On the construction of directed triplewhist tournaments

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Abstract. We show that, for all primes $p \equiv 1 \pmod 4$, $29 \leq p < 10,000$, $p \neq 97, 193, 257, 449, 641, 769, 1153, 1409, 7681$, there exist Z-cyclic triplewhist tournaments on p elements which are also Mendelsohn designs. We also show that such designs exist on v elements whenever v is a product of such primes p.

1. Introduction A whist tournament $Wh(v)$ for $v = 4m + 1$ players is a schedule of games (a, b, c, d) involving two players a, c opposing two other players b, d such that

(1) the games are arranged into $4m + 1$ rounds, each of m games;

(2) each player plays in exactly one game in all but one of the rounds;

(3) each player partners every other player exactly once;

(4) each player opposes every other player exactly twice.

It may be helpful to think of (a, b, c, d) as the cyclic order of the 4 players sitting round a table, with partners sitting opposite each other. See [1, 2] for basic information about whist tournaments, and note in particular that a $Wh(4m + 1)$ exists for all $m \geq 1$.

We are concerned here with two refinements of this structure, called *triplewhist* tournaments and *directedwhist* tournaments. Call the pairs $\{a, b\}$ and $\{c, d\}$ pairs of *opponents of the first kind* and $\{a, d\}$, $\{b, c\}$ pairs of *opponents of the second kind.* Then a *triplewhist* tournament $TWh(v)$ is a $Wh(v)$ in which every player is an opponent of the first (resp. second) kind exactly once with every other player. We further say that b is a's *left hand opponent* and c's *right hand opponent*, and make similar definitions for each of a, c and d. Then a *directedwhist* tournament $DWh(v)$ is a $Wh(v)$ in which each player is a left (resp. right) hand opponent of every other player exactly once. *Directedwhist* tournaments are equivalent to resolvable perfect Mendelsohn designs of block size 4.

If the players are elements in Z_{4m+1}, and if the ith round is obtained from the initial (first) round by adding $i - 1 \pmod{4m + 1}$ to each element, then we

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say that the tournament is *Z-cyclic* (see, e.g. [3–8]). By convention, we always take the initial round to be the round from which 0 is absent. We note that the games (tables)

$$(a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m)$$

form the initial round of a Z-cyclic $TWh(4m+1)$ if

(α) $\bigcup_i \{a_i, b_i, c_i, d_i\} = Z_{4m+1} - \{0\}$,

(β) $\bigcup_i \{\pm(a_i - c_i), \pm(b_i - d_i)\} = Z_{4m+1} - \{0\}$,

(γ) $\bigcup_i \{\pm(a_i - b_i), \pm(c_i - d_i)\} = Z_{4m+1} - \{0\}$,

(δ) $\bigcup_i \{\pm(a_i - d_i), \pm(b_i - c_i)\} = Z_{4m+1} - \{0\}$,

whereas they form the initial round of a Z-cyclic $DWh(4m+1)$ if (α) and (β) hold along with

(ϵ) $\bigcup_i \{b_i - a_i, c_i - b_i, d_i - c_i, a_i - d_i\} = Z_{4m+1} - \{0\}$.

(The differences in (ϵ) are known as the *forward* differences.)

In a recent paper [5] the present authors gave the first examples of whist tournaments which are simultaneously triplewhist and directedwhist tournaments. Such tournaments are denoted by $DTWh(v)$. For the initial round of a Z-cyclic $DTWh(4m+1)$ all of (α)–(ϵ) must be satisfied.

Example 1.1 [5] A Z-cyclic $DTWh(29)$. Take as the initial round games

$$(1,19,10,9),(24,21,8,13),(25,11,18,22),(20,3,26,6),$$
$$(16,14,15,28),(7,17,12,5),(23,2,27,4).$$

It was shown in [5] that $DTWh(p)$ exists for all primes $p \equiv 5 \pmod 8$, $p \geq 29$, and indeed that a $DTWh(v)$ exists whenever v is a product of such primes. In this present paper we consider primes $p = 2^k t + 1$, t odd, $k \geq 3$. We show that Z-cyclic $DTWh(p)$ exist for all such p, $41 \leq p < 10{,}000$, except possibly for nine exceptional values of p. We then extend these results to all products of such primes.

We close this introductory section by presenting the initial round of three Z-cyclic $DTWh(p)$. In each case the reader can check that all of (α)–(ϵ) hold. How these designs were obtained will be explained in Section 2.

Example 1.2 Initial round of a Z-cyclic $DTWh(41)$

$$(1,3,31,34),(36,26,9,35),(10,30,23,12),(32,14,8,22),$$
$$(18,13,25,38),(33,17,39,15),(16,7,4,11),(2,6,21,27),$$
$$(37,29,40,28),(20,19,5,24).$$

Example 1.3 $DTWh(73)$

$$(1,26,47,36),(19,56,17,27),(16,51,22,65),(12,20,53,67),$$
$$(37,13,60,18),(46,28,45,50),(8,62,11,69),(6,10,63,70),$$
$$(55,43,30,9),(23,14,59,25),(4,31,42,71),(3,5,68,35),$$
$$(64.58,15,41),(48,7,66,49),(2,52,21,72),(38,39,34,54)$$
$$(32,29,44,57),(24,40,33,61).$$

Example 1.4 $DTWh(89)$

$$(1,3,88,61),(9,27,80,15),(64,14,25,77),(42,37,47,70),$$
$$(2,6,87,33),(18,54,71,30),(39,28,50,65),(84,74,5,51),$$
$$(4,12,85,66),(36,19,53,60),(78,56,11,41),(79,59,10,13),$$
$$(8,24,81,43),(72,38,17,31),(67,23,22,82),(69,29,20,26),$$
$$(16,48,73,86),(55,76,34,62),(45,46,44,75),(49,58,40,52),$$
$$(32,7,57,83),(21,63,68,35).$$

2. Construction of Z-cyclic $DTWh(p)$ Let $p = 2^k t + 1$ be prime, $k \geq 3$, t odd. Throughout this section we write $d = 2^k$, $m = 2^{k-1}$, $n = 2^{k-2}$. We shall make use of the following two results.

Theorem 2.1 (Liaw [9, Proposition 3.1].) *Let $p = 2^k t + 1$ be prime, t odd, $k \geq 3$ and let W be a primitive root of p. Let $d = 2^k$ and $n = 2^{k-2}$. Then the games*

$$(1, W, -W, W^{1+a}) \text{ times } W^{2i+dj}, \quad 0 \leq i \leq n-1, \ 0 \leq j \leq t-1,$$

form the initial round of a Z-cyclic $TWh(p)$ provided

$$a \equiv 2^{k-1} - 1 \pmod{d}, \tag{2.1}$$

and

$$2(W^{a+1} - 1), \ (W+1)(W^a - 1), \ (W-1)(W^a + 1) \tag{2.2}$$

are all squares in Zp.

Lemma 2.2 *Let y be any residue* (mod 2^k) *and let $s \in \{0,1,\ldots,2^{k-1}-1\}$ be fixed. Then*

$$\bigcup_{i=0}^{2^{k-2}-1} \{y+2i, y+2s+1+2i, y+2^{k-1}+2i, y+2^{k-1}+2s+1+2i\}$$

is a complete set of residues (mod 2^k).

Construction A Consider Liaw's construction of $TWh(p)$ presented in Theorem 2.1. The forward differences (ϵ) are

$$\{W-1, -2W, W(W^a+1), -(W^{a+1}-1)\} \text{ times } W^{2i+dj}.$$

If we write $W-1 = W^e$, $-2W = W^f$, $W(W^a+1) = W^g$, $1-W^{a+1} = W^h$, then, by Lemma 2.2, Condition (ϵ) will be satisfied provided

$$\{e,f,g,h\} = \{y, y+2s+1, y+m, y+2s+1+m\} \tag{2.3}$$

is a subset of Z_d for some $s \in \{0,1,\ldots,m-1\}$, $y \in \{0,1,\ldots,d-1\}$.

In Table A (placed at the end of the paper) we tabulate choices of (p, W, a) which satisfy (2.1), (2.2) and (2.3), thereby establishing the existence of a $DTWh(p)$ for the majority of $p < 10,000$.

Example 2.1 Consider $p = 73$. Here $t = 9$, $k = 3$, $d = 8$, $m = 4$, $n = 2$. As in Table A, we take $W = 26$ and $a = 51$ to obtain initial round games of a $DTWh(73)$ to be

$$(1, 26, 47, 36) \text{ times } 26^{2i+8j}, \quad 0 \le i \le 1, \ 0 \le j \le 8.$$

These are the games exhibited in Example 1.3.

To deal with further values of p, we now consider a more general construction. Consider the initial round games

$$(1, W^c, W^f, -W^e) \text{ times } W^{2i+dj}, \quad 0 \le i < n, \ 0 \le j < t \tag{2.4}$$

where $c = \alpha + k_1 d$ (α odd), $e = k_2 d$, $f = m + \alpha + k_3 d$. (Note that the case $c = \alpha = 1$, $e = 1 + a - mt$, $f = m + 1 + m(t-1) = mt + 1$ yields Construction A as a special case.)

The choice $y = 0$, $\alpha = 2s+1$ in Lemma 2.2 shows that the elements of the games in the initial round (2.4) do indeed constitute the set $Z_p - \{0\}$.

It is straightforward to check that the triplewhist conditions (β)–(δ) are satisfied provided

$$(W^f-1)(W^c+W^e), \ (W^c-1)(W^e+W^f), \ (W^e+1)(W^c-W^f)$$

are all nonsquares.

Observe next that triplewhist conditions (β)–(δ) are preserved under a permutation of the positions in the tables. So we introduce three variants of the proposed general construction (2.4):

$$\begin{aligned} GC1 &: (1, W^c, -W^e, W^f) \text{ times } W^{2i+dj}; \\ GC2 &: (1, -W^e, W^c, W^f) \text{ times } W^{2i+dj}; \\ GC3 &: (1, W^c, W^f, -W^e) \text{ times } W^{2i+dj}. \end{aligned}$$

For all but nine of the primes $p < 10,000$ not covered by Table A, we have found choices of c, e, f for which one of GC1, GC2, GC3 yields a $DTWh(p)$. This data appears in Table B (also placed at the end of the paper) in the form (p, GC#, W, c, e, f).

Example 2.2 Consider $p = 41$. Here $t = 5$, $d = 8$, $m = 4$, $n = 2$. In construction GC1 take $W = 6$, $c = 15$, $e = 8$, $f = 19$ to get initial round games $(1, 6^{15}, -6^8, 6^{19})$ times 6^{2i+8j} i.e. $(1, 3, 31, 34)$ times 6^{2i+8j}, $0 \le i \le 1$, $0 \le j \le 4$. These are the games of Example 1.2.

Example 2.3 Consider $p = 89$. Here $t = 11$, $d = 8$, $m = 4$, $n = 2$. In construction GC1 take $W = 3$, $c = 1$, $e = 0$, $f = 69$ to get initial round games $(1, 3, -1, 3^{69})$ times 3^{2i+8j}, i.e. $(1, 3, 88, 61)$ times 3^{2i+8j}. That is to say the games of Example 1.4.

Combining the results of this section with those of [5], we now have the following theorem.

Theorem 2.3 *If* $p \equiv 1 \pmod 4$ *is prime,* $29 \le p < 10,000$, $p \notin \{97, 193, 257, 449, 641, 769, 1153, 1409, 7681\}$ *then a Z-cyclic* $DTWh(p)$ *exists.*

The excluded cases remain open.

Remark The general constructions GC1–GC3 contain not only Construction A as a special case, but also all of the constructions used in [5] for the case $k = 2$ (i.e. $d = 4$, $m = 2$):

GC1 with $c = 1$, $e = 0$, $f = 3$ gives Construction 1 of [5];

GC1 with $c = 3$, $e = 2(t+1)$, $f = 5 + 2(t-1)$ gives Construction 2 of [5];

GC1 with $c = 3$, $e = 4$, $f = 2t + 3$ gives Construction 3 of [5];

GC1 with $c = 1$, $e = 4$, $f = 2t + 1$ gives Construction 4 of [5];

GC1 with $c = 2t + 3$, $e = 4$, $f = 2t + 1$ gives initial round games $(1, -W^3, -W^4, -W)$ times W^{4j} which is just Construction 6 in [5] with games (tables) written in reverse (anticlockwise) order.

3. Products of primes We now establish

Theorem 3.1 *Let $p_1, p_2, \ldots, p_n$ be primes such that a Z-cyclic $DTWh(p_i)$ exists for all i. Then there exists a Z-cyclic $DTWh(p_1^{\alpha_1} \cdots p_n^{\alpha_n})$ for all choices of $\alpha_i \geq 1$.*

This result follows by repeated application of the following product theorem which is proved by the method recently developed by the authors and P. Leonard [6].

Theorem 3.2 *Suppose that Z-cyclic $DTWh(v)$ and $DTWh(p)$ exist where p is a prime, $p \equiv 1 \pmod 4$. Then a Z-cyclic $DTWh(pv)$ exists.*

Proof We show how to construct the initial round of the required tournament. First take the initial round of a $DTWh(p)$ and multiply each element by v. Next, for each table (a, b, c, d) in the initial round of the $DTWh(v)$, construct the tables

$$(a + iv, b + 2iv, c - iv, d - 2iv), \quad 0 \leq i \leq p - 1.$$

It is then routine to check that these tables include each nonzero element of Z_{pv}, not divisible by v, exactly once, and that (α)–(ϵ) hold. For example, in checking (β), suppose that

$$a_1 + iv - (c_1 - iv) \equiv \pm[b_2 + 2jv - (d_2 - 2jv)] \pmod{pv}.$$

Then $a_1 - c_1 \equiv \pm(b_2 - d_2) \pmod v$, contradicting (β) for the $DTWh(v)$. Similarly, in checking (ϵ), suppose for example that

$$b_1 + 2iv - (a_1 + iv) \equiv \pm[d_2 - 2jv - (c_2 - jv)] \pmod{pv}.$$

Then $b_1 - a_1 \equiv \pm(d_2 - c_2) \pmod v$, contradicting (ϵ) for the $DTWh(v)$.

Example 3.1 A $DTWh(1189)$. Take $p = 29$ and $v = 41$ ($pv = 1189$), and use Examples 1.1 and 1.2. The initial round game of the required $DTWh(1189)$ are

$$(41, 779, 410, 369), (984, 861, 328, 533), (1025, 451, 738, 902),$$
$$(820, 123, 1066, 246), (656, 574, 615, 1148), (287, 697, 492, 205),$$
$$(943, 82, 1107, 164);$$

$$\begin{array}{ll}
(1 + 41i, 3 + 82i, 31 - 41i, 34 - 82i), & 0 \leq i \leq 28; \\
(36 + 41i, 26 + 82i, 9 - 41i, 35 - 82i), & 0 \leq i \leq 28; \\
(10 + 41i, 30 + 82i, 23 - 41i, 12 - 82i), & 0 \leq i \leq 28; \\
(32 + 41i, 14 + 82i, 8 - 41i, 22 - 82i), & 0 \leq i \leq 28; \\
(18 + 41i, 13 + 82i, 25 - 41i, 38 - 82i), & 0 \leq i \leq 28; \\
(33 + 41i, 17 + 82i, 39 - 41i, 15 - 82i), & 0 \leq i \leq 28; \\
(16 + 41i, 7 + 82i, 4 - 41i, 11 - 82i), & 0 \leq i \leq 28; \\
(2 + 41i, 6 + 82i, 21 - 41i, 27 - 82i), & 0 \leq i \leq 28; \\
(37 + 41i, 29 + 82i, 40 - 41i, 28 - 82i), & 0 \leq i \leq 28; \\
(20 + 41i, 19 + 82i, 5 - 41i, 24 - 82i), & 0 \leq i \leq 28.
\end{array}$$

Table A

$k = 3 \quad (p, W, a)$

(73, 26, 51), (137, 40, 99), (233, 34, 155), (281, 23, 67), (313, 17, 147), (409, 29, 291), (457, 26, 443), (521, 3, 331), (569, 6, 163), (601, 7, 27), (617, 3, 275), (809, 11, 347), (857, 3, 19), (937, 10, 531), (953, 10, 643), (1033, 5, 299), (1049, 3, 331), (1097, 3, 163), (1129, 11, 339), (1193, 3, 171), (1289, 6, 1091), (1321, 13, 491), (1433, 7, 331), (1481, 6, 1019), (1609, 7, 83), (1657, 15, 35), (1721, 12, 115), (1753, 7, 83), (1801, 11, 243), (1913, 3, 107), (1993, 5, 219), (2089, 11, 1011), (2137, 10, 299), (2153, 3, 1483), (2281, 7, 43), (2297, 5, 203), (2377, 5, 523), (2393, 3, 27), (2441, 6, 1299), (2473, 5, 515), (2521, 17, 219), (2617, 5, 267), (2633, 3, 1003), (2713, 5, 123), (2729, 3, 11), (2777, 3, 403), (2857, 22, 307), (2953, 13, 1899), (2969, 3, 1163), (3001, 23, 195), (3049, 11, 1347), (3209, 3, 59), (3257, 3, 1379), (3433, 5, 1163), (3449, 3, 315), (3529, 17, 1307), (3593, 3, 707), (3673, 5, 147), (3769, 7, 603), (3833, 3, 2915), (3881, 17, 955), (3929, 3, 1203) (4057, 5, 947), (4073, 3, 2091), (4153, 5, 1571), (4201, 11, 19), (4217, 3, 227), (4297, 5, 75), (4409, 3, 3595), (4441, 21, 739), (4457, 3, 267), (4649, 3, 2419), (4729, 17, 259), (4793, 3, 683), (4889, 3, 315), (4937, 3, 1995), (4969, 11, 379), (5081, 6, 907), (5113, 19, 643), (5209, 17, 531), (5273, 3, 1011), (5417, 3, 643), (5449, 7, 3955), (5641, 14, 1131), (5657, 3, 1499), (5689, 11, 59), (5737, 5, 459), (5801, 3, 371), (5849, 3, 1379), (5881, 31, 4067), (5897, 3, 1307), (6073, 10, 59), (6089, 3, 195), (6121, 7, 211), (6217, 5, 2051), (6329, 3, 1227), (6361, 19, 1491), (6473, 3, 595), (6521, 6, 411), (6553, 10, 995), (6569, 3, 139), (6761, 3, 691), (6793, 10, 571), (6841, 22, 835), (6857, 3, 739), (7001, 3, 1299), (7129, 7, 2283), (7177, 10, 1251), (7193, 3, 283), (7321, 7, 467), (7369, 7, 2131), (7417, 5, 715), (7433, 3, 1011), (7481, 6, 715), (7529, 3, 2595), (7561, 13, 523), (7577, 3, 603) (7673, 3, 195), (7753, 10, 1235), (7817, 3, 2699), (7993, 5, 139), (8009, 3, 179), (8089, 17, 731), (8233, 10, 1139), (8297, 3, 35), (8329, 7, 19), (8377, 5, 99), (8521, 13, 1027), (8537, 3, 43), (8681, 15, 291), (8713, 5, 803), (8761, 23, 787), (8969, 3, 891), (9001, 7, 1131), (9049, 7, 299), (9161, 3, 907), (9209, 3, 2339), (9241, 13, 467), (9257, 3, 3083), (9337, 5, 355), (9433, 5, 1291), (9497, 3, 643), (9689, 3, 1051), (9721, 7, 2867), (9769, 13, 2563), (9817, 5, 643), (9833, 3, 363), (9929, 3, 1043),

$k = 4 \quad (p, W, a)$

(241, 99, 39), (337, 46, 279), (401, 236, 343), (433, 57, 87), (593, 5, 487), (881, 15, 535), (977, 5, 759), (1009, 33, 311), (1201, 29, 1063), (1297, 15, 359), (1361, 24, 487), (1489, 29, 759), (1553, 5, 263), (1777, 10, 407), (1873, 37, 903), (2129, 3, 775), (2161, 69, 1143), (2417, 17, 1159), (2609, 6, 1799), (2801, 3, 471), (2833, 10, 1383), (2897, 5, 1607), (3089, 3, 1287), (3121, 44, 215), (3217, 15, 87), (3313, 35, 1335), (3697, 20, 1767), (3761, 19, 2535), (3793, 19, 3687), (3889, 29, 2311), (4049, 3, 4023), (4177, 11, 151), (4241, 51, 23), (4273, 7, 1367), (4337, 12, 4151), (4561, 11, 823), (4657, 35, 231), (4721, 7, 7), (4817, 10, 1943), (5009, 29, 2887), (5233, 13, 1975), (5297, 17, 695), (5393, 3, 1655), (5521, 11, 2743), (6257, 3, 5111), (6353, 5, 5015), (6449, 6, 3447), (6481, 28, 1639), (6577, 30, 1575), (6673, 10, 6535), (6737, 10, 5575), (6833, 3, 1703), (6961, 13, 5015), (7057, 5, 3255), (7121, 3, 3207), (7537, 7, 1111), (7793, 5, 3495), (8017, 5, 631), (8081, 3, 119), (8209, 7, 6919), (8273, 3, 71), (8369, 12, 23), (8689, 13, 5991), (8753, 3, 5847), (8849, 3, 6103), (9041, 11, 3527), (9137, 3, 5319), (9521, 3, 295), (9649, 7, 103),

$k = 5 \quad (p, W, a)$

(673, 485, 175), (929, 382, 239), (1249, 55, 1135), (1697, 27, 815), (1889, 24, 1487), (2017, 107, 879), (2081, 17, 1327), (2273, 76, 175), (2593, 26, 623), (2657, 6, 1583), (3041, 132, 2351), (3169, 94, 2031), (3361, 31, 2415), (3617, 24, 2895), (4001, 135, 1391), (4129, 131, 3151), (4513, 28, 3951), (5153, 12, 2799), (5281, 91, 3119), (5857, 266, 4143), (6113, 44, 335), (6689, 13, 1103), (7393, 15, 3919), (7457, 31, 911), (7649, 56, 1071), (7841, 51, 4687), (8161, 97, 4847), (8353, 15, 2447), (8609, 15, 2895), (8737, 37, 5199), (8929, 19, 2959), (9377, 24, 3087), (9697, 88, 2447).

$k = 6 \quad (p, W, a)$

(1217, 89, 1119), (2113, 413, 479), (4289, 368, 543), (4673, 872, 4447), (4801, 861, 2207), (5441, 332, 991), (5569, 342, 3231), (5953, 798, 2143), (6337, 1456, 4959), (6977, 54, 1951), (7489, 69, 6751), (7873, 14, 7519), (8513, 96, 5471), (8641, 47, 3295), (9281, 69, 6559).

$k = 8 \quad (p, W, a)$

(7937, 11, 127).

Table B
$(p, GC\#, W, c, e, f)$:

(41, 1, 6, 15, 8, 19), (89, 1, 3, 1, 0, 69), (113, 1, 3, 17, 0, 89), (353, 1, 3, 23, 96, 167), (577, 2, 5, 9, 384, 169), (1601, 3, 3, 39, 64, 647), (2689, 2, 19, 121, 1792, 2105), (2753, 1, 3, 47, 704, 527), (3137, 2, 3, 21, 64, 2293), (3329, 3, 3, 307, 256, 1203), (3457, 3, 7, 215, 768, 3223), (4481, 3, 3, 105, 3200, 169), (4993, 1, 5, 63, 128, 1919), (6529, 1, 7, 39, 640, 871), (7297, 3, 5, 123, 128, 5051), (9473, 1, 3, 183, 1024, 7223), (9601, 3, 13, 69, 640, 9093), (9857, 1, 5, 77, 384, 3853).

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