Two-edge-connected $[2, k]$-factors in graphs.

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Abstract :

Let $\sigma_2(G) = \min\{d_G(u)+d_G(v) \mid u, v \in V(G),\ u, v \notin E(G)\}$ for a non-complete graph G. An $[a, b]$-factor of G is a spanning subgraph F with minimum degree $\delta(F) \geq a$ and maximum degree $\Delta(F) \leq b$. In this note, we give a partially positive answer to a conjecture of M. Kano. We prove the following results:

Let G be a 2-edge-connected graph of order n and let $k \geq 2$ be an integer. If $\sigma_2(G) \geq 4n/(k+2)$, then G has a 2-edge-connected $[2, k]$-factor if k is even and a 2-edge-connected $[2, k+1]$-factor if k is odd. Indeed, if k is odd, there exists a graph G which satisfies the same hypotheses and has no 2-edge-connected $[2, k]$-factor. Nevertheless, we have shown that if G is 2-connected with minimum degree $\delta(G) \geq 2n/(k+2)$, then G has a 2-edge-connected $[2, k]$-factor.

I. INTRODUCTION.

We consider graphs without loops or multiple edges. Let G be a graph of order n, with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(x)$ the degree of the vertex x in G, and by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of G, respectively. Define $\sigma_p(G) = \min\{d_G(u_1) + \ldots + d_G(u_p) \mid \{u_1, \ldots, u_p\}$ stable set $\}$ for a graph with independence number $\alpha(G) \geq p$. Recall that $\lfloor x \rfloor$ denotes the largest integer satisfying $\lfloor x \rfloor \leq x$, and $\lceil x \rceil$ is the smallest integer satisfying $x \leq \lceil x \rceil$.

A spanning subgraph F of G is called an $[a, b]$-factor of G if $a \leq d_F(x) \leq b$ for all $x \in V(G)$. An $[a, a]$-factor is called an $a-$ factor. For an extensive survey of results on $[a, b]$-factors, see [1]. If there is no further requirement on the factor then there is a well-known necessary and sufficient condition for the existence of an $[a,b]$-factor [4] : G has an $[a,b]$-factor if and only if $b|S|-a|T|+\sum_{v\in T} d_{G\setminus S}(v) \geq 0$, for all pairs of disjoint subsets S,T of $V(G)$.

We are interested by the existence of $[a,b]$-factors satisfying further connectivity requirements. The investigation of connected factors was initiated by M. Kano [2]. This topic is closely related to the hamilton cycle problem, as a connected 2-factor is obviously an hamiltonian cycle. On the other hand, we remark that a connected k-factor is a connected k-regular spanning subgraph.

In [2], M. Kano gives the following conjecture :

Conjecture 1 *Let G be a 2-edge-connected graph of order $n \geq k+3$ for an integer $k \geq 2$. If $\sigma_2(G) \geq \frac{4n}{k+2}$, then G has a 2-edge-connected $[2,k]$-factor.*

We prove this conjecture if k is even, and disprove the conjecture if k is odd. If k is odd, we show that the hypothesis implies that G contains a $[2,k+1]$-factor.

II. EXISTENCE OF A 2-EDGE CONNECTED FACTOR IN A GRAPH.

The proof of the following theorem is strongly close to that of a result of S.Brandt (private communication).

Theorem 1 *If the vertices of a 2-edge connected graph are covered by 2-edge connected subgraphs $G_1, \cdots G_r$ with $2r \leq \sum_{i=1}^{r}\Delta(G_i)$ for some $r \geq 2$, then G has a 2-edge connected factor H with $\Delta(H) \leq \sum_{i=1}^{r}\Delta(G_i)$.*

Proof. Let G' be the spanning subgraph of G induced by the set of edges $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_r)$. We call "supergraph" of G' any spanning subgraph J such that $E(G') \subset E(J)$.

Let B_i be a 2-edge connected component of a supergraph of G'. Define the weight of B_i by $w(B_i) = \sum_{G_j \subset B_i} \Delta(G_j)$. Note that every graph G_j is contained in precisely one component and that $\sum_i w(B_i) = \sum_{j=1}^{r} \Delta(G_j)$.

One can verify that every connected component B' of G' is 2-edge connected, and we remark that

$\Delta(G') \leq \sum_{i=1}^{r} \Delta(G_i)$

$2 \leq \Delta(B') \leq w(B')$ for every component B'.

Let H be a subgraph of G of maximum size among the subgraphs satisfying the following properties :

(i) $E(G') \subset E(H)$

(ii) Every connected component B_H is 2-edge connected and satisfies $\Delta(B_H) \leq w(B_H)$.

Then H is connected : otherwise, let B_1 and B_2 be 2 components of H. Since G is 2-edge-connected, by Menger's Theorem there must be two edge-disjoint paths P_1, P_2, each one joining a vertex of B_1 to a vertex of B_2, and having no further vertex with B_1 or B_2 in common.

Now consider the graph $H' = (V(G), E(H) \cup E(P_1) \cup E(P_2))$. Note that every component of H' is 2-edge-connected. B_1 and B_2 belong to the same component B' of H' and B' may contain further components $B_3, \ldots, B_t$ of H. If it is the case, then

$$\Delta(B') \leq \max(\Delta(B_1) + 2 \; , \; \Delta(B_2) + 2 \; , \; 4 + \max_{3 \leq i \leq t} \Delta(B_i))$$

and so, as $\Delta(B_i) \geq 2$ for any i,

$$\Delta(B') \leq \sum_{i=1}^{t} \Delta(B_i) \leq \sum_{i=1}^{t} w(B_i) \leq w(B').$$

If B' contains only B_1 and B_2,

$$\Delta(B') \leq 2 + \max(\Delta(B_1) \; , \; \Delta(B_2)) \leq w(B_1) + w(B_2) = w(B').$$

So, H' contradicts the maximality of H. Hence, H is connected, and by (ii), H is 2-edge connected. □

Corollary : *Let G be a 2-edge-connected graph of order n and $k \geq 2$ an integer. If $\sigma_2(G) \geq 4n/(k+2)$ then G has a 2-edge-connected $[2,k]$-factor if k is even and a 2-edge-connected $[2,k+1]$-factor if k is odd.*

Proof. We first show that $\sigma_p(G) \geq \frac{p}{2}\sigma_2(G)$ for any integer p

- If p is even it is immediate.

- If p is odd, let us consider a stable set $\{x_1, \cdots, x_p\}$ and suppose $d(x_1) = \text{Max } \{d(x_i)\ , 1 \leq i \leq p\}$. So, $d(x_1) + d(x_2) \geq \sigma_2(G)$ and then $d(x_1) \geq \frac{\sigma_2(G)}{2}$. We have

$$\sum_{i=1}^{n} d(x_i) = d(x_1) + \sum_{j=2}^{n} d(x_j) \geq \frac{\sigma_2(G)}{2} + \frac{p-1}{2}\ \sigma_2(G) = \frac{p}{2}\ \sigma_2(G).$$

It follows that $\sigma_p(G) \geq \frac{p}{2}\ \sigma_2(G)$.

Now, as $\sigma_2(G) \geq \frac{4n}{k+2}$, we get that $\sigma_{\frac{k+2}{2}}(G) \geq (k+2)\sigma_2(G)/4 \geq n$ if k is even and $\sigma_{\frac{k+3}{2}}(G) \geq n$ if k is odd. Kouider and Lonc [5] proved that if $\sigma_{h+1}(G) \geq n$ or $\alpha(G) < h+1$, then the vertices of the graph are covered by at most h cycles. Hence there is a collection C of at most $\lceil k/2 \rceil$ cycles covering the vertices of G, and then we apply Theorem 1. □

The bounds on σ_2 are best possible :

In the case k odd, let us set $k = 2h-1$, and consider the graph G_0 of order n, with $n/h \in \mathbb{N}$, consisting of h disjoint complete graphs A_i with $|A_i| = \frac{n}{h}$ for $1 \leq i \leq h-1$, $|A_h| = \frac{n}{h} - 1$, and an extra vertex x adjacent to all other vertices. We get $\sigma_2(G_0) = \frac{2n}{h} - 1 \geq \frac{4n}{k+2}$ as soon as $n \geq \frac{h(2h+1)}{2} = \frac{(k+1)(k+2)}{4}$. Assuming that G_0 has a 2-edge-connected $[2,k]$-factor F, there are at least two edges between x and each A_i in this factor, and thus $d_F(x) \geq 2h = k+1$. So G_0 satisfies the hypothesis of the corollary and has no 2-edge-connected $[2,k]$-factor.

In particular this example shows that Conjecture 1 fails for odd k. Note that $\delta(G_0) = \frac{n}{h} - 1 = \frac{2n}{k+1} - 1 > \frac{2n}{k+2}$ if n is sufficiently large, so even the corresponding minimum degree condition

$\delta \geq 2nk + 2$ does not imply the existence of a 2-edge-connected $[2, k]$-factor. Nevertheless, if the graph is 2-connected, then we can show that the minimum degree condition implies the existence of a $[2, k]$-factor. Note that the previous example G_0 has connectivity 1.

In the case k even, let us set $k = 2h$. The construction above gives an example of graph G_1 which has no 2-edge-connected $[2, k-1]$-factor. So the result we got is sharp.

Let us now turn to 2-connected graphs.

Theorem 2 *Let G be a 2-connected graph and $k \geq 3$ be an integer. Suppose $\delta(G) \geq 2n/(k+2)$. Then G has a 2-edge-connected $[2, k]$-factor.*

Proof. By the corollary of Theorem 1, we are done if k is even. So we may assume tha. We already know that $V(G)$ is covered by at most $\frac{k+1}{2}$ cycles. In the desired factor, we want that the max-degree of the vertices is no more than k. The only case we have then to study is when there are exactly $c = \frac{k+1}{2}$s covering $V(G)$ and when the cycles $C_1, \cdots, C_{\frac{k+1}{2}}$ have at least one common vertex, say x. In view to decrease the degree of x in the factor composed by the last family of cycles, we shall replace the last cycle by a family of paths we define below.

In [3], it is proved that $c \leq \lceil n/\delta \rceil - 1$. On the other hand, the inequality $\delta \geq \frac{2n}{k+2}$ (that is $n \leq \frac{k+1}{2}\delta + \delta/2$) implies $\lceil n/\delta \rceil - 1 \leq \frac{k+1}{2} = c$, and then $c = \lceil n/\delta \rceil - 1 = \frac{k+1}{2}$. So, we have :

$$n = \frac{k+1}{2}\delta + \delta_1 \text{ with } 0 < \delta_1 \leq \frac{\delta}{2} \qquad (1).$$

The previous family of cycles C_i , $1 \leq i \leq \lceil n/\delta \rceil - 1$ is obtained by a recursive construction (see [3], page 765) : the vertices on $\bigcup_{j \leq i-1} C_j$ form two disjoint paths P_i and Q_i, hanging respectively on $\{u_i, x\}$

and $\{x, v_i\}$ to $\bigcup_{j \le i-1} C_j$ and ae end of the construction, x has degree $k+1$ in the case we study.

This construction gives an oriented tree T, each vertex of which corresponds to one of the cycles of the previous family ; this tree T has the cycle C_1 as root and $d_T^+(C_1) = 1$. Let $m(C)$ be the number of vertices of G covered by a cycle C and not covered by the ascendants of C in T, so we have $\sum_i (m(C_i)) = n$. Furthermore, by [3], $m(C_1) \ge 2\delta$, $d_T^+(C) \le 2$ for any C in T.

For any C which is not a leaf in the tree T, following the sketch of the proof of [3], we have : $m(C) \ge \delta$ if C has exace son in T and $m(C) \ge 2\delta - 1$ if C has two sons in T. We will show tha there exists a leaf C_t such that $m(C_t) \le \delta/2$ In fact,

$*$ either T is a path ; for every internal node C, we have $m(C) \ge \delta$. Let C_t be the leaf of T different from the root. Then we have, by (1), $c\delta + \delta_1 = n \ge 2\delta + (c-2)\delta + m(C_t)$, so $m(C_t) \le \delta_1 \le \delta/2$.

$*$ or T is not a path ; consider the set of nodes of T that satisfy $d_T^+(C) = 2$, and the node C_1. Let c_1 be its cardinality; we have $c_1 \ge 2$.

Suppose that for any leaf C, $m(C) > \delta/2$. Remark that the number of leaves in T different from the root is also c_1.

- the number of vertices covered by the c_1 nodes is $\ge (2\delta - 1)c_1 + 1$
- the number of vertices covered by the leaves is $> c_1\delta/2$
- the number of vertices covered by the other nodes is $\ge (c - 2c_1)\delta$.

We then obtain :

$\delta c + \delta/2 \ge n > c_1(2\delta - 1 + \delta/2 - 2\delta) + \delta c + 1$, that is $\delta/2 - 1 > c_1(\delta/2 - 1)$, a contradiction.

We may suppose that $C_t = C_c$.

Now, in the graph G, the $m(C_c)$ new vertices covered by C_c form a family of paths $\{P_j = [a_j, b_j]$, $j = 1, \cdots p\}$ with possibly $a_j = b_j$. Let $I = \cup_{1 \le j \le p} \{a_j, b_j\}$. If this family of paths contains exactly r paths of length different from zero, then $|I| = p + r$. We have $|I| \le m(C_c) \le \delta/2$. Let B be the set of vertices of the paths, and $A = V(G) \setminus B$.

Suppose that we have defined a minimum number of such paths covering the set B. So no extremity of one path is adjacent to the extremity of another path. Let x_q be an element of I. We have :

- if the family of paths contains a path of at least 2 vertices, $d_B(x_q) \leq m(C_c) - |I| + 1$, so $d_A(x_q) \geq \delta - m(C_c) + |I| - 1 \geq 2|I| - 1 \quad (\alpha)$.

- otherwise, each path is a single vertex and $d_B(x_q) = 0$; and so $d_A(x_q) \geq \delta$. As in this case $|I| = |B| = \delta_1 \leq \delta/2$, we get $d_A(x_q) \geq 2|I| \quad (\beta)$.

In any case, we can define a set of edges between I and A by the following way : if the path P_j is reduced to one vertex, we choose two edges between I and A, and if not, we choose one edge between a_j and A (resp. between b and A), in such a way that all the extremities in A are different. This is possible by the previous inequalities (α) and (β).

Finally, the 2-edge-connected $[2, k]$−factor is defined by the cycles $C_1, \cdots, C_{c-1}$ and the subgraph formed by the paths P_j and the previous set of edges. □

References

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