An Algorithm to Analyse the Polynomial Deck of the Line Graph of a Triangle-free Graph

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ABSTRACT. An algorithm is presented in which a polynomial deck, $\mathcal{P}D$, consisting of m polynomials of degree $m-1$, is analysed to check whether it is the deck of characteristic polynomials of the one-vertex-deleted subgraphs of the line graph, H, of a triangle-free graph, G. We show that if two necessary conditions on $\mathcal{P}D$, identified by counting the edges and triangles in H, are satisfied, then one can construct potential triangle-free root graphs, G, and by comparing the polynomial decks of the line graph of each with $\mathcal{P}D$, identify the root graph.

1 Introduction

The polynomial reconstruction conjecture was first posed in [2]. It is a variation of Ulam's and Kelly's reconstruction conjecture [3, 7] and states that the characteristic polynomial $\phi(H)$ of a graph H can be reconstructed from $\mathcal{P}D(H)$, the polynomial deck (p-deck) of H consisting of the characteristic polynomials of the one-vertex-deleted subgraphs (with multiplicities). This conjecture is not settled yet but S. Simic proved it for connected graphs with the smallest eigenvalue bounded below by -2 [6]. These graphs include generalized line graphs.

In [5], A. Schwenk calls the two problems of the reconstruction from the p-deck, $\mathcal{P}D(H)$, of the graph, H, and of the characteristic polynomial, $\phi(H)$, **Problem B** and **Problem D** respectively.

In this article, we present an algorithm, *Alg*, in which a p-deck, $\mathcal{P}D$, consisting of m polynomials of degree $m-1$, is analysed and tested for the possibility of being the p-deck of characteristic polynomials of the one-vertex-deleted subgraphs of the irregular line graph, H, of a triangle-free graph, G. If either of two necessary conditions, P_1 and P_2, on $\mathcal{PD}$, identified by counting the edges and triangles in H, fails, then $\mathcal{PD}$ does not correspond to the p-deck of the irregular line graph, H, of a triangle-free graph, G. Otherwise potential triangle-free root graphs, G, can be constructed and by comparing the p-decks of their line graphs with $\mathcal{PD}$, the root graph can be identified. Because of the result in [6], this algorithm explicitely constructs the unique root graph, G and hence the characteristic polynomial, $\phi(L_G)$, from the legitimate p-deck, $\mathcal{PD}$, thus addressing Problem D for the line graph of a triangle-free graph. The way *Alg* is constructed is such as to find possible counter examples to problem B among the line graphs of triangle-free graphs.

In section 2, we establish the conditions P_1 and P_2, and show how the degree sequence of the root graph, G, of the irregular line graph, L_G, can be determined from a legitimate p-deck $\mathcal{PD}(L_G)$ provided that G is triangle-free. In section 3, we present the algorithm and discuss its possible outputs. We conclude with an example showing the output of *Alg* in section 4.

2 The Line Graph of a Triangle-Free Graph

The graphs considered are finite and simple, i.e. without multiple edges or loops. The line graph of a root graph $G = (\mathcal{V}(G), \mathcal{E}(G))$ is denoted by L_G, and its order is $|\mathcal{E}(G)|$. For a graph, H, with adjacency matrix $A(H)$ $(= A)$ and vertex set $\mathcal{V}(H) = \{w_1, w_2, \dots, w_m\}$, the eigenvalues are the real numbers, λ, such that, if I is the identity matrix, $\lambda I - A$ is not injective. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$, form the spectrum, $Sp(H)$, of H. The characteristic polynomial $\phi(A(H))(= \phi(H))$ which is the product $\prod_{i=1}^{m}(\lambda - \lambda_i)$, is a polynomial $\sum_{i=0}^{m} q_i\lambda^i$ with integer coefficients q_i and can be written as $Det(\lambda I - A) = 0$. The coefficient $q_n = 0$, the constant term q_0 =Det$(-A)$, $-q_{n-2}$ is the number of edges and $\frac{-q_{n-3}}{2}$ is the number of triangles in H.

Definition 2.1 A **Krausz partition** $\mathcal{K}(H)$ *of a line graph* $H = L_G$ *is the set of cliques (maximal complete subgraphs) such that every edge of* L_G *is in exactly one clique and every vertex of* L_G *is in exactly two cliques [4].*

Two cliques, in $\mathcal{K}(H)$, of the line graph, H, of a triangle-free graph, have at most one vertex in common. Thus the set of vertices, adjacent to a given vertex in H, can be partitioned into no more than two complete subgraphs of H.

It is well known that, from the p-deck of characteristic polynomials of vertex-deleted subgraphs of a graph H, one can readily determine, for each vertex w_i, the degree d_i and the number T_i of triangles through w_i. Moreover, if H is a line graph L_G and u, v are adjacent vertices in G of degree

$x_u + 1$, $x_v + 1$ respectively then
(i) the degree d_{uv} of the edge uv in G as a vertex of H is $x_u + x_v$, $x_u \geq x_v$; and
(ii) the number of triangles in H through the vertex uv is

$$\binom{x_u}{2} + \binom{x_v}{2} + T_{uv} \tag{1}$$

where T_{uv} is the number of triangles in G containing edge uv.

Lemma 2.1 *For a two-partition into $x,\ y \in \mathbb{Z}^+ \cup \{0\}$ of $\rho \in \mathbb{Z}^+$, the integer $T = \binom{x}{2} + \binom{y}{2}$ takes distinct values as x runs through the values 0 to $\lfloor \frac{\rho}{2} \rfloor$. Moreover, T determines uniquely the couple (x, y), $x \geq y$.*

Proof: Since $x+y = \rho$, then $T = x^2 - \rho x + \frac{\rho^2}{2} - \frac{\rho}{2}$. Thus T is a quadratic function in x and reaches its minimum value when $x = \frac{\rho}{2}$. Furthermore T decreases steadily as x runs through the values 0 to $\lfloor \frac{\rho}{2} \rfloor$. □

Remark: It is noted that only when $(\rho, x) = (1, 0)$ or when $(\rho, x) = (2, 1)$ is $T = 0$. When $\rho > 2$, $T > 0$.

2.1 Two Conditions P_1 and P_2

Given $\mathcal{P}D$ and supposing it is the p-deck of characteristic polynomials of the one-vertex-deleted subgraphs of the line graph H of a triangle-free graph G, let $\{d_i\}$, $1 \leq i \leq m$, be the degree sequence of H and $\{T_i\}$, $1 \leq i \leq m$, be the number of triangles in H through the vertices $\{w_i\}$ of H.

Definition 2.2 *A p-deck $\mathcal{P}D$ is said to satisfy the* **condition** *P_1 if for each i, $1 \leq i \leq m$, the equations*

$$x + y = d_i \tag{2}$$

and

$$\binom{x}{2} + \binom{y}{2} = T_i \tag{3}$$

have a unique solution (x_i, y_i) of couples of non-negative integers with $x_i \geq y_i$.

It is clear from Lemma 2.1 that for a p-deck that satisfies condition P_1 there is a **unique** two-partition of each d_i. Also the p-deck of a line graph satisfies condition P_1.

Definition 2.3 *Let $\mathcal{PD}$ satisfy condition P_1 with the appropriate set of two-partitions of the vertex degrees $d_i = (x_i + y_i)$ for each i. Then, the* **end-edge-degree sequence of couples** *eed is $\{(x_i+1, y_i+1) : x_i \geq y_i\}$.*

The sequence *eed* not only determines the two cliques that share a particular vertex in $H = L_G$ but also $\mathcal{K}(H)$, the Krausz partition of H. It also determines the degrees of the end vertices of each edge in G.

2.2 Extraction of the Root Graph

Definition 2.4 **The repeated degree sequence, dgr,** *is the list (with repetitions) of the entries in each couple (x_i+1, y_i+1) of eed and is denoted by $\{(z_j + 1)^{t_j}\}$ where t_j is the number of times $z_j + 1$ is repeated in dgr.*

Definition 2.5 *A p-deck $\mathcal{PD}$ is said to satisfy* **condition** *P_2 if for each distinct term $z_j + 1$ in $dgr = \{(z_j+1)^{t_j}\}$, there exists a positive integer m_j such that $t_j = (z_j + 1)m_j$.*

Remark:

1. In the case when $\mathcal{PD}$ is the p-deck of the line graph of a triangle-free graph G, then m_j is equal to the number of edges with an end-vertex of degree $z_j + 1$ in G.
2. When the partition of d_i is $d_i = 2x_i$ so that $x_i = y_i$, the term x_i contributes twice to m_j.

Lemma 2.2 *Let G be a triangle-free graph and let $\mathcal{PD}$ be the p-deck of its line graph. Let $dgr = \{(z_j + 1)^{t_j}\}$ be derived from $\mathcal{PD}$. If there exists $m_j \in \mathbb{Z}^+$ such that $t_j = (z_j + 1)m_j$, then the root graph G of H has degree sequence* $dgg(G) = \{(z_j + 1)^{m_j}\}$.

Proof: A vertex in H is shared by two cliques K_{x_j+1} and K_{y_j+1} in $\mathcal{K}(H)$ and contributes the couple $(x_j + 1, y_j + 1)$ to *eed*. Each of the $z_j + 1$ vertices of a clique K_{z_j+1} contributes the term $z_j + 1$ to *dgr*. So if the clique K_{z_j+1} is repeated m_j times in $\mathcal{K}(H)$, then the term $z_j + 1$ appears $m_j(z_j + 1)(= t_j)$ times in *dgr*. But the number of cliques K_{z_j+1} in $\mathcal{K}(H)$ is the number of vertices of degree $z_j + 1$ in G. Thus $z_j + 1$ is repeated m_j times in *dgg*. □

Remarks:

1. That $\mathcal{PD}(L_G)$ satisfies condition P_2 follows from Lemma 2.2.
2. The p-deck of L_G readily determines $|\mathcal{E}(G)|$ but not the order of G. However, this is easily worked out from the sequence $dgr(L_G)$.

Corollary 2.1 *Let G be a triangle-free graph. If $dgr(L_G) = \{r_i^{(m_i, r_i)}\}$ then the order of G is $\sum m_i$.*

2.3 Conditions Not Sufficient

The condition P_1 alone is not enough to determine a line graph of a triangle-free graph as shown by the graph shown in Figure 1.

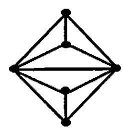

Figure 1. A Beineke Graph

With care, one can construct a class of counter examples $\mathcal{F}$ showing that not even the two conditions P_1 and P_2 together are sufficient to determine a line graph of a triangle-free graph. One such graph in $\mathcal{F}$, is F, of order 1162, shown in Figure 2. This is because at a vertex of degree 9, a decomposition into two cliques of order 6 and 5 gives the same number T of triangles as the decomposition, found in graph F, into three cliques of order 7, 3 and 2.

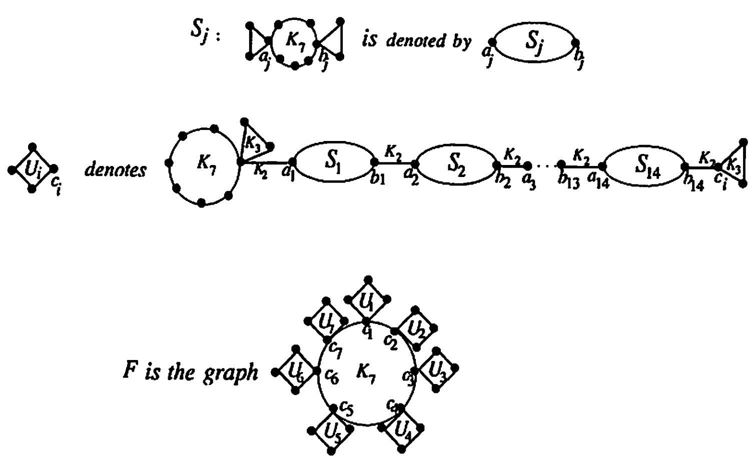

Figure 2. The Graph F

Clearly graph F is not a line graph since the forbidden claw $K_{1,3}$ is an induced subgraph at every vertex of degree 9 but satisfies both conditions P_1 and P_2.

3 Recognition and Reconstruction

Let H be a line graph of a triangle-free graph G. It is recalled that

$$\phi'(H, \lambda) = \sum_{i=1}^{m} \phi(H - w_i, \lambda).$$

By integrating, $\phi(H)$ is determined, save for the constant term which is $\mathrm{Det}(-H)$. When a line graph, L_G, is regular then its root graph, G, is either regular or semiregular bipartite [1], i.e. a bipartite graph in which the vertices in one part have degree k and those in the other part have degree j. The p-deck of a regular graph H immediately reveals the degree ρ of a vertex which is the largest eigenvalue of H so that $\phi(\rho) = 0$. Thus $\mathrm{Det}(-A(H))$ and hence $\phi(H)$ is determined.

For irregular graphs $H (= L_G)$, the algorithm *Alg*, which we now present, reconstructs, from a legitimate p-deck $\{\phi(H - w_i, \lambda)\}$, the characteristic polynomial $\phi(H, \lambda)$, provided G is a triangle-free graph. Though not sufficient, conditions P_1, P_2 act as a filter to recognise the p-deck of the line graph of a triangle-free graph and the exceptional graphs in $\mathcal{F}$. The algorithm *Alg* is constructed in such a way that the root graph G is also identified. The exceptional graphs, denoted by the set $\mathcal{F}$, are eliminated at the last stage of the algorithm when the p-deck of L_G is compared with the original p-deck $\mathcal{PD}$.

3.1 The Algorithm *Alg*

Given a p-deck $\mathcal{PD} = \{\phi_i\}$ of m monic polynomials each of degree $m-1$ with the coefficient of x^{m-2} being zero, *Alg* determines whether $\mathcal{PD}$ is the p-deck of the irregular line graph of order m of a triangle-free graph G and outputs $\phi(L_G)$.

Step 1: Let Σ be the sum of all the polynomials in the p-deck. Then $\phi = \int \Sigma$ is determined.

Step 2: The sequence *dgl* is $\{d_i\}$ where d_i is the difference in the coefficients of $-\lambda^{m-2}$ in ϕ and of $-\lambda^{m-3}$ in ϕ_i. If d_i is a constant for all i, then the procedure is stopped since a possible L_G is not irregular.

Step 3: The sequence *Tri* is $\{T_i\}$ where T_i is half the difference in the coefficients of $-\lambda^{m-3}$ in ϕ and of $-\lambda^{m-4}$ in ϕ_i.

Step 4: If $\mathcal{PD}$ does not satisfy condition P_1, then it is not the legitimate p-deck of the line graph of a triangle-free graph and the procedure is stopped. Otherwise the sequences *eed* and *dgr* are formed. The entries of a couple in *eed* give the degrees of the two end-vertices of an edge in G. So by running through the couples in *eed*, the function ψ is formed, defined by $\psi(d) = b$, where b is the list of degrees of the vertices that would have a neighbour of degree d in G provided that $\mathcal{PD} = \mathcal{PD}(L_G)$.

Step 5: If $\mathcal{PD}$ does not satisfy condition P_2, then it is not the legitimate p-deck of the line graph of a triangle-free graph and the procedure is stopped. Otherwise, a graph L_G (or perhaps an exceptional graph in $\mathcal{F}$) exists satisfying P_1 and P_2. If *dgr* is $\{(z_j + 1)^{t_j}\}$, then *dgg* is derived from *dgr*. For each j, t_j is divided by $(z_j + 1)$ to give the multiplicity of the clique K_{z_j+1} in $\mathcal{K}(L_G)$, which is equal to the multiplicity of the degree $z_j + 1$ in the degree sequence, *dgg*, of G.

Step 6: By means of the function ψ and the degree sequence *dgg*, all possible root graphs G are constructed. For each possible root graph G, the set $\mathcal{S}(G)$ of characteristic polynomials of the one-vertex-deleted subgraphs of the line graph of each G, is calculated and compared with $\mathcal{PD}$.

Step 7: At this stage there are three possible results:

Case 1: If $\mathcal{S}(G) = \mathcal{PD}$ for exactly one graph G, then L_G and $\phi(L_G)$ are determined uniquely.

Case 2: If $\mathcal{S}(G) = \mathcal{PD}$ for at least two non-isomorphic graphs G_1 and G_2, then the two line graphs $H_1 = L_{G_1}$ and $H_2 = L_{G_2}$ are non-isomorphic since there exists a 1-1 mapping between a graph of order greater than four and its line graph. In fact the only line graph that does not have a unique root graph is K_3 whose root graphs are $K_{1,3}$ and K_3 (the latter not being triangle-free).

The pair of graphs H_1 and H_2 obtained would provide **a counter example to the reconstruction problem B** (which has already been proved false [5]).

The constant terms $\mathrm{Det}(-A(H_1))$ and $\mathrm{Det}(-A(H_2))$, which may be determined directly, are equal because according to [6], counter examples to **the reconstruction problem D** are not to be found among graphs with their smallest eigenvalue bounded below by -2, which include line graphs. This means that $\phi(H)$ is unique.

Case 3: Because P_1 and P_2 are not sufficient to recognize an irregular line graph of a tree it may happen that no element of the set $\mathcal{S}(G)$ is the same as $\mathcal{PD}$ so that the procedure is stopped. In this case, $\mathcal{PD}$ is a p-deck that satisfies conditions P_1 and P_2 but is **not** the p-deck of the line graph of a triangle-free graph. Either the p-deck $\mathcal{PD}$ is not legitimate or else we have a rare case when $\mathcal{PD}$ is the p-deck of a graph in $\mathcal{F}$, such as F of Figure 2.

4 Example

We tried *Alg*, using the software **Mathematica**, in programming mode, on several p-decks and most of them yielded one root graph. An example will now be given to illustrate a case when more than one possible root graph is obtained.

Example 4.1

$$
\begin{aligned}
\textit{Let } \mathcal{PD} = \; & \{-1+6x^2-5x^4+x^6, \\
& -1+4x^2-4x^4+x^6, \\
& -1+4x^2-4x^4+x^6, \\
& 2x+4x^2-2x^3-5x^4+x^6, \\
& 2x+4x^2-2x^3-5x^4+x^6, \\
& -1+2x+7x^2-2x^3-6x^4+x^6, \\
& -1+2x+7x^2-2x^3-6x^4+x^6\}
\end{aligned}
$$

Supposing that $\mathcal{PD}$ is the p-deck of a line graph $H = L_G$,
the degree sequence of H is $dgl = \{2,3,3,2,2,1,1\}$,
the sequence of triangles through each vertex is $Tri = \{1,1,1,0,0,0,0\}$,
$eed = \{(1,3),(2,3),(2,3),(2,2),(2,2),(1,2),(1,2)\}$,
$dgr = \{1,1,1,2,2,2,2,2,2,2,2,3,3,3\} = \{1^3,2^8,3^3\}$,
$dgg = \{1^3,2^4,3^1\}$,
$\mathcal{K} = \{3K_1,4K_2,K_3\}$.

$$
\psi : \begin{cases} 1 & \mapsto \{2,2,3\} \\ 2 & \mapsto \{3,3,2,2,2,2,1,1\} \\ 3 & \mapsto \{2,2,1\} \end{cases}
$$

If $\mathcal{PD}$ is the p-deck of the line graph of a triangle-free graph then there are two possible root graphs G_1, G_2 shown in Figure 3.

Figure 3. The graphs G_1, G_2 and their line graphs

The p-deck of $L_{G_2}(= H)$ agrees with $\mathcal{PD}_2$ but that of L_{G_1} does not. So $\mathcal{PD}_2$ is the p-deck of the line graph of the triangle-free graph G_2 with $\phi(H) = -2-5x+4x^2+12x^3-2x^4-7x^5+x^7$.

For an irregular line graph H of a triangle-free graph G, this method proves to be a powerful tool to determine the root graph G, H itself and its characteristic polynomial, $\phi(H)$, from a suitable p-deck $\mathcal{PD}$. It is particularly efficient when in the degree sequence of the triangle-free root graph, dgg, one or more terms larger than 1 have multiplicity one. Its efficiency is inversely proportional to the number of root graphs G whose degrees meet the constraints imposed by the sequence eed. Since this sequence determines the list of degrees of the neighbours of vertices of each distinct degree in the root graph G, it restricts very effectively the number of possible root graphs (very often to just one possibility).

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