existence of a 3-$(v, 4, \lambda)$ directed design is sufficient.

## 1 Introduction

A $t$-$(v, k, \lambda)$ *directed design* is a pair $(\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of $v$ elements, called *points*, and $\mathcal{B}$ is a collection of ordered $k$-tuples of distinct elements of $\mathcal{P}$, called *blocks*, with the property that every ordered $t$-tuple of distinct elements of $\mathcal{P}$ occurs in exactly $\lambda$ blocks (as a subsequence). A $t$-$(v, k, \lambda)$ directed design with no repeated blocks is called *simple*. A $t$-$(v, k, 1)$ directed design is necessarily simple. Background information on directed designs is given in [2] and [3].

We usually specify a directed design by listing its blocks. For example, the following blocks form a 3-$(4, 4, 1)$ directed design:

$$(1,2,3,4),\ (2,1,4,3),\ (3,1,4,2),\ (4,2,3,1),\ (3,2,4,1),\ (4,1,3,2).$$

Here, for example, the block $(1, 2, 3, 4)$ contains the ordered triples $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$ and $(2, 3, 4)$.

A $t$-$(v, k, \lambda)$ directed design is *cyclic* if it has an automorphism which permutes its points in a cycle of length $v$. The base blocks below, developed modulo 6, form a cyclic 3-$(6, 4, 1)$ directed design. This design is given by Soltankhah [13].

$$(0,1,3,5),\ (0,4,2,1),\ (0,3,1,2),\ (0,5,1,4),\ (0,5,2,3).$$

The following result (which is straightforward to prove) gives necessary conditions for the existence of a $t$-$(v, k, \lambda)$ directed design.

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**Result 1.1** *Let $\mathcal{D}$ be a $t$-$(v,k,\lambda)$ directed design. Then $\mathcal{D}$ is an $s$-$(v,k,\lambda_s)$ directed design for $0 \le s < t$ where*

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s} t!}{\binom{k-s}{t-s} s!}.$$

*Hence $\lambda_s$ must be an integer for $s = 0, 1, 2, \ldots, t-1$.*

2-$(v,k,\lambda)$ directed designs have been studied quite extensively. For such designs, the necessary conditions of Result 1.1 reduce to $2\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $2\lambda(v-1) \equiv 0 \pmod{k-1}$. It has been shown [1, 7, 12, 15, 16] that for $k \in \{3,4,5,6\}$ these necessary conditions are sufficient, with two exceptions, namely that no directed designs with parameters 2-$(15,5,1)$ or 2-$(21,6,1)$ exist.

In this paper, we are concerned with 3-$(v,4,\lambda)$ directed designs. For these, the necessary conditions of Result 1.1 reduce to the condition $\lambda v \equiv 0 \pmod 2$. It has been shown, by Soltankhah [13] building on work of Levenshtein [9], that this necessary condition is sufficient for all values of $v$, except possibly $v \equiv 3$ and $11 \pmod{12}$.

Both Levenshtein and Soltankhah make use of the following result involving $t$-$(v,K,\lambda)$ designs. A $t$-$(v,K,\lambda)$ *design* is a pair $(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a set of $v$ elements, called *points*, and $\mathcal{B}$ is a collection of subsets of $\mathcal{P}$, called *blocks*, with the property that the size of every block is in the set $K$ and every $t$-element subset of $\mathcal{P}$ is contained in exactly $\lambda$ blocks. A $t$-$(v,K,\lambda)$ *design* with no repeated blocks is called *simple.*

**Result 1.2 (Replacement Lemma)** *If there exist a $t$-$(v,K,\lambda_1)$ design and a $t$-$(k',k,\lambda_2)$ directed design for each $k' \in K$, then there exists a $t$-$(v,k,\lambda_1\lambda_2)$ directed design. A sufficient condition for the resulting directed design to be simple is that all original designs be simple and either $K = \{k\}$ or $\lambda_1 = 1$.*

**Proof** Replacing each block of the $t$-$(v,K,\lambda_1)$ design with a copy of a directed $t$-$(k',k,\lambda_2)$ design with point set the points of that block gives a $t$-$(v,k,\lambda_1\lambda_2)$ directed design. The claim about simplicity is clear. □

Levenshtein's contribution to the result we mentioned earlier was to prove, using the replacement lemma, that a 3-$(v,4,1)$ directed design exists for all even $v$. His proof is essentially as follows. Hanani [4, 5] has shown that there exists a 3-$(v,4,1)$ design for $v \equiv 2$ or $4 \pmod 6$, and a 3-$(v,\{4,6\},1)$ design for $v \equiv 0 \pmod 6$. Hence, provided that there exist a 3-$(4,4,1)$ directed design and a 3-$(6,4,1)$ directed design, it follows using the replacement lemma that a 3-$(v,4,1)$ directed design exists for all even $v$. These two small designs do indeed exist: we gave them as examples earlier.

In a similar way, Soltankhah [13] uses the replacement lemma to deduce the existence of simple 3-$(v, 4, 2)$ directed designs for $v \equiv 1$ or 5 (mod 12) from the existence of simple 3-$(v, 4, 2)$ designs for these values of $v$. Except for the case $v = 13$, the existence of these latter designs follows from Theorem 1 of Khosrovshahi and Ajoodani-Namini [8]. The argument relies on the existence of a *large set* of mutually disjoint 2-$(u, 3, 1)$ designs; these exist for $u \equiv 1$ or 3 (mod 6), $u \neq 7$ [10, 11, 17]. The missing simple 3-$(13, 4, 2)$ design, corresponding to $u = 7$, appears in Hanani [5].

Soltankhah [13] also uses the replacement lemma to show that there exists a simple 3-$(v, 4, 2)$ directed design for all even $v$. In addition, she proves, using more complicated methods, that a simple 3-$(v, 4, 2)$ directed design exists for $v \equiv 7$ or 9 (mod 12).

Since $\lambda_1$ copies of a 3-$(v, 4, \lambda)$ directed design form a 3-$(v, 4, \lambda_1\lambda)$ directed design, these results imply the result we mentioned earlier; that is, there exists a 3-$(v, 4, \lambda)$ directed design for all $v$ and $\lambda$ satisfying the necessary condition $\lambda v \equiv 0 \pmod 2$, except possibly in the cases $v \equiv 3$ or 11 (mod 12).

In the next section we deal with the two remaining cases.

# 2 Main Theorem

In this section we complete the proof of the following theorem.

**Theorem 2.1** *There exists a simple* 3-$(v, 4, 2)$ *directed design for all* $v$.

This theorem, together with Levenshtein's theorem stating that a 3-$(v, 4, 1)$ directed design exists for all even $v$, immediately gives the following result.

**Theorem 2.2** *There exists a* 3-$(v, 4, \lambda)$ *directed design for all* $v$ *and* $\lambda$ *satisfying the necessary condition* $\lambda v \equiv 0 \pmod 2$.

Our method, which was suggested by Soltankhah [14], is to use the replacement lemma to deduce Theorem 2.1 from the following theorem of Hanani [6].

**Result 2.3** *There exists a* 3-$(v, \{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27, 29, 31\}, 1)$ *design for all* $v$.

Thus we need to show that a simple 3-$(v, 4, 2)$ directed design exists for all values of $v$ in the set $\{4, 5, 6, 7, 9, 11, 13, 15, 19, 23, 27, 29, 31\}$. All these values except $v = 11$, 15, 23 and 27 are covered by the results of Soltankhah [13] that we mentioned earlier. We now exhibit a simple 3-$(v, 4, 2)$ directed design for each of the four remaining values of $v$.

Developing the 9 base blocks below using the automorphism group $\{z \mapsto a^2z + b : a, b \in Z_{11}, a \neq 0\}$ yields a simple 3-(11, 4, 2) directed design.

(0, 1, 2, 4), (0, 1, 2, 5), (0, 1, 3, 5), (0, 1, 6, 9), (0, 1, 7, 8),
(0, 1, 7, 10), (0, 1, 8, 10), (1, 0, 7, 4), (1, 0, 8, 3).

Developing the 91 base blocks below modulo 15 yields a simple 3-(15, 4, 2) directed design.

| | | | | |
|---|---|---|---|---|
| (0, 1, 2, 3), | (0, 1, 3, 4), | (0, 1, 8, 7), | (0, 1, 8, 10), | (0, 1, 9, 14), |
| (0, 1, 10, 11), | (0, 1, 11, 12), | (0, 1, 12, 13), | (0, 1, 13, 14), | (0, 2, 1, 4), |
| (0, 2, 4, 1), | (0, 2, 5, 6), | (0, 2, 5, 7), | (0, 2, 8, 11), | (0, 2, 9, 12), |
| (0, 2, 10, 12), | (0, 3, 1, 5), | (0, 3, 1, 6), | (0, 3, 2, 6), | (0, 3, 2, 10), |
| (0, 3, 7, 9), | (0, 3, 8, 12), | (0, 3, 10, 13), | (0, 3, 11, 8), | (0, 3, 12, 11), |
| (0, 4, 3, 14), | (0, 4, 7, 2), | (0, 4, 7, 3), | (0, 4, 8, 13), | (0, 4, 10, 5), |
| (0, 4, 12, 5), | (0, 4, 13, 1), | (0, 4, 14, 2), | (0, 5, 2, 8), | (0, 5, 4, 6), |
| (0, 5, 4, 8), | (0, 5, 9, 2), | (0, 5, 10, 3), | (0, 5, 11, 14), | (0, 5, 12, 1), |
| (0, 5, 12, 7), | (0, 5, 13, 1), | (0, 5, 14, 11), | (0, 6, 2, 7), | (0, 6, 3, 7), |
| (0, 6, 4, 11), | (0, 6, 10, 2), | (0, 6, 11, 1), | (0, 6, 12, 4), | (0, 6, 12, 14), |
| (0, 6, 14, 13), | (0, 7, 1, 5), | (0, 7, 6, 8), | (0, 7, 8, 1), | (0, 7, 14, 11), |
| (0, 7, 14, 12), | (0, 8, 2, 9), | (0, 8, 3, 9), | (0, 8, 5, 3), | (0, 9, 1, 6), |
| (0, 9, 4, 10), | (0, 9, 8, 14), | (0, 9, 11, 7), | (0, 9, 13, 3), | (0, 10, 6, 5), |
| (0, 10, 7, 3), | (0, 10, 8, 2), | (0, 10, 9, 4), | (0, 10, 12, 6), | (0, 11, 2, 14), |
| (0, 11, 6, 5), | (0, 11, 6, 10), | (0, 11, 7, 13), | (0, 11, 8, 4), | (0, 11, 9, 3), |
| (0, 11, 9, 5), | (0, 12, 7, 4), | (0, 12, 9, 2), | (0, 12, 11, 3), | (0, 12, 14, 10), |
| (0, 13, 8, 6), | (0, 13, 8, 12), | (0, 13, 9, 11), | (0, 13, 10, 4), | (0, 13, 11, 4), |
| (0, 13, 12, 2), | (0, 13, 14, 5), | (0, 14, 7, 13), | (0, 14, 8, 4), | (0, 14, 8, 6), |
| (0, 14, 12, 9). | | | | |

Developing the 21 base blocks below using the automorphism group $\{z \mapsto a^2z + b : a, b \in Z_{23}, a \neq 0\}$ yields a simple 3-(23, 4, 2) directed design.

| | | | | |
|---|---|---|---|---|
| (0, 1, 2, 3), | (0, 1, 3, 4), | (0, 1, 4, 5), | (0, 1, 5, 6), | (0, 1, 6, 10), |
| (0, 1, 8, 11), | (0, 1, 8, 14), | (0, 1, 11, 19), | (0, 1, 12, 22), | (0, 1, 14, 12), |
| (0, 1, 17, 15), | (0, 1, 20, 18), | (1, 0, 3, 14), | (1, 0, 6, 4), | (1, 0, 11, 3), |
| (1, 0, 15, 8), | (1, 0, 17, 4), | (1, 0, 19, 6), | (1, 0, 20, 5), | (1, 0, 21, 22), |
| (1, 0, 22, 18). | | | | |

Developing the 25 base blocks below using the automorphism group $\{z \mapsto a^2z+b : a,b \in \mathrm{GF}(27), a \neq 0\}$ yields a simple 3-(27, 4, 2) directed design. Here $p+qx+rx^2$ with $p,q,r \in \mathrm{GF}(3)$ is represented as $p+3q+9r$. The irreducible polynomial used is $x^3-x-1$.

$$\begin{array}{ccccc}
(0,1,2,3), & (0,1,2,4), & (0,1,4,5), & (0,1,6,9), & (0,1,6,16), \\
(0,1,7,12), & (0,1,7,13), & (0,1,8,13), & (0,1,8,25), & (0,1,10,17), \\
(0,1,10,25), & (0,1,11,15), & (0,1,11,24), & (0,1,15,20), & (0,1,17,20), \\
(0,1,23,24), & (0,1,26,21), & (1,0,7,23), & (1,0,8,15), & (1,0,8,25), \\
(1,0,10,20), & (1,0,12,17), & (1,0,14,25), & (1,0,20,21), & (1,0,24,23).
\end{array}$$

This completes the proof of Theorem 2.1.

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