Excluded Minors for [2,3]-Graph Planarity

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Abstract

A Kuratowski-type approach for [2,3]-graphs, i.e. hypergraphs the cardinality of whose edges not more than 3, is presented, leading to a well-quasi-order in such a context, with a complete obstruction set of six forbidden hypergraphs to plane embedding.

A Kuratowski-type result is presented for finite hypergraphs, the cardinality of whose edges is not more than 3. Motivation goes back to [1, 2] and is related to the representation of graphs that can be interpreted as hypergraphs all of whose edges have cardinality 3. (For example, the graphs $G_{n,4}$ of [1] are nonplanar for $n \geq 19$, n odd).

The statement "graphs to be considered may have loops and multiple edges", used in Graph Minor Theory, is generalized as follows. Given a set V and a positive integer r, an *r-multisubset* M of V is a collection $\{(v_1,\mu_1),(v_2,\mu_2),\ldots,(v_t,\mu_t)\}$ such that: **(1)** $v_j \in V$; **(b)** t and μ_j are positive integers, for $j=1,2,..,t$; **(c)** $\mu_1+\ldots+\mu_t=r$. The integer μ_j is called the *multiplicity* of v_j in M, for $j=1,\ldots,t$. We also write $M=\{v_1,\ldots,v_1,(\mu_1 \text{ times}),\ldots,v_t,\ldots,v_t,(\mu_t\text{times})\}$.

A *finite hypergraph* G is defined as a tuple

$$(s_G, V_G, E_G^2, E_G^3, \ldots, E_G^s, \Phi_G^2, \Phi_G^3, .., \Phi_G^s),$$

where $1 < s = s_G \in \mathcal{Z}$, V_G and E_G^r are finite sets (whose elements are called respectively *vertices* and *r-edges* of G) and Φ_G^r is a correspondence from E_G^r into the family of r-multisubsets of V_G (called

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the *r-incidence correspondence* of G), so if $\Phi^r_G(e) = \{v_1, .., v_r\}$ then $v_1, .., v_r$ in V_G are *incident* with e, *pairwise adjacent* among themselves and called the *endvertices* of e in E^r_G, for $r = 2, \ldots, s$. We define E_G to be the union of all the sets E^r_G, for $r = 2, \ldots, s$ and say that if $e \in E$, then e is an *edge* of G. An r-edge e of G is said to be *proper* if $\Phi^r_G(e)$ is an r-subset of V_G, i.e. if the multiplicities of this multisubset are all 1. The collection of all finite hypergraphs will be denoted by H.

Let H_s be the family of all finite hypergraphs with r-edges, where $2 \leq r \leq s$. Hypergraphs in H_s are said to be $[2, \ldots, s]$-*graphs*. Our result holds for [2,3]-graphs. If u, v, w are vertices of a [2,3]-graph G, then an edge in G may have respectively exactly endvertices: (1) u twice; (2) u and v, once each; (3) u three times; (4) u twice and v once; (5) u, v, w, once each. Edges as in (2) and (5) are proper edges.

In the complex plane, consider the set $\{a_k = e^{2\pi ik/r}; k = 1, \ldots, r\}$ and its convex hull I_r, for $r = 2, 3$. Given a finite [2,3]-graph G, we say that an *embedding* of G in the Euclidean plane P, also called a *plane embedding* of G, consists of:

1. an embedding of V_G in P;

2. for each r-edge f of G with endvertices $u_1, u_2, \ldots, u_r$ (where $2 \leq r \leq 3$), a *simple-curve r-polygon*, (*simple curve* if $r = 2$), namely a continuous map, $\Phi_f : I_r \to P$ such that: (a) $\Phi_f|(I_r \setminus \{a_1, \ldots, a_r\})$ is one-to-one; (b) $\Phi_{f(a_j)} = u_{\sigma(j)}$, for $j = 1, 2, \ldots, r$ and some permutation σ of $\{1, 2, \ldots, r\}$;

and satisfies the condition that the images $\Phi_f(I_r \setminus \{a_1, \ldots, a_r\})$ associated to the r-edges f of G, for $r \leq 2$, and the sets $\{v\}$ associated to the vertices v of G are pairwise disjoint. In particular this implies that the simple curve r-polygons corresponding to all the r-edges of G are set so that each corresponding $\Phi_f(I_r)$ is disjoint from its complement in the union $\cup\{\Phi_f(I_r); f \in E^r_G, 2 \leq r \leq 3\}$. We consider simple curve r-polygons with $r = 3$, namely *simple-curve triangles*. If such a triangle Φ_f has vertices $\Phi_f(a_1), \Phi_f(a_2), \Phi_f(a_3)$, then the *side* of Φ_f between vertices a_1 and a_2 is the restriction of Φ_f to the side (a_1, a_2) of I_3.

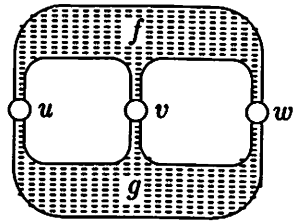

Figure 1: Nontrivial example of a planar [2,3]-graph.

If G has a plane embedding, then G is said to be a *planar hypergraph*. Clearly, planar graphs are also planar hypergraphs. As a nontrivial example of a planar [2,3]-graph G, consider $V_G = \{u, v, w\}$, $E_G^2 = \emptyset$ and $E_G^3 = \{f, g\}$, both f and g having endvertex set V_G. Figure 1 contains a plane embedding representation of G.

A necessary and sufficient condition is found for a [2,3]-graph G to be planar. This condition will be expressed in terms of the following operations defined on G:

1. Deleting a vertex v of G, thus deleting all the edges incident to v;

2. deleting an edge of G;

3. contracting an edge e in G with endvertex set $\{(v_1, \mu_1), (v_2, \mu_2), \ldots, (v_t, \mu_t)\}$, $(t > 1)$ by identifying v_1 and v_2 into a new vertex v and replacing e by e' with endvertex set $\{(v, 1), \ldots, (v_t, \mu_t)\}$; Thus, if e is an edge of G two of whose endvertices are v_1 and v_2, this operation applied to an edge different from e may produce indirectly:

   (a) if e is a 2-edge, a bending of e onto a loop, that is a 2-edge with a vertex v repeated twice (v is the *identification vertex* of v_1 and v_2);

   (b) if e is a 3-edge with vertex set $\{(v_1, 1), (v_2, 1), (v_3, 1)\}$ or $\{(v_1, 1), (v_2, 2)\}$, a bending of e onto a 2-edge whose endvertex set is respectively $\{(v, 2), (v_3, 1)\}$ or $\{(v, 3)\}$.

4. reducing the multiplicity $\mu_i > 1$ of an endvertex v_i of an 3-edge of G to a lower value > 0.

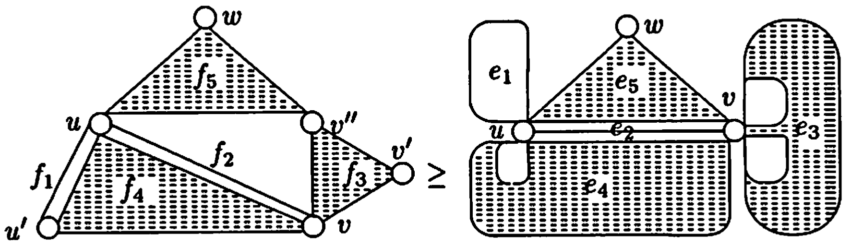

Figure 2: Example of application of operation 3.

Example. By applying operation 3. to the [2,3]-graph depicted on the left of Figure 2 (with vertex set formed by vertices u, u', v, v', v'', w and edge set formed by edges $f_1 = \{u, u'\}$, $f_2 = \{u, v\}$, $f_3 = \{v, v', v''\}$, $f_4 = \{u, u', v\}$, $f_5 = \{u, v, w\}$), we obtain the graph H depicted on the right of the same figure, having exactly the edges (1) to (5) given above as cases of edges of a [2,3]-graph. More precisely, the passage from the graph on the left of Figure 2 to the one on its right is produced by identification of the vertices u and u' and of the vertices v, v' and v'', with the corresponding accompanying changes in edge structure. Here, the graph representation on the right of the figure may be thought of as being obtained by continuous geometric deformation in the plane from the graph representation on the left of the figure. This deformation either keeps each side of a simple-curve triangle representing a 3-edge as an finite curve with separate ends or bends it into a closed curve (or loop). The interior of each representation of a 3-edge f_i, ($i = 3, 4, 5$), in the left of Figure 2 is kept topologically equivalent after such a deformation, becoming the interior of the corresponding representation of the 3-edge e_i in the right of the figure. This agrees with the use made of curved r-polygons in the definition of a plane embedding of a [2,3]-graph. ▪

The operations 1.-4. expressed above make H into a quasi-order, i.e. a class with a reflexive transitive relation, [3]. In our case, this relation will be expressed by saying that given two finite hypergraphs G and G', G is a hyperminor of G', written $G \leq G'$, if $G = G'$ or if G can be obtained from G' by means of a sequence of operations $\sigma_1, \ldots, \sigma_t$, where σ_j is one of the operations 1.-4., for j $= 1, \ldots, t$, meaning that such a sequence of operations takes G' onto a finite hypergraph isomorphic to G. Thus, $(H, \leq)$ is a quasi-order.

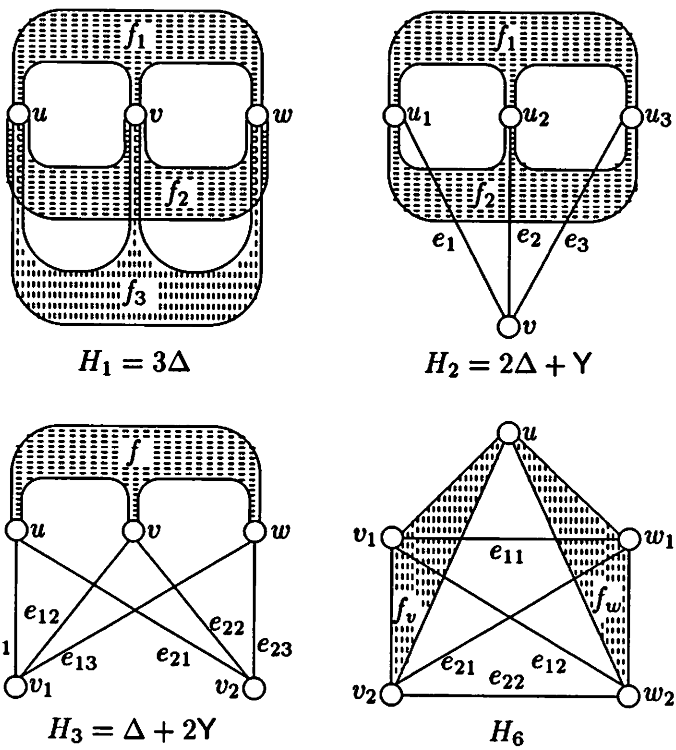

Figure 3: Four of the claimed obstructions.

It is not known whether there exists a finite *obstruction set* O_H *to plane embedding* in this quasi-order, i.e. a set of nonplanar hypergraphs, each of which does not have proper non-planar subhypergraphs and such that each nonplanar hypergraph has an element of O_H as a hyperminor. (If such an obstruction set existed, then $(H, \leq)$ would be seen to be a well-quasi-order in the sense of [3]). However, we do establish a restriction result for the quasi-order $(H_3, \leq)$, where the corresponding obstruction set O_{H_3} will be composed by six [2,3]-graphs. They are:

I. $V_G = \{u, v, w\}$; $E_G^2 = \emptyset; E_G3{=}\{f_1, f_2, f_3\}$, where f_j has endvertices u, v, w, for $j = 1, 2, 3$; we denote this: $G = G_1 = 3T$;

II. $V_G = \{u_1, u_2, u_3, v\}; E_G2 = \{e_1, e_2, e_3\}$, where e_j has endvertices u_j and v, for $j = 1, 2, 3; E_G3 = \{f_1, f_2\}$, where f_k has endvertices u_j, for $k = 1,2$ and $j = 1,2,3$; we denote this: $G = G_2 = 2T + S$;

III. $V_G = \{u_1, u_2, u_3, v_1, v_2\}; E_G2 = \{e_k, j; k = 1,2; j = 1,2,3\}$, where $e_{k,j}$ has endvertices v_k and u_j, for $k = 1,2$ and $j = 1,2,3$; $E_G^3 = \{f\}$, where f has endvertices u_j, for $j = 1,2,3$; we denote this: $G = T + 2S = G_3$;

IV. This is the standard graph $K_{3,3}$; we denote this: $G_4 = K_{3,3} = 3S$;

The four graphs just given (see Figure 3) may be built from a vertex set $\{u_1, u_2, u_3\}$ by the attachment of three [2,3]-graphs each one of which is isomorphic either to T (defined as having only one 3-edge with endvertices u_1, u_2, u_3) or to S (defined as having vertices u_1, u_2, u_3, an extra vertex v and exactly three 2-edges with respective endvertices u_j and v, for $j = 1,2,3$).

V. This is the standard graph K$_5$; We thus take $G_5 = K_5$;

VI. $V_G = \{u, v_1, v_2, w_1, w_2\}; E_G^1 = \{e_{k,j}; k, j = 1,2\}$, where $e_{k,j}$ has endvertices v_k and w_j, for $k, j \in \{1,2\}$. (These four edges form a 4-cycle); $E_G^2 = \{f_v, f_w\}$, where f_y has endvertices u, y_1, y_2, for $y = v, w$; We denote $G = G_6$.

Let H_3' be the family of finite vertex-colored graphs satisfying the following two conditions:

1. A graph in H_3' has its vertex set partitioned into two colored classes: a class V_B whose elements are said to be *black vertices* and a class V_W whose elements are said to be *white vertices*;

2. each black vertex has degree 3 and its three neighbors are pairwise different white vertices.

We define a map $\Theta_3 : H_3 \to H_3'$ as follows: Let $G \in H_3$ be a [2,3]-graph. Let $\Theta_3(G)$ be a graph in H_3' with vertex parts $V_W = V_G$ and $V_B = E_G^3$ and edge set $E_{\Theta_3(G)} = E_{G^2} \cup F$, where

$$F = \{(v, f); v \in V_G, f \in E_G^3 \text{ and } v \text{ is an endvertex of } f\}.$$

Then clearly $\Theta_3(G)$ is well defined. Moreover, Θ_3 is a bijection. Figure 3 illustrates the images through Θ_3 of the six claimed obstructions given above, where the black vertices are represented by "•" and the white ones by "o". These six 2-colored graphs have underlying graphs $K_5, K_{3,3}$ or subdivisions of $K_{3,3}$.

Let G be a [2,3]-graph. Assuming that the plane P is provided with the usual cartesian metric, if G is planar, then a plane embedding of G can be found in which the area of each simple curve triangle representing a 3-edge of G is arbitrarily small. Such an embedding may be obtained as follows:

Let $\Theta_3(G)$ be embedded in the plane. Then each edge e of $\Theta_3(G)$ with endevertices say u and v is given in this embedding by means of a simple curve $\Phi = \Phi_{e,u,v} : [0,1] \to P$ with $\Phi(0) = u$ and $\Phi(1) = v$. Let y be a black vertex of $\Theta_3(G)$ and let u, v, w be its white vertices adjacent by means of respective edges c, d, e. Consider the composition simple curves

$$\tau_{u,v} = \Phi_{d,v,y} \circ (\Phi_{c,u,y})^{-1}, \tau_{v,w} = (\Phi_{e,w,y}) \circ (\Phi_{d,v,y})^{-1}$$

and

$$\tau_{w,u} = (\Phi_{c,u,y}) \circ (\Phi_{e,w,y})^{-1}.$$

Given an arbitrarily small real number $\delta = \delta_y > 0$, let $t_{u,v} : [0,1] \to P$ be such that

1. $0 < |t_{u,v}(x) - \tau_{u,v}(x)| < \delta$, for every $x \in (0,1)$;
2. $t_{u,v}(x) = \tau_{u,v}(x)$, for $x = 0, 1$;
3. $t_{u,v}([0,1]) \cap \Phi_{e,w,y}((0,1]) = \emptyset$.

We can define similarly simple curves $t_{v,w}, t_{w,u}$, in such a way that a compact connected region $t(y) = t_\delta(y)$ with positive area is determined, whose frontier is given by $t_{u,v}[0,1] \cup t_{v,w}[0,1] \cup t_{w,u}[0,1]$, and having the vertex y of $\Theta_3(G)$ in its interior.

So, G may be embedded in the plane if there is a plane embedding of $\Theta_3(G)$, for in such a case, the star $st(y)$ of each black vertex y of $\Theta_3(G)$ in its plane embedding may be slightly fattened to a simple curve triangle $t(y) = t_\delta(y)$ with y in the interior of $t(y)$ and such that the vertices of $t(y)$ are the white neighbors of y. Moreover, by taking

δ small enough, this fattening of the stars of the black vertices of $\Theta_3(G)$ may be performed so that each fattened star $t(y)$ has empty intersection with its complement in G. This complement coincides with the complement of the thin star $st(y)$ in $\Theta_3(G)$.

This argument shows that an obstruction set O_{H_3} for the plane embedding problem of [2,3]-graphs by means of the quasi-order structure of $(H_3, \leq)$ must be constituted by [2,3]-graphs whose images via Θ_3 are nonplanar graphs, 2-colored as indicated.

Clearly, $\Theta_3(G) =$ white K_5 and $\Theta_3(G) =$ white $K_{3,3}$ can be included, corresponding these to $G = G_5$ and $G = G_4$ above, respectively. Also $K_{3,3}$ with one, two and three independent black vertices, and the remaining vertices white, will provide graphs $\Theta_3(G)$ for adequate obstructions G in O_{H_3}, being these exactly $G = G_3, G_2, G_1$, respectively.

Other [2,3]-graphs in O_{H_3} must yield through Θ_3 at most subdivivisions of K_5 or $K_{3,3}$. Let $K_{3,3}$ be vertex-colored with only two black vertices u and v joined by an edge e and the remaining vertices white. By subdividing e into two edges by means of new white vertex, we get $\Theta_3(G_6)$, and so G_6 can be included in O_{H_3}.

If we assume that a subdivision of $K_{3,3}$ is 2-colored so that there is a path of length 4 through three black vertices and two white ones or an 8-cycle with alternate black and white vertices, then this does not lead to an obstruction in O_{H_3}, for it is not difficult to find that the resulting [2,3]-graph has G_2 as a hyperminor. (A graph G that has associated $\Theta_3(G)$ as a subdivision of $K_{3,3}$ with an 8-cycle as above reduces to a graph G' that has its associated $\Theta_3(G')$ with exactly a path of length 4 as above; but this G' reduces to G'' with its own associated $\Theta_3(G'')$ having exactly a path of length 2 with black endvertices).

Any subdivision of K_5 is readily excluded as a candidate $\Theta_3(G)$ for a new obstruction $G \in O_{H_3}$, for either it is all white and so it has a white K_5 as a minor; or it has a black vertex of degree 4, which is not specified in the definition of $H'_3 = \Theta_3(H)$; or it has a black vertex of degree 3, going back to the analyzed cases with underlying $K_{3,3}$. This shows that O_{H_3} can be taken exactly as constituted by the graphs G_j , for $j = 1, 2, 3, 4, 5, 6$. This establishes our main goal.

Theorem 1 *A* $[2, 3]$*-graph is nonplanar if and only if it has as a* *hyperminor an element of* O_{H_3}.

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