Support Sizes of Indecomposable Threefold Triple Systems

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Abstract

In this paper, we determine the spectrum of support sizes of indecomposable threefold triple systems of order v for all $v > 15$.

1 Introduction

A λ–fold triple system of order v, denoted $TS(v, \lambda)$, is a pair $(V, \mathbf{B})$ where V is a v–set and $\mathbf{B}$ is a collection of 3–subsets (called blocks or triples) of V such that each 2–subset of V is contained in precisely λ blocks of $\mathbf{B}$. It is well known [5] that the necessary and sufficient conditions for the existence of a $TS(v, \lambda)$ are

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod 2 \\ \lambda v(v-1) &\equiv 0 \pmod 6 \end{aligned} \tag{1}$$

Let $(V, \mathbf{B})$ be a $TS(v, \lambda)$. If there exists a subcollection $\mathbf{B}_1$ of $\mathbf{B}$ such that $(V, \mathbf{B}_1)$ is a $TS(v, \lambda_1)$ for some λ_1, $1 \leq \lambda_1 < \lambda$, then $(V, \mathbf{B})$ is called decomposable. Otherwise it is called indecomposable.

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It can be easily seen that the number of blocks contained in a $TS(v, \lambda)$ is

$$b = \lambda v(v-1)/6 \qquad (2)$$

In the definition of $TS(v, \lambda)$, repeated blocks are permitted. For a $TS(v, \lambda)$ $(V, \mathbf{B})$, let $\mathbf{B}^*$ be the set of all the distinct blocks of $\mathbf{B}$ and let $b^* = |\mathbf{B}^*|$. $\mathbf{B}^*$ is called the support of $(V, \mathbf{B})$ and b^* is called the support size. Let $m_v = [v(v-1)/6]$, then obviously

$$m_v \le b^* \le b \qquad (3)$$

A triple system is called simple if it contains no repeated blocks. In this case we have $b^* = b$.

Designs with various support sizes have interesting applications in statistics [4] and have been studied extensively. For given positive integers v and λ, let

$SS(v, \lambda) = \{b^* \mid \exists TS(v, \lambda) \text{ with support size } b^*\}$,
$ISS(v, \lambda) = \{b^* \mid \exists \text{ indecomposable } TS(v, \lambda) \text{ with support size } b^*\}$.

The support size problem for triple systems, i.e., the problem of determining the set $SS(v, \lambda)$ has been almost completely solved [1]. But, to our knowledge, the only systematic work done regarding the support size problem for indecomposable triple systems is the following result for $\lambda = 2$:

Theorem 1.1 *[8] For $v \equiv 0, 1 \pmod 3$, $v \ge 15$,*

$$ISS(v,2) = \begin{cases} \{m_v+6, m_v+8, \cdots, 2m_v\}, & \text{if } v \equiv 1,3 \pmod 6, \\ \{s_v, s_v+2, \cdots, 2m_v\}, & \text{if } v \equiv 0,4 \pmod{12}, \\ \{s_v+1, s_v+2, \cdots, 2m_v\}, & \text{if } v \equiv 6,10 \pmod{12}. \end{cases}$$

where $m_v = v(v-1)/6$, $s_v = v(v+2)/6$.

In this paper, we consider the support size problem for indecomposable threefold triple systems. It follow from (1) that there exists a $TS(v, 3)$ if and only if

$$v \equiv 1 \pmod 2 \qquad (4)$$

The support size problem for threefold triple systems was completely solved:

Theorem 1.2 *[2] Let $m_v = [v(v-1)/6]$. If $v \equiv 1, 3 \pmod 6$, then*

$$SS(v,3) = \begin{cases} \{m_v, m_v+4, m_v+6, \cdots, 3m_v\}, & \text{if } v \ge 13, \\ \{1\}, & \text{if } v = 3, \\ \{7, 11, \cdots, 21\} \setminus \{12, 16\}, & \text{if } v = 7, \\ \{12, 18, 20, \cdots, 36\}, & \text{if } v = 9. \end{cases}$$

If $v \equiv 5 \pmod 6$, *then*

$$SS(v,3) = \{m_v + 7, m_v + 10, \cdots, 3m_v + 1\}$$

For $v \equiv 5 \pmod 6$, every $TS(v,3)$ is indecomposable since in this case there is no $TS(v,1)$ exists. So we have

$$ISS(v,3) = SS(v,3) = \{m_v + 7, m_v + 10, \cdots, 3m_v + 1\}$$

for all $v \equiv 5 \pmod 6$. Thus we need only to consider the problem for $v \equiv 1,3 \pmod 6$. Our main purpose in this paper is to determine the spectrum of support sizes for indecomposable threefold triple systems for all $v > 15$.

2 Necessary Conditions

Obviously, $ISS(v,3) \subseteq SS(v,3)$. It follows from Theorem 1.2 that $m_v + k \notin ISS(v,3)$ if $k \in \{1,2,3,5\}$. The purpose of this section is to prove that for $v \equiv 1,3 \pmod 6$, if $k \in \{0,4,6,7,8,9,10,11\}$, then $m_v + k \notin ISS(v,3)$.

Let $(V, \mathbf{B})$ be a $TS(v,3)$ and b_1, b_2 and b_3 be non-negative integers. The vector (b_1, b_2, b_3) is called the fine structure of $(V, \mathbf{B})$ if $\mathbf{B}$ contains exactly b_i i–times repeated blocks for $i = 1,2,3$. For any $TS(v,3)$ with support size $m_v + k$ and fine structure (b_1, b_2, b_3), if $v \equiv 1,3 \pmod 6$, then

$$\begin{aligned} b^* &= b_1 + b_2 + b_3 = m_v + k \\ b &= b_1 + 2b_2 + 3b_3 = 3m_v \end{aligned} \tag{5}$$

For a $TS(v,3)$ $(V, \mathbf{B})$, let $\mathbf{B}_i$ denote the set of all distinct i–times repeated blocks of $\mathbf{B}$, $1 \leq i \leq 3$.

Lemma 2.1 *For* $v \equiv 1,3 \pmod 6$, *if* $(V, \mathbf{B})$ *is a* $TS(v,3)$ *with fine structure* (b_1, b_2, b_3), *then* $b_1 \geq b_2$. *If* $b_1 = b_2$, *then* $(V, \mathbf{B})$ *is decomposable.*

Proof. Each 2–subset of V contained in a block of $\mathbf{B}_2$ must also be contained in a block of $\mathbf{B}_1$ and so $b_1 \geq b_2$.

If $b_1 = b_2$, then each 2–subset of V is contained in a block of $\mathbf{B}_2$ if and only if it is contained in a block of $\mathbf{B}_1$, then $(V, \mathbf{B}_1 \cup \mathbf{B}_3)$ is a $TS(v,1)$ and so $(V, \mathbf{B})$ is decomposable. □

Lemma 2.2 *[3] For* $v \equiv 1,3 \pmod 6$, *if* (b_1, b_2, b_3) *is the fine structure of a* $TS(v,3)$, *where* $b_1 = 3s - 2t$, $b_2 = t$ *and* $b_3 = m_v - s$, *then* $0 \leq t \leq s \leq m_v$, $s \notin \{1,2,3,5\}$ *and* $(t,s) \notin \{(1,4),(2,4),(3,4),(1,6),(2,6),(3,6),(5,6),(2,7),(5,7),(1,8),(3,8),(5,8)\}$.

Lemma 2.3 *For $v \equiv 1, 3 \pmod 6$, if $k \in \{0, 4, 6, 7\}$, then $m_v + k \notin ISS(v, 3)$.*

Proof. Let $(V, \mathbf{B})$ be an indecomposable $TS(v, 3)$ with support size $m_v + k$ and fine structure (b_1, b_2, b_3) with $b_1 = 3s - 2t$, $b_2 = t$ and $b_3 = m_v - s$. It follows from (5) that

$$b_1 = 2k - s, \quad b_2 = t = 2s - k. \tag{6}$$

Since $(V, \mathbf{B})$ is indecomposable, then $b_1 > b_2$, by Lemma 2.1. It follows from (6) and the fact $b_2 \geq 0$ that

$$k/2 \leq s < k \tag{7}$$

and so $k = 0$ is impossible.

By (6) and (7), if $k = 4$, then $(t, s) = (0, 2), (2, 3)$; if $k = 6$, then $(t, s) = (0, 3), (2, 4)$ or $(4, 5)$; if $k = 7$, then $(t, s) = (1, 4), (3, 5)$ or $(5, 6)$. By Lemma 2.2, all of these cases are impossible.

This completes the proof. □

A partial triple system $PTS(v, \lambda)$ is a pair $(X, \mathbf{B})$ where X is a v–set and $\mathbf{B}$ is a collection of 3–subsets (called blocks or triples) of X such that each 2–subset of X is contained in at most λ blocks.

For $v \equiv 1, 3 \pmod 6$, let $(V, \mathbf{B})$ be a $TS(v, 3)$, $\mathbf{A} = \mathbf{B}_1 \cup 2\mathbf{B}_2$ and X be the following subset of V:

$$X = \{x \in V \mid \exists B \in \mathbf{A} \text{ such that } x \in B\}.$$

Let $w = |X|$, then $(X, \mathbf{A})$ is a $PTS(w, 3)$ with the propertythat each 2–subset of X is contained in either 0 or 3 blocks of $\mathbf{A}$. For $x \in X$, the number of blocks of $\mathbf{A}$ containing x is called the degree of x and denoted $d(x)$. It can be easily verified that if $\mathbf{A} \neq \phi$, then

$$\begin{array}{c} d(x) \equiv 0 \pmod 3 \\ 6 \leq d(x) \leq 3(w-1)/2 \\ w \leq |\mathbf{A}|/2, \; 3m_w \geq |\mathbf{A}|. \end{array} \tag{8}$$

Let $(X, \mathbf{A})$ be a $PTS(w, 3)$. $(X, \mathbf{A})$ is said to be a $PTS(w, 3)$ of degree type

$$\pi = ((h_1)_{r_1}, (h_2)_{r_2}, \cdots, (h_s)_{r_s})$$

if there are s positive integers $r_1, r_2, \cdots, r_s$ with $r_1 + r_2 + \cdots + r_s = w$ and s positive integers $h_1, h_2, \cdots, h_s$ with $0 < h_1 < h_2 \cdots < h_s$, such that there are exactly r_i elements of X of degree h_i, $1 \leq i \leq s$.

If $(V, \mathbf{B})$ is a $TS(v, 3)$ with fine structure (b_1, b_2, b_3) and $(X, \mathbf{A})$ is of degree type $((h_1)_{r_1}, \cdots, (h_s)_{r_s})$, then

$$\sum_{i=1}^{s} r_i h_i = \sum_{x \in X} d(x) = 3b_1 + 6b_2. \tag{9}$$

A 2-subset of X is called an i-pair if it is contained in exactly i blocks of $\mathbf{B}_1$.

Lemma 2.4 *For $x \in X$, let n_x be the number of distinct 3-pairs containing x. If x appears in exactly t_i blocks of $\mathbf{B}_i$, $i = 1, 2$. Then*

$$n_x = 2(t_1 - t_2)/3 \tag{10}$$

Proof. Each block containing x contains 2 pairs containing x. Each pair contained in a block of $\mathbf{B}_2$ is contained in a unique block $B \in \mathbf{B}_1$ and B contains no 3-pairs. So we have $2t_1 = 3n_x + 2t_2$ and the conclusion follows. □

Lemma 2.5 *For $v \equiv 1, 3 \pmod 6$, let $(V, \mathbf{B})$ be a $TS(v, 3)$ with fine structure (b_1, b_2, b_3). If $b_1 - b_2 = 3$ and the 3 3-pairs form a block B of $\mathbf{B}$, then $(V, \mathbf{B})$ is decomposable.*

Proof. In fact, $(V, \{B\} \cup \mathbf{B}_2 \cup \mathbf{B}_3)$ is a $TS(v, 1)$ and so $(V, \mathbf{B})$ is decomposable. □

Lemma 2.6 *Let (X, A) be the above defined $PTS(w, 3)$. If X contains exactly 3 3-pairs, then*

(i) If x is contained in a 3-pair, then $d(x) \geq 9$;

(ii) There are at least 3 elements $x \in X$ with $d(x) \geq 9$;

(iii) $w \geq 9$. If $d(x) \leq 9$ for all $x \in X$, then $w < 12$ and $(V, \mathbf{B})$ is decomposable;

(iv) Each $x \in X$ is contained in at least 2 distinct blocks of $\mathbf{B}_2$.

Proof. Let $X = \{1, 2, \cdots, w\}$. For any $x \in X$, it follows from Lemma 2.4 that n_x must be even. Since there are only 3 3-pairs, then $n_x = 0$ or 2 for each $x \in X$. Thus, without loss of generality, we may assume that the 3-pairs are $(1, 2), (1, 3), (2, 3)$ and either.

(1) $\{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,6\}, \{1,3,7\}, \{2,3,8\}, \{2,3,9\}\} \subseteq \mathbf{B}_1$

or

(2) $\{\{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,7\}, \{1,3,8\}, \{1,3,9\}, \{2,3,10\}, \{2,3,11\}, \{2,3,12\}\} \subseteq \mathbf{B}_1$.

In case (1), each of the pairs $\{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{2,4\}, \{2,5\}, \{2,8\}, \{2,9\}, \{3,6\}, \{3,7\}, \{3,8\}$ and $\{3,9\}$ must be contained in some blocks of $\mathbf{B}_2$, and so $d(x) \geq 9$ for $x \in \{1,2,3\}$. In case (2), it can be proved similarly that $d(x) \geq 12$ for $x \in \{1,2,3\}$. Obviously, $w \geq 9$ in case (1) and $w \geq 12$ in case (2). If $w < 12$, then case (2) is impossible and so the 3 3–pairs $\{1,2\}, \{1,3\}$ and $\{2,3\}$ form a block $\{1,2,3\}$, then $(V, \mathbf{B})$ is decomposable by Lemma 2.5. Thus we have proved (i), (ii) and (iii).

For $x \in X$, in both cases (1) and (2), we have

$$n_x = \begin{cases} 2 & \text{if } x \in \{1,2,3\}, \\ 0 & \text{if } x \in X \setminus \{1,2,3\}. \end{cases} \tag{11}$$

If $x \in \{1,2,3\}$, then $d(x) \geq 9$. By (10) and (11), we have

$$t_1 - t_2 = 3, \ t_1 + 2t_2 = d(x) \geq 9$$

and so $t_2 \geq 2$. If $x \in X \setminus \{1,2,3\}$, then $d(x) \geq 6$. It follows from (10) and (11) that $t_2 \geq 2$.

This completes the proof. □

E.J.Morgan [7] enumerated all non-isomorphic $TS(7,3)$ systems; there are ten in total. An examination of them shows the following result.

Lemma 2.7 *There exists a simple indecomposable* $TS(7,3)$. *Any* $TS(7,3)$ *with repeated blocks is decomposable.*

Lemma 2.8 *If* $v \equiv 1,3 \pmod 6$, *then* $m_v + 8 \notin ISS(v,3)$.

Proof. Let $(V, \mathbf{B})$ be an indecomposable $TS(v,3)$ with support size $m_v + 8$ and fine structure $(b_1, b_2, b_3) = (3s - 2t, t, m_v - s)$. It follows from (6), (7) and Lemma 2.2 that

$$t = 2s - 8, \ 4 \leq s < 8, \ s \neq 5.$$

(i) If $s = 7$, then $(b_1, b_2, b_3) = (9, 6, m_v - 7)$. By (8), we have $w \leq |\mathbf{A}|/2 = 21/2 < 12$. By Lemma 2.6 (iii), $(V, \mathbf{B})$ must be decomposable. A contradiction.

(ii) If $s = 6$, then $(b_1, b_2, b_3) = (10, 4, m_v - 6)$ and $|\mathbf{A}| = 18$, and so $7 \leq w \leq 9$ by (8).

case 1. $w = 7$. In this case, by (8), we have $6 \leq d(x) \leq 9$ for each $x \in X$. So the degree type of $(X, \mathbf{A})$ is $\pi = (6_3, 9_4)$. If $d(x) = 9$, since $w = 7$, then each 2–subset of X containing x is contained in 3 blocks of $\mathbf{A}$. Let $X = \{1,2,3,4,5,6,7\}$ and assume $d(1) = d(2) = d(3) = d(4) = 9$. Then each 2–subset of X, except $\{5,6\}, \{5,7\}$ and $\{6,7\}$, is contained in 3 blocks of $\mathbf{A}$ and none of $\{5,6\}, \{5,7\}$ and $\{6,7\}$ appear in any block of $\mathbf{A}$. Thus $(X, \mathbf{A} \cup \{\{5,6,7,\}, \{5,6,7\}, \{5,6,7\}\})$ is a $TS(7,3)$ with repeated blocks and so is decomposable by Lemma 2.7. So $(V, \mathbf{B})$ is also decomposable.

case 2. $w = 8$. Let $X = \{1,2,3,4,5,6,7,8\}$. In this case, for each $x \in X$, $6 \le d(x) \le 9$. So the degree type of $(X, \mathbf{A})$ is $\pi = (6_6, 9_2)$. Suppose $d(1) = 9$,since $w = 8$ and any 2–subset of X appears in either 0 or 3 blocks of $\mathbf{A}$, then there is a unique element of X, say 2, such that $\{1,2\}$ appears in no blocks of $\mathbf{A}$.

If $d(2) = 9$, let p be the number of distinct pairs $\{a, b\}$ such that $\{1, a, b\} \in \mathbf{B}_2$. Then there are exactly $9 - 2p$ distinct pairs $\{x, y\}$ such that $\{1, x, y\} \in \mathbf{B}_1$. It follows that we must have $\{2, a, b\} \in \mathbf{B}_1$ and $\{2, x, y\} \in \mathbf{B}_2$. Then $p = 3$ and $|\mathbf{B}_2| \ge 6 > b_2$ which is impossible.

If $d(2) = 6$. Suppose $d(3) = 9$ and let d be the unique element of X such that $\{3, d\}$ appears in no blocks of $\mathbf{A}$. Obviously $d \neq 2$. Put $d = 4$. There exist $x, y \in X$ such that $\{1,4,x\}, \{1,3,y\} \in \mathbf{B}_1$. Then we must have:

$$\begin{aligned} \mathbf{A} \;=\; & \{\{13x\},\{13x\},\{13y\},\{14x\},\{14y\},\{14y\},\\ & \{1z_1z_2\},\{1z_1z_2\},\{1z_1z_2\},\{23x\},\{23y\},\{23y\},\\ & \{24y\},\{24x\},\{24x\},\{3z_1z_2\},\{3z_1z_2\},\{3z_1z_2\}\} \end{aligned}$$

This is impossible since $\mathbf{A}$ can not contain 3–times repeated blocks.

case 3. $w = 9$. In this case, the degree type of $(X, \mathbf{A})$ is $\pi = (6_9)$. Let $X = \{1,2,3,4,5,6,7,8,0\}$. Obviously $(X, \mathbf{A})$ can not contain a $TS(5,3)$. So we may suppose, without loss of generality, that $\mathbf{A}$ contains the following 12 blocks:

$$\begin{array}{cccccc} \{124\}, & \{125\}, & \{125\}, & \{135\}, & \{134\}, & \{134\}, \\ \{624\}, & \{624\}, & \{625\}, & \{635\}, & \{635\}, & \{634\}, \end{array}$$

But then $\{789\}$ must appears 6 times in $\mathbf{A}$ which is impossible.

(iii) If $s = 4$, then $(b_1, b_2, b_3) = (12, 0, m_v - 4)$ and so $|\mathbf{A}| = 12$, $w = 6$ and the degree type of $(X, \mathbf{A})$ is $\pi = (6_6)$. Let $X = \{1,2,3,4,5,6\}$. Since each 2–subset of X contains in either 0 or 3 blocks of $\mathbf{A}$, a simple calculation shows that there are 3 2–subsets of X not contained in any block of $\mathbf{A}$. Assume $\{1,6\}$ is one of them. Then $\mathbf{A}$ must contain the following 6 blocks.

$$\{123\},\{124\},\{125\},\{134\},\{135\},\{145\}.$$

The other 2 2–subsets not contained in $\mathbf{A}$ must be of the form $\{x, 6\}$, say $\{2,6\}$ and $\{3,6\}$. Then $\mathbf{A}$ must contain $\{2,3,4\}$, $\{2,3,5\}$, $\{2,4,5\}$ and $\{3,4,5\}$. Since these 10 blocks form a $TS(5,3)$ on $X \setminus \{6\}$, then we can not find the last 2 blocks for $\mathbf{A}$.

This completes the proof of the lemma. □

Lemma 2.9 *If* $v \equiv 1,3 \pmod 6$, *then* $m_v + 9 \notin ISS(v,3)$.

Proof. Let $(V, \mathbf{B})$ be an indecomposable $TS(v,3)$ with support size $m_v + 9$ and fine structure $(b_1, b_2, b_3) = (3s - 2t, t, m_v - s)$. Then $t = 2s - 9$, $5 \le s \le 8$. By Lemma 2.2, $(t,s) \neq (1,5), (3,6), (5,7)$. So we need only to consider the case $(t,s) = (7,8)$. In this case $(b_1, b_2, b_3) = (10, 7, m_v - 8)$. So we have $|\mathbf{A}| = 24$ and $9 \le w \le 12$. If $w < 12$, then $(V, \mathbf{B})$ is decoposable by Lemma 2.6 (iii). If $w = 12$, then $d(x) = 6$ for each $x \in X$. This is impossible since by Lemma 2.6 (ii), there exists $x \in X$ with $d(x) \ge 9$.

This completes the proof. □

Lemma 2.10 *Let* $v \equiv 1,3 \pmod 6$, *if* $(V, \mathbf{B})$ *is a* $TS(v,3)$ *with fine structure* $(b_1, b_2, b_3) = (12, 9, m_v - 10)$ *or* $(11, 8, m_v - 9)$, *then* $(V, \mathbf{B})$ *is decomposable.*

Proof. Let $X = \{1, 2, \cdots, w\}$ and let $\{1,2\}, \{1,3\}$ and $\{2,3\}$ be the 3 3–pairs of X. By Lemma 2.5, we need only to consider the case $\{1,2,3\} \notin \mathbf{B}_1$ and so we may assume that $w \ge 12$ and $\mathbf{B}_1$ contains the following

$$\begin{aligned} \mathbf{B}_1' : \quad & \{\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,7\},\{1,3,8\},\\ & \{1,3,9\},\{2,3,10\},\{2,3,11\},\{2,3,12\}\}. \end{aligned}$$

¿From (10) and by Lemma 2.6 (iv) necessarily we have

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{B}_1' \cup \{\{4,7,10\},\{5,8,11\},\{6,9,12\}\},\\ \mathbf{B}_2 &= \{\{1,4,7\},\{1,5,8\},\{1,6,9\},\{2,4,10\},\{2,5,11\},\\ &\quad \{2,6,12\},\{3,7,10\},\{3,8,11\},\{3,9,12\}\}. \end{aligned}$$

Let

$$\begin{aligned} \mathbf{C} &= \{\{1,2,4\},\{1,3,7\},\{1,5,8\},\{1,6,9\},\{2,3,10\},\\ &\quad \{3,8,11\},\{3,9,12\},\{2,5,11\},\{2,6,12\},\{4,7,10\}\}, \end{aligned}$$

then $\mathbf{C} \subset \mathbf{A}$ and $(V, \mathbf{C} \cup \mathbf{B}_3)$ is a $TS(v,1)$ and so $(V, \mathbf{B})$ is decomposable. This completes the proof. □

For each $(X, \mathbf{A})$, let G be the following graph on vertex set X: For $a, b \in X$, $a \neq b$, we join a and b by an edge of G if and only if $\{a,b\}$ is a 3–pair of X. G is called the single-edge graph of $(X, \mathbf{A})$.

For each $(X, \mathbf{A})$, let M be the following graph on X: For $a, b \in X$, $a \neq b$, we join a and b by an edge of M if and only if $\{a, b\}$ is contained in no blocks of $\mathbf{A}$. M is called the "minimizing graph" of $(X, \mathbf{A})$.

Obviously, any vertex of G is of even degree and any vertex of M is of degree $w - 1 - 2d(x)/3$.

Lemma 2.11 *Let* $v \equiv 1, 3 \pmod 6$. *If* $(V, \mathbf{B})$ *is a* $TS(v, 3)$ *with fine structure* $(b_1, b_2, b_3) = (13, 7, m_v - 9)$ *or* $(12, 6, m_v - 8)$, *then* $(V, \mathbf{B})$ *is decomposable.*

Proof. There are exactly 6 3–pairs of X and so there are only 3 possiblilities for the single-edge graph G:

case 1. Two disjoint triangles $\{1, 2, 3\}, \{4, 5, 6\}$.
In this case, if $\mathbf{C} = \{\{1, 2, 3\}, \{4, 5, 6\}\} \subset \mathbf{B}_1$, then $(V, \mathbf{C} \cup \mathbf{B}_2 \cup \mathbf{B}_3)$ is a $TS(v, 1)$ and so $(V, \mathbf{B})$ is decomposable.
If $\{1, 2, 3\} \notin \mathbf{B}_1$, then $|\mathbf{B}_1| \geq 16 > b_1$ which is impossible.

case 2. Two triangles with one common vertex:

$$\{1, 2, 3\}, \{1, 4, 5\}.$$

In this case, if $\{\{1, 2, 3\}, \{1, 4, 5\}\} \subset \mathbf{B}_1$, then $(V, \mathbf{B})$ is decomposable, as in case 1. If $\{1, 2, 3\} \notin \mathbf{B}_1$, then we may suppose that $\mathbf{B}_1$ contains the following 12 blocks.

$$\{1,2,4\},\ \{1,2,5\},\ \{1,2,a\},\ \{1,3,4\},\ \{1,3,5\},\ \{1,3,b\},$$
$$\{2,3,c\},\ \{2,3,d\},\ \{2,3,e\},\ \{4,5,1\},\ \{4,5,f\},\ \{4,5,g\}.$$

and $|\{a, b, c, d, e\}| = 5$, $|\{f, g\}| = 2$, $\{a, b\} \cap \{f, g\} = \phi$ (otherwise the pair $\{1, 4\}$ or $\{1, 5\}$ would appear in $\mathbf{B}_2$). We may let $c = f = 6$ and $d = g = 7$. Then it can be seen that

$$\begin{aligned}
\mathbf{B}_1 &= \{\{1,2,4\},\{1,2,5\},\{1,2,a\},\{1,3,4\},\\
&\quad \{1,3,5\},\{1,3,b\},\{2,3,6\},\{2,3,7\},\\
&\quad \{2,3,e\},\{4,5,1\},\{4,5,6\},\{4,5,7\},\{a,b,e\}\}.\\
\mathbf{B}_2 &= \{\{6,2,4\},\{6,3,5\},\{7,2,5\},\{7,3,4\},\\
&\quad \{1,a,b\},\{2,a,e\},\{3,b,e\}\}
\end{aligned}$$

Let

$$\begin{aligned}
\mathbf{C} &= \{\{1,2,4\},\{1,3,5\},\{2,3,6\},\{4,5,6\},\{3,4,7\},\\
&\quad \{2,5,7\},\{1,a,b\},\{2,a,e\},\{3,b,e\}\}.
\end{aligned}$$

Then $\mathbf{C} \subset \mathbf{A}$ and $(V, \mathbf{C} \cup \mathbf{B}_2 \cup \mathbf{B}_3)$ is a $TS(v, 1)$ and so $(V, \mathbf{B})$ is decomposable.

case 3. A cycle with 6 vertices: $\{1,2,3,4,5,6\}$. It is easily seen that we have either

$$\begin{aligned}\mathbf{B}_1 &= \{\{126\},\{123\},\{12a\},\{342\},\{345\},\{34a\},\\ &\quad \{561\},\{564\},\{56c\},\{16x\},\{23c\},\{45x\}\}\\ \mathbf{B}_2 &= \{\{a13\},\{a24\},\{c26\},\{c35\},\{x15\},\{x46\}\}\end{aligned}$$

or

$$\begin{aligned}\mathbf{B}_1 &= \{\{126\},\{123\},\{12a\},\{342\},\{345\},\{34b\},\\ &\quad \{561\},\{564\},\{56b\},\{16x\},\{23x\},\{45a\}\},\\ \mathbf{B}_2 &= \{\{a15\},\{a24\},\{b35\},\{b46\},\{x13\},\{x26\}\}.\end{aligned}$$

In either case, $(V,\mathbf{B})$ is decomposable.

This completes the proof. □

Lemma 2.12 *Let* $v \equiv 1,3 \pmod 6$. *If* $(V,\mathbf{B})$ *is a* $TS(v,3)$ *with fine structure* $(b_1,b_2,b_3) = (13,4,m_v-7)$ *or* $(15,3,m_v-7)$, *then* $(V,\mathbf{B})$ *is decomposable.*

Proof. We have $|\mathbf{A}| = 21$ and so it follows from (8) that $7 \le w \le 10$.

If $w = 7$, then $(X,\mathbf{A})$ is a $TS(7,3)$ with repeated blocks and so is decomposable by Lemma 2.7. Then $(V,\mathbf{B})$ is also decomposable.

For the degree type of $(X,\mathbf{A})$, it follows from (8) that

$$\pi = \begin{cases} (6_3,9_5) & \text{if } w=8\\ (6_6,9_3) \text{ or } (6_7,9_1,12_1) & \text{if } w=9\\ (6_9,9_1) & \text{if } w=10.\end{cases}$$

(i) $w=8$. Let $X=\{1,2,3,4,5,6,7,8\}$, $N=\{1,2,3,4,5\}$, $S=\{6,7,8\}$ such that $d(x)=9$ for each $x\in N$ and $d(y)=6$ for each $y\in S$. For each $x\in X$, the degree of x in the minimizing graph M of $(X,\mathbf{A})$ is 1 if $x\in N$ and is 3 if $x\in S$ and so there are only two possibilities for M.

case 1.

$$\{\{1,2\},\{3,6\},\{4,7\},\{5,8\},\{6,7\},\{6,8\},\{7,8\}\}=M.$$

Then the 21 blocks of $\mathbf{A}$ are of the following forms:

$$\begin{array}{llllll}\{13\cdot\}, & \{13\cdot\}, & \{13\cdot\}, & \{14\cdot\}, & \{14\cdot\}, & \{14\cdot\},\\ \{1\cdot\cdot\}, & \{1\cdot\cdot\}, & \{1\cdot\cdot\}, & \{23\cdot\}, & \{23\cdot\}, & \{23\cdot\},\\ \{24\cdot\}, & \{24\cdot\}, & \{24\cdot\}, & \{2\cdot\cdot\}, & \{2\cdot\cdot\}, & \{2\cdot\cdot\},\\ \{34\cdot\}, & \{34\cdot\}, & \{34\cdot\}. & & & \end{array}$$

Since $d(5) = 9$, it is impossible to arrange 5 into 9 of the above 21 blocks such that each pair containing 5 appears in either 0 or 3 blocks.

case 2.

$$\{\{1,6\},\{2,6\},\{3,7\},\{4,7\},\{5,8\},\{6,8\},\{7,8\}\} = M.$$

Each of 1, 2, 3 , 4 and 8 can not meet the 3 blocks of A containing $\{6,7\}$ and this is impossible.

(ii) $w = 9$. Let $X = \{1,2,3,4,5,6,7,8,9\}$. If $\pi = (6_6, 9_3)$, then let $N = \{1,2,3\}$, $S = \{4,5,6,7,8,9\}$ such that

$$d(x) = \begin{cases} 9, & \text{if } x \in N \\ 6, & \text{if } x \in S \end{cases}$$

For each $x \in X$, let $d'(x)$ denote the degree of x in the minimizing graph M of $(X, \mathbf{A})$. Then

$$d'(x) = \begin{cases} 2, & \text{if } x \in N, \\ 4, & \text{if } x \in S. \end{cases}$$

case 1. There exists $x \in N$, say $x = 1$, such that $\{\{1,2\},\{1,3\}\} \subset M$. In this case, if $\{2,3\} \in M$, then there are 9 blocks of $\mathbf{A}$ containing z for each $z \in \{1,2,3\}$ and so $|\mathbf{A}| \geq 27 > 21$ which is impossible. If $\{2,3\} \notin M$, then $\mathbf{A}$ contains 9 blocks of form $\{1,a,b\}$, 6 blocks of form $\{2,c,d\}$, 6 blocks of form $\{3,e,f\}$ and 3 blocks of form $\{2,3,g\}$ where $a,b,c,d,e,f,g \in S$. It follows that $|\mathbf{A}| \geq 24$ which is also impossible.

case 2. There exist two elements of N, say 1 and 2 , and an element of S , say 4 , such that $\{\{1,2\},\{1,4\}\} \subset M$. Then for each $x \in S \setminus \{4\}$, the pair $\{1,x\}$ appears in 3 blocks of $\mathbf{A}$. There is a unique $z \in S \setminus \{4\}$ such that $\{2,z\}$ is not contained in any block of $\mathbf{A}$. There are 18 blocks of $\mathbf{A}$ conaining 1 or 2. Since $d(3) = 9$, $d(4) = d(z) = 6$ and $|\mathbf{A}| = 21$, then the last 3 blocks of $\mathbf{A}$ must be $\{3,4,z\}$, $\{3,4,z\}$ and $\{3,4,z\}$, but this is impossible.

case 3. There exist $a,b \in N$ and $c \in S$, say $a = 1$, $b = 2$ and $c = 4$, such that $\{\{1,4\},\{2,4\}\} \subset M$ and $\{1,2\} \notin M$. Then $\mathbf{A}$ contains 3 blocks of form $\{1,2,a\}$, 6 blocks of form $\{1,b,c\}$, 6 blocks of form $\{2,d,e\}$ and 6 blocks of form $\{4,f,g\}$. Since $\{1,2\} \notin M$ and $\{1,4\} \in M$, then $\{1,2,4\} \notin \mathbf{B}$, and so there exists $m \in S \setminus \{4\}$ such that $\{1,2,m\} \in \mathbf{A}$. It follows that the number of blocks of $\mathbf{A}$ of form $\{1,m,x\}$ or $\{2,m,y\}$ is 4 or 5. As $d(m) = 6$, $\{4,m\}$ must be contained in 3 blocks of $\mathbf{A}$ and so $d(m) \geq 7$, but this is impossible.

With the above discussions, from now on, without loss of generality, we may suppose that

$$\{\{1,4\},\{1,5\},\{2,6\},\{2,7\},\{3,8\},\{3,9\}\} \subset M.$$

case 4. $\{\{4,6\},\{4,7\}\} \subset M$. In this case, if $\{4,5\} \in M$, then A contains a 3–repeated block $\{4,3,2\}$ which is impossible. So $\{4,5\} \notin M$ and we may suppose that $\{4,8\} \in M$. It follows that $\{4,x\} \notin M$ if $x \in \{2,3,5,9\}$ and so the blocks of A containing $\{4,3\}$ must be $\{4,3,2\}$, $\{4,3,5\}$ and $\{4,3,x\}$ where $x \in \{2,5\}$. The remaining blocks of A conaining 4 must be $\{4,9,2\}$, $\{4,9,y\}$, where $\{x,y\} = \{2,5\}$. It follows from $d(9) = 6$ and $\{1,9\} \notin M$ that $\{9,1,2\},\{9,1,5\} \in$ A. But this is impossible since $\{1,5\} \in M$.

case 5. From now on, without loss of generality, we may assume that

$$\{\{1,4\},\{1,5\},\{2,6\},\{2,7\},\{3,8\},\{3,9\},\{4,5\},\{4,6\},$$
$$\{4,8\},\{6,7\},\{8,9\},\{5,7\},\{5,9\},\{6,x\},\{7,y\}\} = M,$$

where $\{x,y\} = \{8,9\}$.
Since we must have $\{\{5,2\},\{5,6\}\} \cap M = \phi$, and $d(5) = 6$, then A contains 3 blocks of form $\{5,2,x\}$ and 3 blocks of form $\{5,6,y\}$ where $x,y \notin \{2,6\}$.
Now we have either $\{6,8\} \in M$ or $\{6,9\} \in M$. Since $\{5,8\} \notin M$, then $\{5,8\}$ must be contained in 3 blocks of A, then we have $\{\{5,2,8\},\{5,6,8\}\} \subset$ A and so $\{6,8\} \notin M$ and $\{\{6,9\},\{7,8\}\} \subset M$. The blocks of A containing 5 are

$$\{5,2,3\},\{5,2,8\},\{5,2,a\},\{5,6,3\},\{5,6,8\},\{5,6,b\}$$

where $\{a,b\} = \{3,8\}$. The blocks containing 8 are

$$\{8,5,2\},\{8,5,6\},\{8,5,c\},\{8,1,2\},\{8,1,6\},\{8,1,d\}$$

where $\{c,d\} = \{2,6\}$. The blocks containing 2 are

$$\{2,5,3\},\quad \{2,5,3\},\quad \{2,5,8\},\quad \{2,1,9\},\quad \{2,1,8\},$$
$$\{2,1,8\},\quad \{2,4,3\},\quad \{2,4,9\},\quad \{2,4,9\}$$

or

$$\{2,5,3\},\quad \{2,5,8\},\quad \{2,5,8\},\quad \{2,1,8\},\quad \{2,1,9\},$$
$$\{2,1,x\},\quad \{2,4,3\},\quad \{2,4,9\},\quad \{2,4,y\}.$$

where $\{x, y\} = \{3, 9\}$.
It follows that the 21 blocks of A are

$$(I)\quad \begin{array}{llll} \{1,2,8\}, & \{1,2,8\}, & \{1,2,9\}, & \{1,6,3\}, \\ \{1,6,3\}, & \{1,6,8\}, & \{1,7,3\}, & \{1,7,9\}, \\ \{1,7,9\}, & \{4,2,3\}, & \{4,2,9\}, & \{4,2,9\}, \\ \{4,7,3\}, & \{4,7,3\}, & \{4,7,9\}, & \{5,2,3\}, \\ \{5,2,3\}, & \{5,2,8\}, & \{5,6,3\}, & \{5,6,8\}, \\ \{5,6,8\}, & & & \end{array}$$

or

$$(II)\quad \begin{array}{llll} \{1,2,3\}, & \{1,2,8\}, & \{1,2,9\}, & \{1,6,3\}, \\ \{1,6,8\}, & \{1,6,8\}, & \{1,7,3\}, & \{1,7,9\}, \\ \{1,7,9\}, & \{4,2,3\}, & \{4,2,9\}, & \{4,2,9\}, \\ \{4,7,3\}, & \{4,7,3\}, & \{4,7,9\}, & \{5,2,3\}, \\ \{5,2,8\}, & \{5,2,8\}, & \{5,6,3\}, & \{5,6,3\}, \\ \{5,6,8\}, & & & \end{array}$$

or

$$(III)\quad \begin{array}{llll} \{1,2,8\}, & \{1,2,9\}, & \{1,2,9\}, & \{1,6,3\}, \\ \{1,6,8\}, & \{1,6,8\}, & \{1,7,3\}, & \{1,7,3\}, \\ \{1,7,9\}, & \{4,2,3\}, & \{4,2,3\}, & \{4,2,9\}, \\ \{4,7,3\}, & \{4,7,9\}, & \{4,7,9\}, & \{5,2,3\}, \\ \{5,2,8\}, & \{5,2,8\}, & \{5,6,3\}, & \{5,6,3\}, \\ \{5,6,8\}. & & & \end{array}$$

So we have $b_1 = b_2 = 7$ in case (I) or case (III), $b_1 = 9$ and $b_2 = 6$ in case (II). But all of these cases are impossible since $(b_1, b_2, b_3) \in \{(13, 4, m_v - 7), (15, 3, m_v - 7)\}$. Now if $\Pi = (6_7, 9_1, 12_1)$, then let $D = \{1\}$, $N = \{2\}$, $S = \{3, 4, 5, 6, 7, 8, 9\}$, and

$$d(x) = \begin{cases} 12, & \text{if } x = 1, \\ 9, & \text{if } x = 2, \\ 6, & \text{if } x \in S. \end{cases}$$

Then for the degree $d'(x)$ of x in M, we have

$$d'(x) = \begin{cases} 0, & \text{if } x = 1, \\ 2, & \text{if } x = 2, \\ 4, & \text{if } x \in S. \end{cases}$$

Without loss of generality, we may suppose that $\{\{2,3\}, \{2,4\}\} \subset M$. Then it follows from $d(3) = d(4) = 6$ that $\{3, 4\} \notin M$

and A contains 3 blocks $\{3,4,x\}$, $\{3,4,x\}$, $\{3,4,y\}$ where $x, y \in S \setminus \{3,4\}$. Since $\{\{2,3\},\{2,4\}\} \subset M$, then we have

$$\{\{1,3,x\},\{1,4,x\},\{1,3,y\},\{1,4,y\},\{1,3,y\},\{1,4,y\}\} \subset \mathbf{A}$$

and so $\{1,y\}$ is contained in at least 4 blocks of **A**, a contradiction.

(iii) $w = 10$. Let $X = \{1,2,\cdots,10\}$, then the degree type of $(X,\mathbf{A})$ is $(6_9, 9_1)$ and

$$d'(x) = \begin{cases} 3, & \text{if } d(x) = 9, \\ 5, & \text{if } d(x) = 6. \end{cases}$$

At most 12 distinct pairs are contained in the blocks of $\mathbf{B}_2$ and since $|\mathbf{B}_1| \geq 13$ then without loss of generality, we may suppose that $\{1,2,3\} \in \mathbf{A}$ and $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are 3–pairs. Since there is only an element $a \in X$ with $d(a) = 9$,then $(X,\mathbf{A})$ contains a $TS(5,3)$ $(Y,\mathbf{B})$ with $Y = \{1,2,\cdots,5\}$. It is easy to see that if $a \notin Y$ then $\{a,y\} \in M$ for each $y \in Y$ which is impossible and if $a \in Y$ then **A** contains 3–repeated block $\{a,x,z\}$ where $\{x,z\} \subset \{6,7,8,9,10\}$ which is also impossible.
This completes the proof of the lemma. □

Lemma 2.13 *If* $v \equiv 1,\ 3 \pmod 6$, *then* $m_v + 10 \notin ISS(v,3)$.

Proof. Let $(V,\mathbf{B})$ be a $TS(v,3)$ with $b^* = m_v + 10$ and fine structure (b_1,b_2,b_3). By Lemma 2.2

$$(b_1,b_2,b_3) \notin \{(14,2,m_v-6),(15,0,m_v-5)\}.$$

So we have

$$(b_1,b_2,b_3) \in \{(10,10,m_v-10),(11,8,m_v-9),(12,6,m_v-8),(13,4,m_v-7)$$

By Lemma 2.1, and Lemmas 2.10 – 2.12, for all the above 4 cases $(V,\mathbf{B})$ is decomposable. This completes the proof. □

Lemma 2.14 *If* $v \equiv 1,\ 3 \pmod 6$ *then* $m_v + 11 \notin ISS(v,3)$.

Proof. Let $(V,\mathbf{B})$ be a $TS(v,3)$ with $b^* = m_v + 11$ and fine structure (b_1,b_2,b_3). By Lemma 2.2,

$$(b_1,b_2,b_3) \notin \{(14,5,m_v-8),(16,1,m_v-6)\}.$$

So we have

$$(b_1, b_2, b_3) \in \{(11, 11, m_v - 11), (12, 9, m_v - 10), (13, 7, m_v - 9), (15, 3, m_v - 7)\}.$$

By Lemma 2.1 and Lemmas 2.10 – 2.12, for all the above cases, $(V, \mathbf{B})$ is decomposable. The proof is completed. □

Combining Lemmas 2.2, 2.3, 2.8, 2.9, 2.13, 2.14 gives our main theorem in this section.

Theorem 2.15 *If $m_v + k \in ISS(v, 3)$, then $12 \le k \le 2m_v$.*

3 Recursive constructions

In this section we give several recursive constructions for indecomposable triple systems with various support sizes.

Let $(X, \mathbf{A})$ be a $TS(u, \lambda)$ and $(Y, \mathbf{B})$ be a $TS(v, \lambda)$. If $Y \subseteq X$ and $\mathbf{B}$ is a subcollection of $\mathbf{A}$, then $(Y, \mathbf{B})$ is callled a subsystem of $(X, \mathbf{A})$, or $(Y, \mathbf{B})$ is embedded in $(X, \mathbf{A})$.

The following lemma will be useful for our constructions and the proof is obvious.

Lemma 3.1 *If a $TS(u, \lambda)$ contains an indecomposable $TS(v, \lambda)$ as a subsystem, then the $TS(u, \lambda)$ is also indecomposable.*

An incomplete triple system $TS(u, v, \lambda)$ is an ordered triple $(X, Y, \mathbf{A})$ where X is a u–set and Y is a v–subset of X, and $\mathbf{A}$ is a collection of 3–subsets (called blocks or triples) of X such that

(i) each pair of distinct elements of X, not both from Y, is contained in exactly λ blocks of $\mathbf{A}$,

(ii) each pair of distinct elements of Y is contained in no blocks of $\mathbf{A}$.

Let $(X, Y, \mathbf{A})$ be a $TS(u, v, \lambda)$, the umber of distinct blocks of $\mathbf{A}$ is called the support size of the $TS(u, v, \lambda)$. Let

$$SS(u, v, \lambda) = \{k \mid \exists TS(u, v, \lambda) \text{ with support size } k\}.$$

Lemma 3.2 *If $k_1 \in SS(u, v, \lambda)$ and $k_2 \in ISS(v, \lambda)$, then $k_1 + k_2 \in ISS(u, \lambda)$.*

Proof. Let $(X, Y, \mathbf{A})$ be a $TS(u, v, \lambda)$ with support size k_1, $(Y, \mathbf{B})$ be an indecomposable $TS(v, \lambda)$ with support size k_2. Then $\mathbf{A} \cap \mathbf{B} = \phi$ and so $(X, \mathbf{A} \cup \mathbf{B})$ is a $TS(u, \lambda)$ with support size $k_1 + k_2$

and contains $(Y, \mathbf{B})$ as a subsystem. By Lemma 3.1, $(X, \mathbf{A} \cup \mathbf{B})$ is indecomposable. This completes the proof. □

To give our main construction for incomplete triple systems with given support sizes, we need the following result:

Lemma 3.3 *[9] Let $u, v \equiv 1$ or $3 \pmod 6$, $u \geq 2v+1$, $v \geq 7$. Let $n = u-v$ and K_n be the complete graph of order n with vertex set Z_n. Then we can always choose $(n-v-1)n/2$ edges from K_n to form $(n-v-1)n/6$ triples, and the remaining edges form a cyclic subgraph of K_n of degree v which can be partitioned into v 1-factors.*

Main construction. *Let $u, v \equiv 1$ or $3 \pmod 6$, $u = 2v + k$, $v \geq 7$ and $k \geq 1$. If $0 \leq s \leq 2v$, $s \neq 1$, then*

$$(k-1)(v+k)/6 + (v+s)(v+k)/2 \in SS(2v+k, v, 3).$$

Proof. Let $Y = \{\infty_1, \infty_2, \cdots, \infty_v\}$ and $X = Z_{v+k}$. By Lemma 3.3, we may choose $(k-1)(v+k)/2$ edges from K_{v+k} to form a set $\mathbf{B}_0$ of $(k-1)(v+k)/6$ triples and the remaining edges can be partitioned into v 1-factors $F_1, F_2, \cdots, F_v$.
Define 2 permutations σ and τ on the set $\{1, 2, \cdots, v\}$ such that

$$\begin{cases} i = \sigma(i) = \tau(i), & \text{for } 1 \leq i \leq r_1, \\ i = \sigma(i) \neq \tau(i), & \text{for } r_1 + 1 \leq i \leq r_1 + r_2, \\ i \neq \sigma(i) \neq \tau(i) \neq i, & \text{for } r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + r_3 \end{cases}$$

and

$$\begin{cases} r_1 + r_2 + r_3 = v, \\ r_1 + 2r_2 + 3r_3 = s + v, \\ r_1, r_2, r_3 \geq 0 \end{cases}$$

This is always possible for any $s \in \{0, 2, 3, \cdots, 2v\}$. Now let

$$G_i = F_i \cup F_{\sigma(i)} \cup F_{\tau(i)}, \quad 1 \leq i \leq v.$$

Then G_i is a 3-factor. Lelt

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in G_i\}, \quad 1 \leq i \leq v$$

and

$$\mathbf{B} = 3\mathbf{B}_0 \cup \{\bigcup_{1 \leq i \leq v} \mathbf{B}_i\}.$$

Then $(X, Y, \mathbf{B})$ is a $TS(u, v, 3)$ with support size $(k-1)(v+k)/6 + (v+s)(v+k)/2$. □

Combining Lemma 3.2 with the main construction gives the following theorem:

Theorem 3.4 *Let* $u, v \equiv 1$ *or* $3 \pmod 6$, $u = 2v + k$, $v \geq 7$ *and* $k \geq 1$. *If* $m_v + t \in ISS(v, 3)$ *and* $0 \leq s \leq 2v$, $s \neq 1$, *then*

$$m_{2v+k} + t + s(v + k)/2 \in ISS(2v + k, 3).$$

Let $k = 1$ or 3 in Theorem 3.4, we then have the $2v + 1$ construction and the $2v + 3$ construction:

Corollary 1 *[2] Let* $v \equiv 1$ *or* $3 \pmod 6$, $v \geq 7$. *If* $m_v + t \in ISS(v, 3)$ *and* $0 \leq s \leq 2v$, $s \neq 1$, *then*

$$m_{2v+1} + t + s(v + 1)/2 \in ISS(2v + 1, 3).$$

Corollary 2 *Let* $v \equiv 3 \pmod 6$, $v \geq 9$. *If* $m_v + t \in ISS(v, 3)$ *and* $0 \leq s \leq 2v$, $s \neq 1$, *then*

$$m_{2v+3} + t + s(v + 3)/2 \in ISS(2v + 3, 3).$$

We may easily generalize our main construction and Theorem 3.4 for arbitrary λ with $\lambda \leq v$:

Theorem 3.5 *Let* $u, v \equiv 1$ *or* $3 \pmod 6$, $u = 2v + k$, $v \geq 7$, $k \geq 1$ *and* $\lambda \leq v$. *Let* $0 \leq s \leq (\lambda - 1)v$, $s \neq 1$. *Then*

(i) $(k - 1)(v + k)/6 + (v + s)(v + k)/2 \in SS(2v + k, v, \lambda)$.

(ii) If $m_v + t \in ISS(v, \lambda)$, *then*

$$m_{2v+k} + t + s(v + k)/2 \in ISS(2v + k, \lambda).$$

In order to give constructions for $TS(v, \lambda)$s with support sizes close to λm_v, the following lemma is also needed:

Lemma 3.6 *Let* u, v *and* λ *be positive integers,* $u, v \equiv 1$ *or* $3 \pmod 6$ *and* $u \geq 2v + 1$, $\lambda \leq v - 2$. *If* $m_v + t \in ISS(v, \lambda)$, *then*

$$\lambda m_u - (\lambda - 1)m_v + t \in ISS(u, \lambda).$$

Proof. It is proved in [9] that, for such parameters u, v and λ, there exists a simple $TS(u, v, \lambda)$, i.e. a $TS(u, v, \lambda)$ with support size $\lambda(m_u - m_v)$. The conclusion then follows from Lemma 3.2. □

A group divisible design $GD(k, n; v)$ is ordered triple $(X, \mathbf{G}, \mathbf{A})$ where X is a v–set, $\mathbf{G}$ is a set of n–subsets (called groups) of

X, G partitions X, and A is a set k–subsets (called blocks) of X such that each pair of distinct elements of X from distinct groups appears in a unique block, and each pair of elements of X from a same group appears in no block.
In some particular cases, we also need the following construction using group divisible designs.

Theorem 3.7 *(i) If there exists a $GD(3, n; v)$ such that*

$$m_n + t_1 \in ISS(n, \lambda), \quad m_n + t_2, m_n + t_3 \in SS(n, \lambda),$$

then

$$m_{3v} + t_1 + t_2 + t_3 \in ISS(3v, \lambda).$$

(ii) If there exists a $GD(3, n; v)$ such that

$$m_{n+1}+t_1 \in ISS(n+1, \lambda), \quad m_{n+1}+t_2, m_{n+1}+t_3 \in SS(n+1, \lambda).$$

then

$$m_{3v+1} + t_1 + t_2 + t_3 \in ISS(3v_1, \lambda).$$

Proof. Let $(X, \mathbf{G}, \mathbf{A})$ be a $GD(3, n; v)$, let $\mathbf{G} = \{G_1, G_2, G_3\}$.

(i) Form an indecomposable $TS(n, \lambda)$ with support size $m_n + t_1$ on G_1, a $TS(n, \lambda)$ with support size $m_n + t_2$ on G_2, and a $TS(n, \lambda)$ with support size $m_n + t_3$ on G_3, and let each block of A λ–times repeated. This gives an indecomposable $TS(3v, \lambda)$ on X with support size $m_{3v} + t_1 + t_2 + t_3$.

(ii) Let ∞ be a new element. Form an indecomposable $TS(n + 1, \lambda)$ with support size $m_{n+1} + t_1$ on $G_1 \cup \{\infty\}$, form a $TS(n + 1, \lambda)$ with support size $m_{n+1} + t_2$ on $G_2 \cup \{\infty\}$, and a $TS(n_1, \lambda)$ with support size $m_{n+1} + t_3$ on $G_3 \cup \{\infty\}$, and let each block of A λ–times repeated. This gives an indecomposable $TS(3v + 1, \lambda)$ on $X \cup \{\infty\}$ with support sizes $m_{3v+1} + t_1 + t_2 + t_3$.

This completes the proof. □

4 Support sizes for small v

In this section, we give constructions for indecomposable threefold triple systems with given support sizes which will be used in proving our main theorem of the paper.
$ISS(7, 3)$ was determined by Lemma 2.7; we restate the result in the following form:

Lemma 4.1 *[7]* $ISS(7,3) = \{21\}$.

Lemma 4.2 $ISS(9,3) = \{24, 25, \cdots, 36\}$.

Proof. Since $m_9 = 12$, then, by Theorem 2.15, if $m_9 + k \in ISS(9,3)$, then $k \geq 12$, and so

$$24 \leq m_9 + k = b^* \leq b = 36.$$

For each $b^* \in \{31, \cdots, 36\}$, an indecomposable $TS(9,3)$ with support size b^* can be found in [6]. For each $b^* \in \{24, \cdots, 30\}$, we form an indecomposable $TS(9,3)$ with support size b^* in the following:

(i) $b^* = 24$, $X = Z_6 \cup \{a, b, c\}$.

$\mathbf{B}_1$: $013, 014, 035, 124, 235, 245, a15,$
$a34, b01, b24, b35, c03, c14, c25.$

$\mathbf{B}_2$: $b04, b15, b23, c05, c12, c34, a13, a45.$

$\mathbf{B}_3$: $abc, a02.$

(ii) $b^* = 25$. $X = Z_6 \cup \{a, b, c\}$.

$\mathbf{B}_1$: $a01, a24, a35, b01, b24, b35, c03, c14,$
$c25, 013, 124, 235, 340, 451, 502.$

$\mathbf{B}_2$: $a02, a13, a45, b23, b40, b51, c12, c34, c50.$

$\mathbf{B}_3$: $abc.$

(iii) $b^* = 26$. $X = Z_6 \cup \{a, b, c\}$.

$\mathbf{B}_1$: $a01, a24, a35, b01, b25, b34, c03, c14,$
$c24, c34, c35, 013, 124, 235, 340, 451, 502.$

$\mathbf{B}_2$: $a02, a13, a45, b04, b15, b23, c05, c12.$

$\mathbf{B}_3$: $abc.$

(iv) $b^* = 27$. $X = Z_9$.

$\mathbf{B}_1$: $\{i, i+1, i+2\}, \{i, i+1, i+4\}, i \in Z_9.$

$\mathbf{B}_2$: $\{i, i+2, i+5\}, i \in Z_9.$

$\mathbf{B}_3$: $\phi.$

(v) $b^* = 28$. $X = Z_8 \cup \{\infty\}$.

$\mathbf{B}_1$: $\{i, i+2, \infty\}, i \in Z_8,$
$\{i, i+4, \infty\}, i = 0, 1, 2, 3,$
$\{i, i+1, i+4\}, i \in Z_8.$

$\mathbf{B}_2$: $\{i, i+1, i+3\}, i \in Z_8.$

$\mathbf{B}_3$: $\phi.$

(vi) $b^* = 29$. $X = Z_9$.

$\mathbf{B}_1$: 012, 123, 234, 456, 567, 678, 780, 801, 325, 346, 014, 125, 026, 347, 458, 560, 671, 782, 803, 025, 045, 046.

$\mathbf{B}_2$: 136, 247, 358, 571, 682, 703, 814.

$\mathbf{B}_3$: ϕ.

(vii) $b^* = 30$, $X = Z_8 \cup \{\infty\}$.

$\mathbf{B}_1$: $\{i, i+1, i+4\}, i \in Z_8$,
$\{i, i+2, \infty\}, i \in Z_8, i \neq 1$,
$\{0,1,3\}, \{0,1,6\}, \{0,2,3\}, \{0,2,6\}, \{0,4,\infty\}$,
$\{1,5,\infty\}, \{1,6,\infty\}, \{2,3,\infty\}, \{3,7,\infty\}$.

$\mathbf{B}_2$: $\{i, i+2, i+3\}, i \in Z_8, i \neq 0, 6$.

$\mathbf{B}_3$: ϕ.

This completes the proof. □

Lemma 4.3 *If* $0 \leq t \leq 8$, *then* $36 + 4t \in SS(13, 5, 3)$.

Proof. Let $u = 13$, $v = 5$, $X = Z_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$. Form the following 7 1–factors on Z_8:

$$F_1 = \{01, 23, 45, 67\}, \quad F_2 = \{12, 34, 56, 70\},$$
$$F_3 = \{02, 46, 13, 57\}, \quad F_4 = \{06, 24, 35, 71\},$$
$$F_5 = \{03, 25, 47, 61\}, \quad F_6 = \{14, 36, 50, 72\},$$
$$F_7 = \{04, 15, 26, 37\}.$$

Let

$$\mathbf{B}_0 = \{\{i, i+2, i+4\} \mid i \in Z_8\}.$$

Now we form an $ITS(13, 5, 3)$ $(X, \mathcal{B})$ with support size $36 + 4t$ for each t, $0 \leq t \leq 8$. Let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i$ where $\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\}$, $1 \leq i \leq 5$ and $\mathbf{A}_i$, $1 \leq i \leq 5$ are shown in the following table:

t	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$
0	$F_1F_1F_1$	$F_2F_2F_2$	$F_5F_5F_5$	$F_6F_6F_6$	$F_3F_4F_7$
1	$F_1F_1F_1$	$F_2F_2F_2$	$F_5F_5F_5$	$F_6F_6F_7$	$F_3F_4F_6$
2	$F_1F_1F_3$	$F_2F_2F_2$	$F_5F_5F_5$	$F_6F_6F_7$	$F_1F_4F_6$
3	$F_1F_1F_3$	$F_2F_2F_4$	$F_5F_5F_5$	$F_6F_6F_7$	$F_1F_2F_6$
4	$F_1F_1F_3$	$F_2F_2F_4$	$F_2F_5F_5$	$F_6F_6F_7$	$F_1F_5F_6$
5	$F_1F_2F_3$	$F_2F_2F_4$	$F_1F_5F_5$	$F_6F_6F_7$	$F_1F_5F_6$
6	$F_1F_2F_3$	$F_1F_2F_4$	$F_1F_5F_5$	$F_6F_6F_7$	$F_2F_5F_6$
7	$F_1F_2F_3$	$F_1F_2F_4$	$F_5F_5F_6$	$F_5F_6F_7$	$F_1F_2F_6$
8	$F_1F_2F_5$	$F_1F_2F_6$	$F_1F_5F_6$	$F_2F_5F_6$	$F_3F_4F_7$

This completes the proof. □

Lemma 4.4 *If* $0 \le t \le 7$, *then* $38 + 4t \in SS(13,5,3)$.

Proof. Let $X = Z_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$. Let

$$F_0 = \{02, 13, 45, 67\}, F_3 = \{01, 23, 46, 57\},$$

and let F_1, F_2, F_4, F_5, F_6, F_7 be as in Lemma 4.3. Let $\mathbf{B}_0 = \{\{i, i+2, i+4\} \mid i \in Z_8\}$. For $1 \le i \le 5$, let $\mathbf{A}_i$ shown in the following table.

t	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$
0	$F_1F_1F_3$	$F_2F_2F_2$	$F_5F_5F_5$	$F_6F_6F_6$	$F_0F_4F_7$
1	$F_1F_1F_3$	$F_2F_2F_2$	$F_5F_5F_5$	$F_6F_6F_7$	$F_0F_4F_6$
2	$F_1F_1F_3$	$F_2F_2F_2$	$F_5F_5F_6$	$F_6F_6F_7$	$F_0F_4F_5$
3	$F_1F_3F_5$	$F_2F_2F_2$	$F_5F_5F_6$	$F_6F_6F_7$	$F_0F_1F_4$
4	$F_1F_3F_5$	$F_1F_2F_2$	$F_5F_5F_6$	$F_6F_6F_7$	$F_0F_2F_4$
5	$F_1F_3F_5$	$F_1F_2F_2$	$F_5F_5F_6$	$F_4F_6F_7$	$F_0F_2F_6$
6	$F_1F_3F_5$	$F_1F_2F_2$	$F_4F_5F_6$	$F_5F_6F_7$	$F_0F_2F_6$
7	$F_1F_3F_5$	$F_1F_2F_4$	$F_2F_5F_6$	$F_5F_6F_7$	$F_0F_2F_6$

Let

$$\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i,$$
$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\}$$

then for each t, $0 \le t \le 7$, $(X, \mathbf{B})$ is $ITS(13,5,3)$ with support size $48 + 4t$ for each t with $0 \le t \le 7$. This completes the proof. □

Lemma 4.5 *If* $0 \le t \le 12$, *then* $43 + 2t \in SS(13,5,3)$.

Proof. Let $X = Z_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$. Let F_1, F_2, F_3, F_4, F_5, F_6, F_7 be as in Lemma 4.3, and let $\mathbf{B}_0 = \{\{i, i+1, i+2\} \mid i \in Z_8\}$.

(i) For $0 \le s \le 4$, let

$$\mathbf{A}_1 = F_3F_3F_4$$
$$\mathbf{A}_2 = F_5F_5F_5$$

and let $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ be shown in the following table:

s	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$
0	$F_1F_2F_4$	$F_6F_6F_6$	$F_7F_7F_7$
1	$F_1F_2F_6$	$F_4F_6F_6$	$F_7F_7F_7$
2	$F_1F_6F_7$	$F_4F_6F_6$	$F_2F_7F_7$
3	$F_1F_2F_6$	$F_4F_6F_7$	$F_6F_7F_7$
4	$F_1F_6F_7$	$F_2F_6F_7$	$F_4F_6F_7$

Let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i$ where

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\},\ 1 \leq i \leq 5$$

$(X, \mathbf{B})$ is an ITS(13,5,3) such that

$\mathbf{P} = \{\{\infty_1, 0, 2\}, \{\infty_1, 3, 5\}\{\infty_2, 0, 3\}, \{\infty_2, 2, 5\}\} \subset \mathbf{B},\ 0 \leq s \leq 4$

and so $(X, \mathbf{B}')$ is an $ITS(13, 5, 3)$ with support size $43 + 4s$, $0 \leq s \leq 4$ where

$\mathbf{B}' = (\mathbf{B} \backslash \mathbf{P}) \cup \{\{\infty_1, 0, 3\}, \{\infty_1, 2, 5\}\{\infty_2, 0, 2\}, \{\infty_2, 3, 5\}\}.$

(ii) For $0 \leq s \leq 4$, let

$$\begin{aligned} \mathbf{A}_1 &= F_1F_2F_7, \\ \mathbf{A}_2 &= F_3F_3F_4, \end{aligned}$$

and let $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ be shown in the following table:

s	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$
0	$F_4F_5F_5$	$F_6F_6F_6$	$F_5F_7F_7$
1	$F_4F_5F_5$	$F_5F_6F_6$	$F_6F_7F_7$
2	$F_4F_5F_6$	$F_5F_6F_6$	$F_5F_7F_7$
3	$F_5F_5F_6$	$F_5F_6F_7$	$F_4F_6F_7$
4	$F_5F_6F_7$	$F_5F_6F_7$	$F_4F_5F_6$

Let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i$ where

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\},\ 1 \leq i \leq 5$$

$(X, \mathbf{B})$ is an $ITS(13, 5, 3)$ such that

$\mathbf{P} = \{\{\infty_1, 1, 2\}, \{\infty_1, 3, 4\}, \{\infty_2, 1, 3\}, \{\infty_2, 2, 4\}\} \subset \mathbf{B},\ 0 \leq s \leq 4$

and so $(X, \mathbf{B}')$ is an $ITS(13, 5, 3)$ with support size $49 + 4s$, $0 \leq s \leq 4$ where

$\mathbf{B}' = (\mathbf{B} \backslash \mathbf{P}) \cup \{\{\infty_1, 1, 3\}, \{\infty_1, 2, 4\}, \{\infty_2, 1, 2\}, \{\infty_2, 3, 4\}\}.$

(iii) Let

$$\begin{aligned}\mathbf{A}_1 &= F_2F_7F_7,\\ \mathbf{A}_2 &= F_3F_3F_4,\\ \mathbf{A}_3 &= F_5F_5F_5,\\ \mathbf{A}_4 &= F_6F_6F_6,\\ \mathbf{A}_5 &= F_1F_4F_7,\end{aligned}$$

Let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i$ where

$$\mathbf{B}_0 = \{\{i, i+2, i+5\} \mid i \in Z_8\}.$$

and

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\},\ 1 \le i \le 5$$

$(X, \mathbf{B})$ is an $ITS(13, 5, 3)$ such that
$\mathbf{P} \subset \mathbf{B}$ with $\mathbf{P}$ as in (ii) and so $(X, \mathbf{B}')$ is an $ITS(13, 5, 3)$ with support size 45, where

$$\mathbf{B}' = (\mathbf{B}\backslash\mathbf{P}) \cup \{\{\infty_1, 1, 3\}, \{\infty_1, 2, 4\}, \{\infty_2, 1, 2\}, \{\infty_2, 3, 4\}\}.$$

(iv) Let

$$\begin{aligned}\mathbf{A}_1 &= F_1F_2F_3,\\ \mathbf{A}_2 &= F_3F_4F_4,\\ \mathbf{A}_3 &= F_5F_6F_7,\\ \mathbf{A}_4 &= F_5F_6F_7,\\ \mathbf{A}_5 &= F_5F_6F_7,\end{aligned}$$

and let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i$ where

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\},\ 1 \le i \le 5$$

$(X, \mathbf{B})$ is an $ITS(13, 5, 3)$ such that

$$\begin{aligned}\mathbf{P} = \{ \quad & \{\infty_2, 0, 6\}, \{\infty_2, 2, 4\}, \{\infty_2, 0, 2\}, \{\infty_2, 3, 5\},\\ & \{\infty_3, 0, 4\}, \{\infty_3, 2, 6\}, \{\infty_3, 0, 3\}, \{\infty_3, 2, 5\}\} \subset \mathbf{B},\end{aligned}$$

and so $(X, \mathbf{B}')$ is an $ITS(13, 5, 3)$ with support 67, where

$$\begin{aligned}\mathbf{B}' \quad = \quad & (\mathbf{B} \setminus \mathbf{P}) \cup \{\{\infty_2, 0, 4\}, \{\infty_2, 2, 6\}, \{\infty_2, 0, 3\}, \{\infty_2, 2, 5\}\},\\ & \{\infty_3, 0, 6\}, \{\infty_3, 2, 4\}, \{\infty_3, 0, 2\}, \{\infty_3, 3, 5\}\} \subset \mathbf{B},\end{aligned}$$

(v) Let $F_8 = \{02, 13, 45, 67\}$,$F_9 = \{01, 23, 46, 57\}$,

$$\begin{aligned} \mathbf{A}_1 &= F_3F_4F_9, \\ \mathbf{A}_2 &= F_2F_5F_6, \\ \mathbf{A}_3 &= F_2F_7F_8, \\ \mathbf{A}_4 &= F_1F_2F_4, \\ \mathbf{A}_5 &= F_1F_7F_7, \end{aligned}$$

Let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i$ where

$$\mathbf{B}_0 = \{\{i, i+2, i+4\} \mid i \in Z_8\}.$$

and

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\},\ 1 \leq i \leq 5$$

$(X, \mathbf{B})$ is an $ITS(13, 5, 3)$ such that

$$\mathbf{P} = \{\{\infty_1, 4, 6\}, \{\infty_1, 1, 7\}, \{\infty_2, 4, 7\}, \{\infty_2, 1, 6\}\} \subset \mathbf{B},$$

and so $(X, \mathbf{B}')$ is an $ITS(13, 5, 3)$ with support size 63, where

$$\mathbf{B}' = (\mathbf{B}\backslash\mathbf{P}) \cup \{\{\infty_1, 4, 7\}, \{\infty_1, 1, 6\}, \{\infty_2, 4, 6\}, \{\infty_2, 1, 7\}\}.$$

This completes the proof. □

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Lemma 4.6 *If* $46 \leq k \leq 78$, $k \neq 47$, *then* $k \in ISS(13, 3)$.

Proof. By theorem 1.2, there is a $TS(5, 3)$ with support size 10, which must be indecomposable and so, by Lemma 3.2 and Lemmas 4.3 – 4.5, if $46 \leq k \leq 78$, $k \notin \{47, 49, 51\}$, then $k \in ISS(13, 3)$. Let $X = Z_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$. Let F_1, F_2, F_4, F_5, F_6, F_7 be as in Lemma 4.3, and let

$$\begin{aligned} F_0 &= \{01, 23, 46, 57\}, \\ F_9 &= \{02, 13, 45, 67\}, \\ \mathbf{A}_1 &= F_0F_1F_1, \\ \mathbf{A}_2 &= F_5F_5F_5, \\ \mathbf{A}_3 &= F_3F_4F_7, \\ \mathbf{A}_4 &= F_2F_2F_2, \\ \mathbf{A}_5 &= F_6F_6F_6, \end{aligned}$$

Let $\mathbf{B} = \bigcup_{i=0}^{5} \mathbf{B}_i \cup \mathbf{C}$ where

$$\mathbf{B}_0 = \{\{i, i+2, i+4\} \mid i \in Z_8\}.$$

$$\mathbf{B}_i = \{\{\infty_i, a, b\} \mid \{a, b\} \in \mathbf{A}_i\},\ 1 \le i \le 5$$

and $(\mathbf{Y}, \mathbf{C})$ is the $TS(5,3)$ with $Y = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$. $(X, \mathbf{B})$ is an $ITS(13, 5, 3)$ such that

$$\mathbf{P} = \{\{\infty_1, \infty_2, \infty_3\}, \{\infty_1, 4, 6\}, \{7, \infty_2, 4\}, \{7, \infty_3, 6\}\} \subset \mathbf{B},$$

$$\mathbf{Q} = \mathbf{P} \cup \{\{\infty_1, 0, 1\}, \{\infty_1, 2, 3\}, \{\infty_3, 0, 2\}, \{\infty_3, 1, 3\}\} \subset \mathbf{B}.$$

It is easy to see that $(X, \mathbf{B}')$ and $(X, \mathbf{B}'')$ are two $ITS(13, 5, 3)$ with support sizes 49 and 51 respectively, where

$$\mathbf{B}' = (\mathbf{B} \setminus \mathbf{P}) \cup \{\{\infty_1, \infty_2, 4\}, \{\infty_1, \infty_3, 6\}, \{7, \infty_2, \infty_3\}\{7, 4, 6\}\}$$

and

$$\begin{aligned}\mathbf{B}'' \ &= \ (\mathbf{B} \setminus \mathbf{Q}) \cup \{\{\infty_1, 0, 2\}, \{\infty_1, 1, 3\}, \{\infty_3, 0, 1\}, \{\infty_3, 2, 3\}, \\ &\quad \{\infty_1, \infty_2, 4\}, \{\infty_1, \infty_3, 6\}, \{7, \infty_2, \infty_3\}, \{7, 4, 6\}\}.\end{aligned}$$

This completes the proof. □

5 Main result

To prove our main theorem the following lemmas are also needed.

Lemma 5.1 *If* $u = 19$ *or 21, then*

$$ISS(u, 3) = \{m_u + 12, m_u + 13, \cdots, 3m_u\}.$$

Proof. Let $u = 2v + k$, $v = 9$ and $k = 1$ or 3, the conclusion follows from Theorem 2.15 and Theorem 3.4. □

Lemma 5.2 $ISS(25, 3) = \{m_{25} + 12, m_{25} + 13, \cdots, 3m_{25}\}$.

Proof. Let $25 = 2v + k$ where $v = 9$, $k = 7$, it follows from Theorem 3.4 that

$$ISS(25, 3) \supseteq \{m_{25} + 12, m_{25} + 13, \cdots, 3m_{25}\} \setminus \{125, 126, 127\}.$$

Let $n = 8$, $v = 24$. There exists a $GD(3, 8; 24)$ and $m_9 + t \in ISS(9, 3)$ for $12 \le t \le 24$ and $m_9 \in SS(9, 3)$. Let $t_1 = 12$, $t_2 \in \{13, 14, 15\}$ and $t_3 = 0$ in Theorem 3.7 (i), it follows that

$$m_{25} + 12 + t_2 = 112 + t_2 \in ISS(25, 3).$$

This completes the proof. □

Lemma 5.3 *If* $u \in \{27, 31, 33, 37\}$, *then*

$$ISS(u,3) = \{m_u + 12, m_u + 13, \cdots, 3m_u\}.$$

Proof. In Theorem 3.4, let $v = 13$ and

$$u = 2v + k, \quad k \in \{1, 5, 7, 11\}.$$

Since

$$ISS(13,3) \supseteq \{m_{13} + 22, m_{13} + 23, \cdots, 3m_{13}\}.$$

by Lemma 4.6, then for $u = 26 + k$, $k \in \{1, 5, 7, 11\}$, we have

$$ISS(u,3) \supset \{m_u + 22, m_u + 23, \cdots, 3m_u\}.$$

Now let $u = 2v + k$ where $v = 9$, $k \in \{9, 13, 15, 19\}$ in Theorem 3.4, it follows that

$$ISS(u,3) \supset \{m_u + 12, m_u + 13, \cdots, m_u + 24\}$$

and then

$$ISS(u,3) = \{m_u + 12, m_u + 13, \cdots, 3m_u\}$$

for $u \in \{27, 31, 33, 37\}$. □

Now we are in a position to prove our main Theorem:

Theorem 5.4 *Let* $m_v = [v(v-1)/6]$. *If* $v \equiv 1, 3 \pmod 6$, $v > 15$, *then*

$$ISS(v,3) = \{m_v + 12, m_v + 13, \cdots, 3m_v\}.$$

If $v \equiv 5 \pmod 6$, $v \geq 5$, *then*

$$ISS(v,3) = \{m_v + 7, m_v + 10, \cdots, 3m_v + 1\}.$$

Proof. For $v \equiv 5 \pmod 6$, since any $TS(v,3)$ is indecomposable, then $ISS(v,3) = SS(v,3)$ and the conclusion follows from Theorem 1.2. For the case $v \equiv 1, 3 \pmod 6$, the theorem is proved in Lemmas 5.1 – 5.3 $15 < v \leq 37$.

For $v \geq 39$, let

$$v = \begin{cases} 2(6t+1)+1, & \text{if } v = 12t+3, \\ 2(6t+3)+1, & \text{if } v = 12t+7, \\ 2(6t+3)+3, & \text{if } v = 12t+9, \\ 2(6t+3)+7, & \text{if } v = 12t+13. \end{cases}$$

The conclusion then follows from Theorem 2.15 and Theorem 3.4.
□

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References

[1] C.J.Colbourn & C.C.Lindner, Support sizes of triple systems, *JCT* (A) **61** (1992), 193–210.

[2] C.J.Colbourn & E.S.Mahmoodian, The spectrum of support sizes for threefold triple systems, *Discrete Math.* **83** (1990), 9–19.

[3] C.J.Colbourn, R.Mathon, A.Rosa & N.Shalaby, The fine structure of threefold triple system: $v \equiv 1$ or $3 \pmod 6$, *Discrete Math.* **92** (1991), 49–64.

[4] W.Foody & A.Hedayat, On theory and applications of *BIB* designs with repeated blocks, *Ann. Statistics* **5** (1977), 923–945.

[5] H.Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[6] R.Mathon & D.Lomas, A census of 2–(9,3,3) designs, *Australasian J. of Combinatorics* **5** (1992) 145–158.

[7] E.J.Morgan, Some small quasi-multiple designs, *Ars Combinaoria* **3** (1977), 233–250.

[8] A.Rosa, Repeated blocks in indecomposable twofold triple systems, *Discrete Math.* **65** (1987), 261–276.

[9] H.Shen, Embeddings of simple triple systems, *Science in China* (A) **35** (1992), 283–191.