THE RAMSEY NUMBERS FOR A QUADRILATERAL VS. ALL GRAPHS ON SIX VERTICES.

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ABSTRACT. The Ramsey numbers $r(C_4, G)$ are determined for all graphs G of order six.

1. INTRODUCTION

For graphs G and H the Ramsey number $r(G, H)$ is the least number N such that in each two coloring (R, B) = (red,blue) of the edges of K_N there is a red copy of G or a blue copy of H. All triangle-graph Ramsey numbers for connected graphs of order six were found by Faudree, Rousseau and Schelp [7]. Subsequently exact values of $r(C_3, G)$ have been determined for $|G| \leq 8$. See Radziszowski [17] for a survey. Much less research has been done on $r(C_4, G)$. It has been shown by J.A. Bondy and P. Erdős [1] that $r(C_n, K_r) = (n-1)(r-1)+1$, if $n \geq r^2 - 2$ and Clancy (see [2]) has found all Ramsey numbers for a quadrilaterals vs. all graphs on five vertices. Also $r(C_4, K_6) \geq 18$ has been proved by Exoo (see [6]) and the authors have proved that $r(C_4, K_6) = 18$ [15]. There are 112 connected graphs and 44 disconnected graphs on six vertices. A few of the first numbers could be found using Ramsey numbers of C_4 vs. trees and books(see [8] and [16]). Most of the later Ramsey numbers are found using [2],[12],[4], [13] and [11].

2. MAIN RESULTS

We look for the numbers $r(C_4, G)$ where G has exactly six points. Also let G_i be the graphs consistent with the notation of [2]. Let $V = \{v_1, v_2, ..., v_p\}$ denote a set of vertices. Let $[v]^2 = (R, B)$ be a two coloring in which we ascribe to each edge of the complete graph of order p a color, either red or blue. This two-coloring defines two edge-induced graphs of order p and we use $< R >$ and $< B >$ as symbols for these graphs. The statement $K_p \to (F, G)$ means that if $|V| = p$, then for every possible two-coloring (R,B) of $[V]^2$, either $< R >$ contains (a subgraph isomorphic

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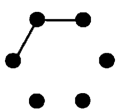
to) F or $< B >$ contains G. The Ramsey number $r(F, G)$ is the smallest natural number p such that $K_p \to (F, G)$.

The quadrilateral Ramsey number for graphs of order 6

$r(C_4, H) = 18$ if and only if $H = K_6 = M_1^c$.
$r(C_4, H) = 16$ if and only if $H = K_6 - e = M_2^c$.

$r(C_4, H) = 14$ if and only if H^c is one of the graphs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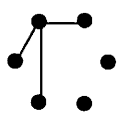
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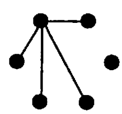
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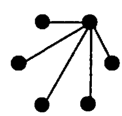
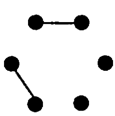

M_{31}

$r(C_4, H) = 13$ if and only if H^c is one of the graphs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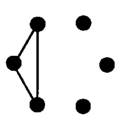
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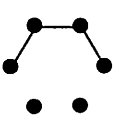
$r(C_4, H) = 11$ if and only if H^c is one of the graphs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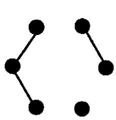
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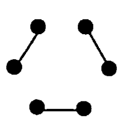
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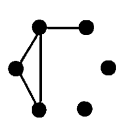
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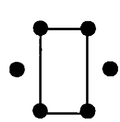
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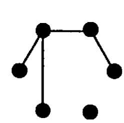
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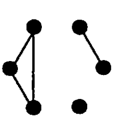
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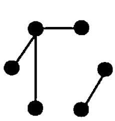
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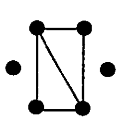
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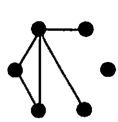
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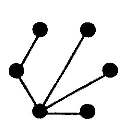
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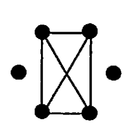
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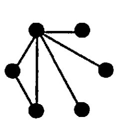
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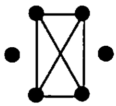
$r(C_4, H) = 10$ if and only if H is one of the graphs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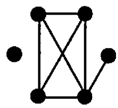
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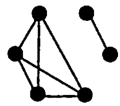
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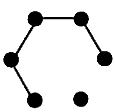
or H^c is one of the graphs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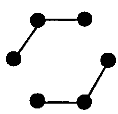
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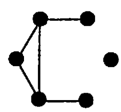
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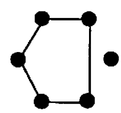
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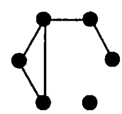
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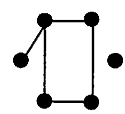
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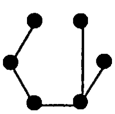
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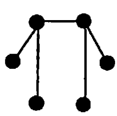
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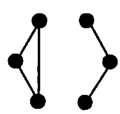
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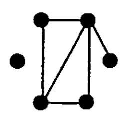
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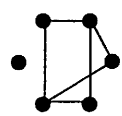
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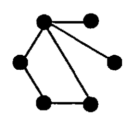
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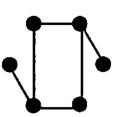
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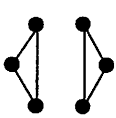
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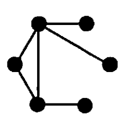
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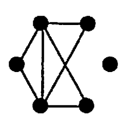
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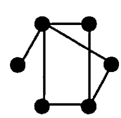
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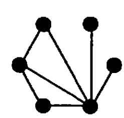
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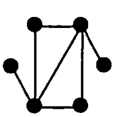
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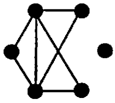
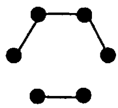
$r(C_4, H) = 9$ if and only if H is one of the graphs


M_{58}

or H^c is one of the graphs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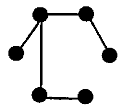
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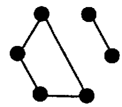
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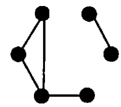
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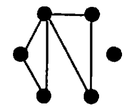
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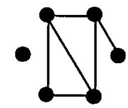
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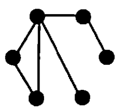
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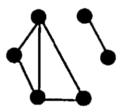
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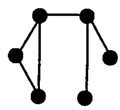
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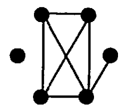
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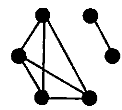
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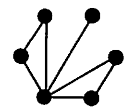
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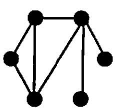
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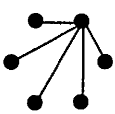
$r(C_4, H) = 8$ if and only if H is one of the graphs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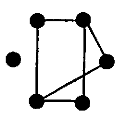
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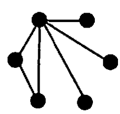
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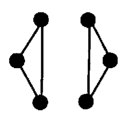
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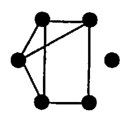
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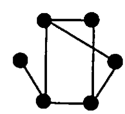
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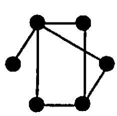
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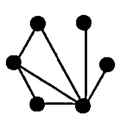
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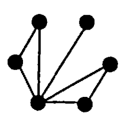
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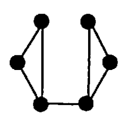
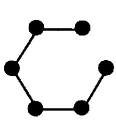
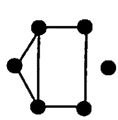

M_{77}

or H^c is one of the graphs


M_{25}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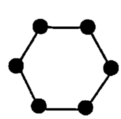
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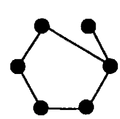
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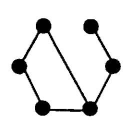
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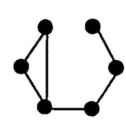
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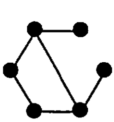
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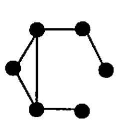
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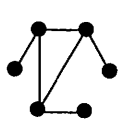
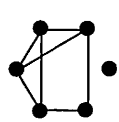

M_{51}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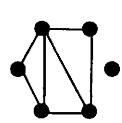
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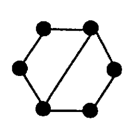
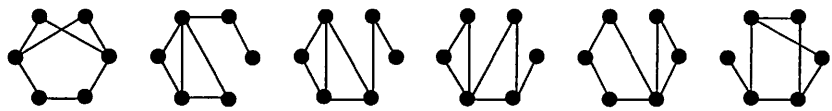
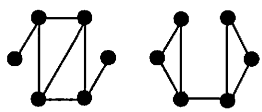

M_{59}

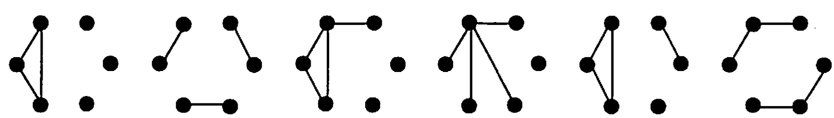


M_{61} M_{62} M_{63} M_{64} M_{67} M_{68}

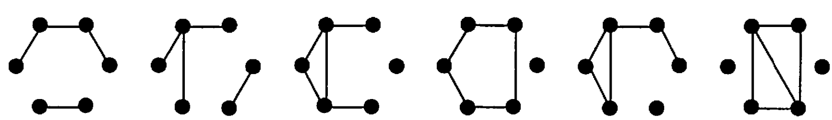


M_{75} M_{77}

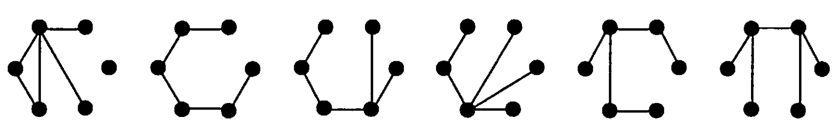
$r(C_4, H) = 7$ if and only if H is one of the graphs



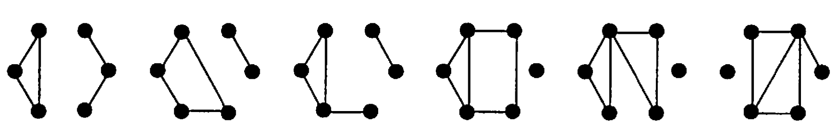
M_6 M_9 M_{10} M_{13} M_{15} M_{16}



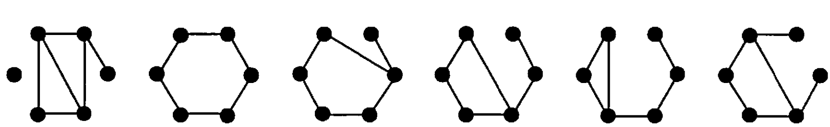
M_{17} M_{18} M_{19} M_{20} M_{21} M_{23}



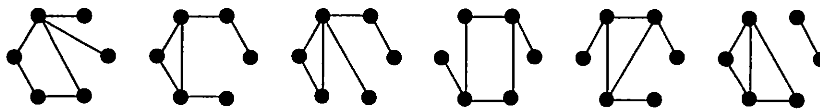
M_{24} M_{25} M_{26} M_{27} M_{28} M_{29}



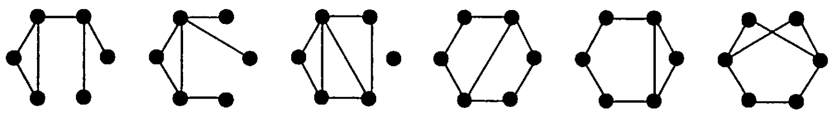
M_{30} M_{32} M_{33} M_{35} M_{36} M_{37}



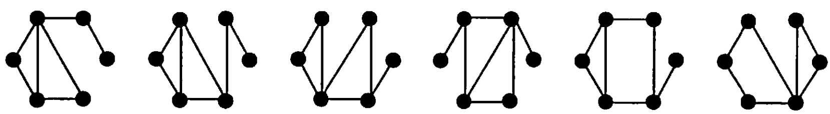
M_{38} M_{40} M_{41} M_{42} M_{43} M_{44}



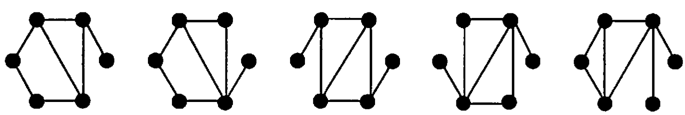
M_{45} M_{46} M_{47} M_{49} M_{51} M_{52}



M_{53} M_{54} M_{56} M_{59} M_{60} M_{61}

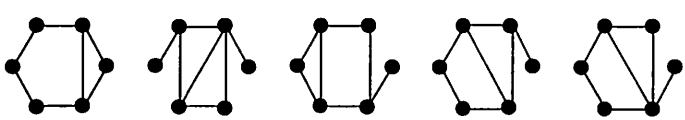


M_{62} M_{63} M_{64} M_{65} M_{66} M_{67}



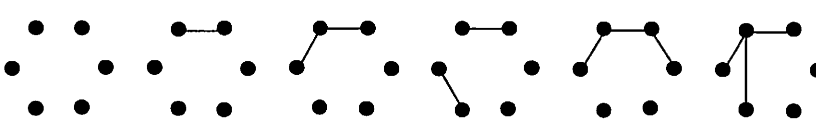
M_{70} M_{74} M_{75} M_{76} M_{78}

or H^c is one of the graphs

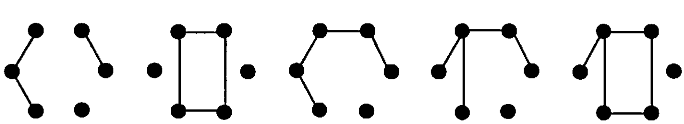


M_{60} M_{65} M_{66} M_{70} M_{74}

$r(C_4, H) = 6$ if and only if H is one of the graphs



M_1 M_2 M_3 M_4 M_5 M_7



M_8 M_{11} M_{12} M_{14} M_{22}

Remark: The proof of the main result follows by the following sequence of lemmas.

Notation: Let $H_i = M_i^c$ for each $i \in [78]$.

Lemma 1. *$r(C_4, H_1) = 18$ and $r(C_4, H) = 11$ if H equals H_5, H_{24}, H_{27}, H_{14}.*

Proof. The proof of the first equality is given in detail in [15]. See figure 5 for a C_4-free graph on 17 vertices with maximum independent set of size 5, which would be give us $r(C_4, H_1) \geq 18$)

Let's next consider $r(C_4, H_5)$. As $K_5 \setminus e$ is a subgraph of H_5 by [2] we find that $r(C_4, H_5) \geq 11$. So we must show that $K_{11} \to (C_4, H_5)$. Let $V = \{v_1, v_2, ..., v_{11}\}$ and suppose on the contrary that, there exists a two coloring (R, B) of $[V]^2$ such that $< R >$ contains no C_4 and $< B >$ contains no H_5. But then there will be a blue $K_5 - e$ say without loss of generality on $X = \{v_1, v_2, ..., v_5\}$ with $e = (v_1, v_2)$. Let $Y = V \setminus X$ and $W = \{v_3, v_4, v_5\}$. Then there can be at most three vertices in Y adjacent in $< R >$ to at least two vertices of W. So we get two vertices in Y adjacent in $< B >$ to at least two vertices in W. But one of these vertices of Y must be adjacent in $< B >$ to v_1 or v_2 (in order to avoid a red C_4). Thus we get a blue H_5 as required. Finally the later three equalities are true as $r(C_4, H_5) = 11$ and $r(C_4, K \setminus e) = 11$ (see [2]). □

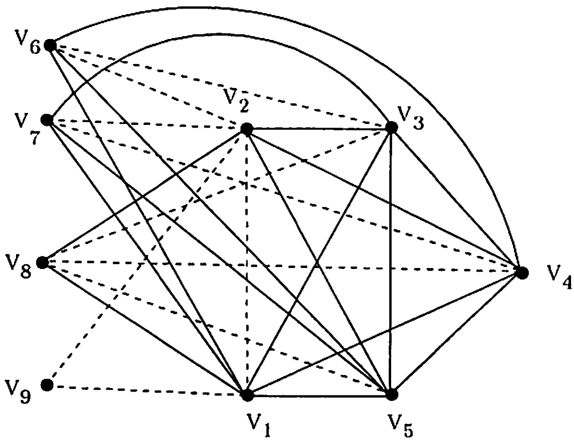
Lemma 2. *$r(C_4, H_8) = 11$, $r(C_4, H_{15}) = 11$ and $r(C_4, H_{18}) = 11$.*

Proof. $K_5 \backslash e$ is a subgraph of H_8 and H_{18} (see [2]) and $R_{10.1}$ doesn't contain H_{15} in the complement (see figure 4(b)), it suffices to show that $K_{16} \to (C_4, H_8)$. Clearly there exists a blue $K_5 \setminus e$ say on $V = \{v_1, v_2, ..., v_5\}$ but no K_5 (Since if there was a blue K_5 it would force a blue H_8). Let $e = (v_1, v_2)$ and $X = \{v_6, v_7, ..., v_{11}\}$. Define a partition $X = \{X_{RR}, X_{RB}, X_{BR}, X_{BB}\}$ according to whether the pair of edges (xv_1, xv_2) is in $R \times R$, $R \times B$, $B \times R$ or $B \times B$. Clearly $|X_{RR}| \leq 1$ and $|X_{BB}| \leq 1$.

Claim : $|X_{RB}| \leq 2$.

Proof of Claim: Suppose the claim is false. Then we can find three vertices say v_6, v_7, v_8 belonging to X_{RB}. But in order to avoid a red C_4 without loss of generality v_6, v_7 and v_6, v_8 edges are blue. But then we would get a blue H_8 containing $v_2, v_3, ..., v_7$ unless say v_6 (or v_7)is adjacent in $< R >$ to v_3,v_4,v_5 but then $\{v_2, v_3, ..., v_7\}$ forms a blue K_5. Hence the claim.

By symmetry we also get $|X_{BR}| \leq 2$. Next as $|X| = 11$ we would get that $|X_{RR}| = 1$, $|X_{BR}| = 2$, $|X_{RB}| = 2$, $|X_{BB}| = 1$. Suppose that $v_6, v_7 \in X_{BR}$, $v_8 \in X_{BB}$ and $v_9 \in X_{RR}$. Since there is no red C_4 or blue H_8, so without loss of generality we would get the following diagram (see figure 2).



- - - - - indicate red edges,other lines indicate blue edges
(Figure 2)

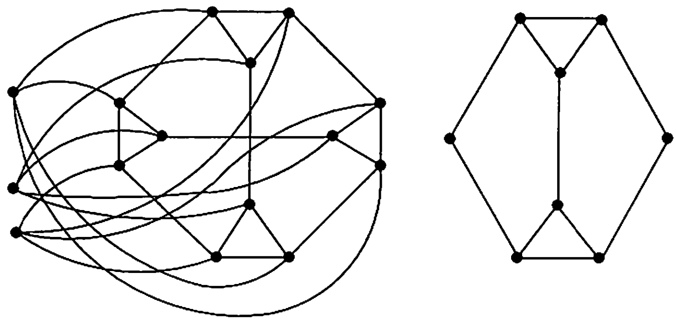
Next to avoid a blue H_8, (v_6, v_7) will be forced to be red. Also to avoid a red C_4 (v_7, v_9) will be forced to be blue. Note v_9 cannot be adjacent to v_3 or v_4 in red as it would force a red C_4. So we would get a blue H_8 containing vertices $v_1, v_3, v_4, v_5, v_7, v_9$. □

Lemma 3. *$r(C_4, H_3) = 14$, $r(C_4, H_7) = 14$ and $r(C_4, H_{13}) = 14$.*

Proof. As K_5 is a subgraph of H_{11} and $r(C_4, K_5) = 14$. It suffices to show $K_{14} \rightarrow (C_4, H_3)$. First we can find a blue K_5 as $r(C_4, K_5) = 14$. But out of the 9 vertices at least one vertex is adjacent in blue to at least three vertices of the K_5 forcing a blue H_3 as required. □

Lemma 4. *$r(C_4, H_2) = 16$.*

Proof. As H_2 is not a subgraph of $R^C_{15.1}$ (see Figure 3(a)), it suffices to show $K_{16} \rightarrow (C_4, H_2)$ Let $V = \{v_1, v_2, ...v_{16}\}$ and suppose on the contrary that, there exists a two coloring (R, B) of $[V]^2$ such that $< R >$ contains no C_4 and $< B >$ contains no H_2. But then there will be a blue K_5 say on $X = \{v_1, v_2, ..., v_5\}$. (since $r(C_4, K_5) = 14$). Let $Y = V \setminus X$. But out of the remaining 11 vertices there must be at least one vertex adjacent in $< R >$ to at most one vertex of X. (since there can be no red C_4). Thus we end up getting a blue H_2 as required. □

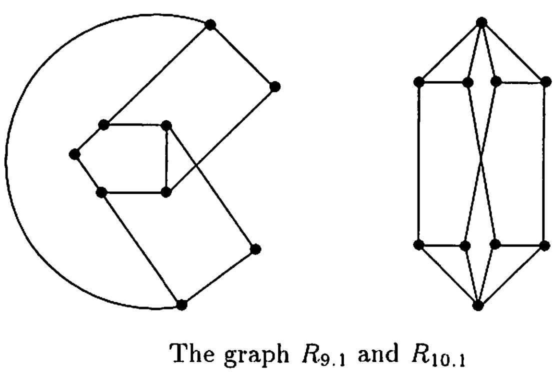


$R_{15.1}$ and $R_{8.1}$
(Figure 3)

Lemma 5. *$r(C_4, H_4) = 13$ and $r(C_4, H_6) = 13$.*

Proof. As H_4 and H_6 not a subgraph of $R^C_{12.1}$ (see figure 6), it suffices to show $K_{13} \to (C_4, H_4)$ and $K_{13} \to (C_4, H_6)$. First note that a blue K_5 (say on vertices $v_1, v_2, \ldots.., v_5$) must exist as all C_4-free graphs on 13 vertices with independence number 4 (namely $R_{13.1}$ (see [15])) contains a H_4 and a H_6 in its complement. Hence there can be at most two vertices in the remaining 8 vertices adjacent in red to three or more vertices of K_5. Thus there must be at least six vertices adjacent in red to two vertices of K_5. But this would directly give us $K_{13} \to (C_4, H_6)$. So we are left to show that $K_{13} \to (C_4, H_4)$. First it should be noted that there cannot be four such vertices adjacent in red to say (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5) and (v_5, v_1) respectively (as it would force a red C_4). Next using $t(5) = 6$ (see [3]) without loss of generality would get these six vertices are adjacent in red to (v_1, v_2), (v_2, v_3), (v_3, v_1), (v_1, v_4), (v_4, v_5) and (v_5, v_1). But then one of the remaining vertices will be in a red C_4. □

Lemma 6. *$r(C_4, H_{16}) = 10$, $r(C_4, H_{30}) = 10$ and $r(C_4, H_{50}) = 10$.*



The graph $R_{9.1}$ and $R_{10.1}$
(Figure 4)

Proof. As $R_{9.1}$ (see figure 4) doesn't contain H_{50} in its complement, it suffices to show $K_{10} \rightarrow (C_4, H_{16})$. First note that a blue $K_5 \setminus e$ (say $X = \{v_1, v_2, ..., v_5\}$ with $e = (v_1, v_2)$) must exist as all graphs on ten vertices without $K_5 \setminus e$ contains H_{16} (see [11])). Also we can assume the edge e is red as otherwise we would get a blue H_{16}. Next there can be at most two vertices outside X adjacent to three or more vertices of X in red. So there must be at least three or more vertices outside of X adjacent to exactly two vertices of X. Let denote these set of vertices by Y. Then if y_1 and y_2 are adjacent to v_1 , v_3 and v_2 , v_4 respectively or v_1 , v_2 and v_2 , v_4 Then we would get a red C_4. But this would force a $|Y| \leq 2$, which is the required contradiction. □

Lemma 7. *$r(C_4, H) = 10$ if H equals H_{29} , H_{54} , H_{45} , H_{26} and further $r(C_4, H) = 11$ if H equals H_{10} , H_{23} , H_{11} , H_{34}.*

Proof. This first set of equalities follows from $r(C_4, K_4) = r(C_4, H_{16}) = 10$ (see [4]) and the next from $r(C_4, H_{34}) = 11$ (see [16]) and $r(C_4, H_5) = 11$. □

Lemma 8. *If G is a C_4-free graph on 8 vertices whose complement doesn't contain a G_{21}, then G is isomorphic to $R_{8.1}$ (see figure 3)*

Proof. This is a result proved in [13]. □

Lemma 9. *$r(C_4, G)$ is equal to 8 if G equals M_{50} , M_{77} , H_{59} , H_{44} , H_{68} , H_{40} , H_{42} , H_{43} , H_{46} , H_{25} , H_{77} , H_{63} , H_{41} , H_{61} , H_{67}.*

Proof. As $R_{7.1}$ (which is $R_{8.1}$ with a divalent vertex deleted, see figure 3(b) for $R_{8.1}$) is a C_4-free graph which doesn't contain a G_7 , G_{17} , M_{50} or H_{61} in its complement, it suffices to show $K_8 \rightarrow (C_4, H_{25})$. First note that a blue G_{21} (say on $X = \{v_1, v_2, ..., v_5\}$ with $e_1 = (v_1, v_2)$ and $e_2 = (v_3, v_4)$) exists by the previous lemma. Let $Y = \{v_6, v_7, v_8\}$ be the remaining three vertices. But then each $y \epsilon Y$ must be adjacent in blue to at most two vertices in the four cycle of G_{21} to avoid a blue H_{25}. But then this would force a red C_4 as required. □

Lemma 10. *$r(C_4, M_{31}) = 8$ and $r(C_4, G) = 7$ for any forest G on six vertices (without isolated vertices). In particular $r(C_4, G) \geq 7$ for any graph on six vertices without isolated vertices.*

Proof. We get the first equality directly from [2] and further we also get that $r(C_4, G) = 7$ for any tree G on six vertices (without isolated vertices). But since K_{15} is C_4-free graph on six vertices we also get that $r(C_4, H) \geq 7$ for any connected graph H on six vertices. These two results combined gives us the second equality. □

Lemma 11. *Let* $r = r(C_4, G)$. *Then*

$$(1)\quad r = \begin{cases} 7 & \text{if } G \text{ equals } M_{49}, M_{65}, M_{74}, M_{41}, H_{65}, H_{74}, M_{64}, \\ & M_{76}, M_{75}, M_{51}, M_{42}, M_{45}, M_{62}, M_{78}, \\ & M_{70}, M_{52}, M_{43}, M_{44}, M_{66}, M_{47}, M_{46}, \\ & M_{53}, M_{54}, M_{63}, M_{32}; \\ 8 & \text{if } G \text{ equals } M_{71}, H_{64}, H_{62}, M_{48}, M_{72}, M_{73}, H_{56}, \\ & M_{68}; \\ 9 & \text{if } G \text{ equals } H_{47}, H_{57}, H_{36}, H_{78}. \end{cases}$$

Proof. First let's show $r(C_4, M_{49}) = 7$. Clearly $r(C_4, M_{49}) \geq 7$ as H_{49} is connected. So we must show $K_7 \rightarrow (C_4, M_{49})$. First note that a blue G_{15} (say on $X = \{v_1, v_2, \ldots.., v_5\}$ with v_1, v_2, v_3 inducing the only blue triangle of G_{15}) exists as $r(C_4, G_{15}) = 7$(see [2]). Next the only way to avoid a blue M_{49} is for both the remaining two vertices to be adjacent in red to v_1 and v_2, but then this would give a red C_4 as required.
It should be noted that each of these graphs can be obtained by a graph on five vertices with a pendant edge added in two possible ways. Thus using the exact same argument as above with $r(C_4, G)$ is equal to 7 if G equals G_{17}(see [2]) and equal to eight if G equals G_7 , G_{17} , M_{31} (see [16], [8]) and equal to nine if G equals G_{21} , G_{18} one would get the above result. □

Lemma 12. $r(C_4, G)$ *if equal to 7 if* G *equals* H_{66} , M_{67} , M_{59} , M_{60}.

Proof. To show the first four equalities it suffices to show $K_7 \rightarrow (C_4, H_{66})$ (since $r(C_4, H_{40}) = 7$). First there must be a blue G_{12} (consisting of $\{v_1, v_2, v_3\}$, $\{v_1, v_3, v_4\}$, $\{v_1, v_4, v_5\}$ triangles) since $r(C_4, G_{16}) = 7$. Next v_2 (or v_5) can be adjacent in blue to exactly one vertex of the remaining two vertices. Also they must be adjacent in blue to different vertices to avoid a blue H_{66}. So without loss of generality (v_6, v_2) ,(v_7, v_5) are blue edges if v_6, v_7 are the remaining two vertices. Next to avoid a blue H_{66}, (v_6, v_1) ,(v_6, v_3) , (v_7, v_1) ,(v_7, v_4) would have to be blue edges and to avoid a red C_4, (v_7, v_3) would have to be blue edge. But this would force a blue H_{66} as required. □

Lemma 13. $r(C_4, H) = 7$ *if* H *equals* H_{70} , M_{33} , M_{30} , H_{60} ,M_{61}.

Proof. Let's first consider the first three equalities. As M_{33} and M_{30} are connected it suffices to show $K_7 \rightarrow (C_4, H_{70})$. First there must be a blue G_{12} (consisting of two triangles v_1, v_2, v_3 and v_1, v_4, v_5 with v_1 the common vertex) since $r(C_4, G_{12}) = 7$. Also let v_6, v_7 be the other two vertices. Without loss of generality v_6 is adjacent to v_2 in blue. Next to avoid a blue H_{70} , v_6 must be adjacent in red to v_4, v_5. But then without loss of generality we may assume that v_7 is adjacent to v_5 in blue. Next in order to avoid a blue H_{70}, v_7 must be adjacent to v_2, v_3 in red. And finally to avoid a red C_4, without loss of generality v_7 must be adject to v_4 in blue. Thus we have two possible cases.

Case 1: (v_3, v_4) is red. But then in order to avoid a red C_4, (v_6, v_7) and (v_2, v_4) must be blue. But this would force a blue H_{70}.

Case 2: (v_3, v_4) is blue. But then in order to avoid a blue H_{10}, the edge (v_2, v_5) must be red, next to avoid a red C_4, the edge (v_6, v_7) must be blue. Further to avoid a blue H_{70} the edge (v_6, v_3) must be red. Next to avoid a red C_4 the edge (v_2, v_4) must be blue. But this would give a blue H_{70}.

To show the next two equalities use $r(C_4, H_{66}) = 7$, $r(C_4, C_5) = 7$ to prove $K_7 \to (C_4, H_{60})$, $K_7 \to (C_4, M_{61})$ respectively and proceed similarly. □

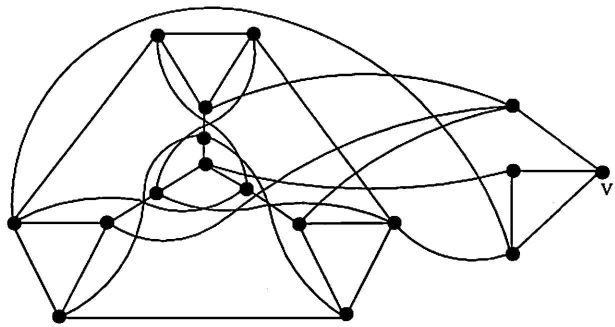
Lemma 14. *$r(C_4, G) = 10$ if G equals H_{76}, M_{57}, H_{49}, H_{71}, M_{34}, H_{72}, H_{58}, M_{69}, H_{37}, H_{39}, H_{22}, H_{19}, H_{12}, H_{20}, H_{21}.*

Proof. First let's consider $r(C_4, H_{12})$. To show that $r(C_4, H_{12}) = 10$ it suffices to show that $K_{10} \to (C_4, H_{12})$ (since $r(C_4, K_4) = 10$). Let $V = \{v_1, v_2, \ldots\ldots, v_{10}\}$ and suppose on the contrary that there exists a two coloring (R, B) of $[V]^2$ such that $< R >$ contains no C_4 and $< B >$ contains no H_{12}. First note that a blue $K_5 \setminus e$ (say $X = \{v_1, v_2, \ldots\ldots, v_5\}$ with $e = (v_1, v_2)$) must exist as all graphs on ten vertices without $K_5 \setminus e$ contains H_{12} (see [11])). Let $Y = V \setminus X$ and $W = \{v_3, v_4, v_5\}$. Then there can be at most three vertices in Y adjacent in $< R >$ to at least two vertices of W. So we get two vertices in Y is adjacent in $< B >$ to at least two vertices in W. But one of these vertices of Y must be adjacent in $< B >$ to v_1 or v_2 (in order to avoid a red C_4). Thus we get a blue H_{12} as required. The other equalities will follow by using $r(C_4, K_4)$ (see [2]), $r(C_4, H_{16})$, $r(C_4, H_{12})$ are all equal to ten and H_{20} is not contained in the complement of three disjoint triangles. □

Lemma 15. *Let $r = r(C_4, G)$. Then*

$$(2)\quad r = \begin{cases} 6 & \text{if } G \text{ equals } M_1, M_2, M_3, M_4, M_5, M_7, M_8, M_{11}, \\ & \qquad M_{12}, M_{14}, M_{22}; \\ 7 & \text{if } G \text{ equals } M_6, M_{10}, M_{13}, M_{15}, M_{19}, M_{20}, M_{21}, \\ & \qquad M_{23}, M_{24}, M_{35}, M_{36}, M_{37}, M_{38}, M_{56}; \\ 8 & \text{if } G \text{ equals } M_{39}, M_{55}; \\ 9 & \text{if } G \text{ equals } M_{58}, H_{73}; \\ 11 & \text{if } G \text{ equals } H_{48}; \\ 14 & \text{if } G \text{ equals } H_{31}. \end{cases}$$

Proof. These directly follows from [2] , [12] , [4] , [13] and [11]. It should be note that $R_{17.1} \setminus N(v)$ is a C_4-free graph on 13 vertices with maximum independent set of size 4 which would give us $r(C_4, H_{31}) \geq 14$ (see figure 5 for $R_{17.1}$). □



The graph $R_{17.1}$
(Figure 5)

Lemma 16. $r(C_4, H_{69}) = 9$.

Proof. As $R_{8.1}$ doesn't contain H_{69} in its complement, it suffices to show $K_9 \rightarrow (C_4, H_{69})$. First note that a blue $K_5 \setminus 2K_2$ (say $X = \{v_1, v_2, \ldots.., v_5\}$ with $e_1 = (v_1, v_3)$ $e_2 = (v_2, v_4)$) must exist as all graphs on nine vertices without $K_5 \setminus 2K_2$ contains H_{69} (see [11]). Also we can assume the edges e_1, e_2 are red as otherwise we would get a blue H_{69}. Next there must be one vertex outside X adjacent to v_1, v_2 or v_2, v_3 or v_3, v_4 or v_4, v_1 in blue. Also to avoid a blue H_{69} it must be adjacent in red to the remaining two vertices of $\{v_1, v_2, \ldots.., v_4\}$. Thus say v_6 is this vertex adjacent in blue to v_1, v_2 and adjacent in red to v_3, v_4. Next out of the remaining three vertices one vertex must be adjacent in blue to v_1 or v_2. Say this vertex is v_7 and it is adjacent in blue to v_2. But to avoid a red C_4 and a blue H_{69}, v_7 also must be adjacent in blue to v_3. Arguing in this manner we can show that without loss of generality that $\{v_6, v_7, \ldots.., v_9\}$ are adjacent to $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_1\}$ in blue and $\{v_3, v_4\}$, $\{v_1, v_4\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$ in red respectively. Further to avoid a red C_4 and a blue H_{69} we can show that any two vertices of $\{v_6, v_7, v_8, v_9\}$ must be adjacent to each other in blue. Also two vertices out of out of $\{v_6, v_7, \ldots.., v_9\}$ (say v_6) must be adjacent to v_5 in blue. But then this would give a blue H_{69} as required. □

Lemma 17. $r(C_4, G) = 9$ *if* G *equals* H_{53} , H_{28} , H_{38} , H_{52} , H_{33}.

Proof. The first equality directly follows from $r(C_4, G_{18}) = 9$ (see [4]) as any graph on 9 vertices must contain a G_{18} but in order to avoid a red C_4 one of the remaining four vertices must be adjacent in blue to two, divalent vertices of G_{18}. The second equality directly follows from $r(C_4, G_{21}) = 9$ (see [4]) as any graph on 9 vertices must contain a G_{21} but in order to avoid a red C_4 one of the remaining four vertices must be adjacent in blue to two, trivalent vertices of G_{18}. To show the third equality it suffices to show that $K_9 \to (C_4, H_{38})$ (since $r(C_4, G_{18}) = 9$ as shown in [4]). Let G be a graph on 9 vertices containing no blue H_{38} then clearly first it must contain a G_{18} and further in order to avoid a red C_4, both the degree four vertices of G_{18} must be adjacent to at least three of the remaining four vertices in red. But this would force a red C_4. as required. The later two equalities directly follows from the above lemma and $r(C_4, P_3)$ is equal to 4. □

Lemma 18. *$r(C_4, H) = 9$ if H equals H_{32} , H_{17} and $r(C_4, H_9) = 11$.*

Proof. First let's show that $r(C_4, H_{32}) = 9$. Since H_{32} is not a subgraph of $R_{8.1}$ complement (see figure 3) it suffices to show $r(C_4, H_{32}) \leq 9$. Suppose there is a C_4-free graph on 9 vertices containing no H_{32} in its complement. First note that there cannot be a degree one vertex as it would force a H_{32} (since $r(C_4, G_{17}) = 8$). Also as the maximum number of edges of a C_4-free graph G on nine edges is 13 there must be a degree 2 vertex (see [3] for $t(9) = 13$). Also the degree two vertex cannot be adjacent to a vertex of degree two or to a vertex of degree three, satisfying the additional condition that that these two vertices are in a triangle, as it would force a H_{32} (since $r(C_4, 2K_2) = 5$). So we have the following two cases. Let's denote the degree two vertex by v and its neighbors by w, r.

Case 1: If at least one of the neighbors of the degree two vertex is adjacent to two vertices of $V(G) \setminus \{v, w, r\}$. Say w is adjacent to s, t. Then in order to avoid a H_{32} the remaining four vertices must contain a $K_{1,3}$ and r will have to be adjacent to the degree three vertex and a degree one vertex of $K_{1,3}$. But then by the same argument the four vertices not in the closure of the neighborhood of v, r will also contain a $K_{1,3}$ which will give us a C_4.

Case 2: If both neighbors of the every degree two vertex are adjacent to three vertices of the complement of the closure of its neighborhood, then clearly (w, r) is not a edge as if this was the case one of there neighbors will be forced to have degree one. So w is adjacent to w_1, w_2, w_3 and r is adjacent to r_1, r_2, r_3. But to avoid a C_4 at least two vertices of $w_1, w_2, w_3, r_1, r_2, r_3$ will be forced to have degree two but then as the assumption is valid for each vertex of degree two we would get a C_4 as required.

Thus to show $r(C_4, H_{17}) = 9$ it suffices to show $K_9 \to (C_4, H_{17})$. By the above part there must be a a blue H_{32}. But this would extend to a blue H_{17} unless we would have a red C_4 as required.

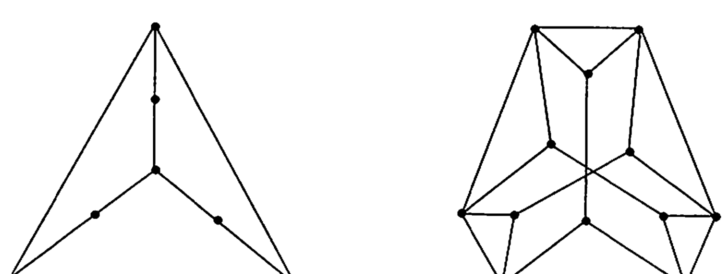
Next let's show that $r(C_4, H_9) = 11$. The proof of this is very similar to the previous counting argument but a bit more detailed. Since H_9 is not a subgraph of $R_{10.1}$ complement (see figure 4) it suffices to show $r(C_4, H_9) \leq 11$. Suppose there is a C_4-free graph G on 11 vertices containing no H_9 in its complement. Suppose there is no vertex of degree two then since all C_4-free graphs on 11 vertices and 18 edges contain a H_9 in its complement (see [3]) we would get all vertices of G must be of degree 3 except for one vertex of degree four. By a counting argument this degree four vertex (say w) must contain an edge in its neighborhood (say (r,s)). But then as there are 6 vertices not adjacent to both r and s and $r(C_4, C_4) = 6$ (see [9]) we would get a H_9 as required. So there must be a vertex of degree 2 in G say v with neighbors p and q. By the above p and q must be adjacent to at least three vertices in $V(G) \setminus \{v, p, q\}$. Then we would have one of the following cases.

Case 1: If without loss of generality say p is adjacent to three vertices in $V(G) \setminus \{v, p, q\}$ say p_1, p_2, p_3 then by the above argument the five vertices not adjacent to v, p must be have no C_4 in it or its complement. But since there are only two such graphs namely C_5 or G_{19} a direct computation will show this would force a C_4 contrary to the assumption.

Case 2: If without loss of generality say p and q are both adjacent to four vertices each say p_1, p_2, p_3, p_4 and q_1, q_2, q_3, q_4 respectively. Clearly it should be noted that all vertices $X = \{p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4\}$ will be forced to have at most degree three in order to avoid a C_4. If say there is a vertex (say x) of degree 2 in X by a earlier remark as the neighbor of x in X has degree at most three we would get a H_9 as required. So we may assume that all vertices of X has degree three each, but this would force G to be a C_4-free graphs on 11 vertices and 18 edges. But as all C_4-free graphs on 11 vertices and 18 edges contain a H_9 in its complement (see [3]), we would get the required contradiction. □

Lemma 19. *$r(C_4, M_{40}) = 7$, $r(C_4, H) = 8$ if H equals H_{75} , H_{55} , H_{51} , H_{35}.*

Proof. See [14] and [9] for $r(C_4, M_{40}) = 7$. Since H_{75} is not a subgraph of $R_{7.2}$ complement (see figure 6) and $r(C_4, H_{46}) = 8$ we would directly get $r(C_4, H_{75}) = 8$ as required. Clearly $r(C_4, H_{55}) \geq 8$ as $r(C_4, M_{31}) = 8$. So it suffices to show that $K_8 \rightarrow (C_4, H_{55})$. By a previous lemma as $R_{8.1}$ contains a H_{55} in its complement we can assume there exists a blue G_{21} say on $X = \{v_1, v_2, \ldots\ldots, v_5\}$ with the center denoted by v_1).



The graph $R_{7.2}$ and $R_{12.1}$
(Figure 6)

First it should be noted v_1 cannot be adjacent in blue to any vertices outside of X as it would force a red C_4 or a blue H_{55}. So all vertices of $X^c = \{v_6, v_7, v_8\}$ must be adjacent in red to v_1 and thus in order to avoid a red C_4 there must be a blue P_3 say $\{v_6, v_7, v_8\}$ in X^c. Next it should be noted that if say v_6 (or v_8) is adjacent in red to say v_2 then it would force v_7 and v_8 to be adjacent in blue to v_2 forcing a blue H_{55}. Hence by symmetry v_6 and v_8 will have to be adjacent in blue to v_2, v_3, v_4, v_5 giving a blue H_{55} as required.

To show $r(C_4, H_{51}) = 8$ as $R_{7.1}$ (which is $R_{8.1}$ with a divalent vertex deleted, see figure 3(b) for $R_{8.1}$) is a C_4-free graph which doesn't contains a H_{51} in its complement it suffices to show $K_8 \to (C_4, H_{51})$. First note that a blue G_{21} (say on $X = \{v_1, v_2, ..., v_5\}$ with $e_1 = (v_2, v_3)$ and $e_2 = (v_4, v_5)$) exists by a previous lemma as $R_{8.1}$ contains a H_{51} in its complement. Let $Y = \{v_6, v_7, v_8\}$ be the remaining three vertices. But each vertex of $\{v_2, v_3, v_4, v_5\}$ must be adjacent to at most one vertex of Y in order to avoid a red C_4 or a blue H_{51}. But then each vertex of $\{v_2, v_3, v_4, v_5\}$ would be adjacent in red to at least two vertices of Y forcing a red C_4 as required.

By the above result in order to show $r(C_4, H_{35}) = 8$ it suffices to show $K_8 \to (C_4, H_{35})$. By above part there must be a blue H_{55}. But this would extend to a blue H_{35} unless we would have a red C_4 as required. □

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