Clique Representations and Dimension-k Chordal Graphs

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Abstract

The well-known clique tree representation for chordal graphs is extended to multidimensional representations for arbitrary graphs in which the number of vertices in the representation, minus the number of edges, plus the number of distinguished cycles, minus the number of distinguished polyhedra, and so on, always equals one. This approach generalizes both chordal graphs and cycle spaces of graphs. It also leads to a 'dimension' parameter that is shown to be no greater than the boxicity, chromatic number, and tree-width parameters.

1 Introduction and background

This paper shows how to start from any graph G and construct a clique representation graph H whose vertices are the *maxcliques*—maximal complete subgraphs—of G and that has distinguished cycles, polyhedra, and higher dimensional features related to vector spaces over $\mathbb{Z}_2$ that generalize the familiar cycle space of G. The termination of this series determines a dimension for H, and thereby a parameter of G. The graphs G whose clique representations have dimension one are precisely the *chordal graphs*—graphs with no induced cycles larger than triangles—and the graphs with clique representations of arbitrary dimension can be viewed as being generalizations of chordal graphs.

The remainder of this section sketches the relevant vector space concepts, assuming only a very basic introduction to these notions such as is given in [10].

Let $V(H)$ and $E(H)$ denote, respectively, the vertex set and edge set of a graph H. The power set of $V(H)$ can be made into a $\mathbb{Z}_2$-vector space with basis $V(H)$ by identifying each $\{v_1, \ldots, v_\ell\} \subseteq V(H)$ with the formal sum $v_1 + \cdots + v_\ell$. Similarly, the power set of $E(H)$ can be made into a $\mathbb{Z}_2$-vector space with basis $E(H)$ by identifying each $\{E_1, \ldots, E_\ell\} \subseteq E(H)$ with the formal sum $E_1 + \cdots + E_\ell$. For each $E_i = vw \in E(H)$, the function f_1 defined by $f_1(E_i) = v + w$ extends uniquely to a linear transformation F_1 from the vector space on the power set of $E(H)$ to the vector space on the power set of $V(H)$. Since the set of all $f_1(E_i)$'s where E_i is an edge of a particular spanning forest of H is a basis for the image of F_1, the rank of F_1 is $|V(H)| - \beta_0(H)$ where $\beta_0(H)$ denotes the number of components in H. The null space of F_1 is called the *circuit space* of H and has dimension $|E(H)| - |V(H)| + \beta_0(H)$.

In terms of the graph H, a *circuit* is a set $\{E_1, \ldots, E_\ell\} \subseteq E(H)$ such that, viewing each edge as a 2-element set of vertices, the symmetric difference $E_1 \oplus \cdots \oplus E_\ell = \emptyset$ (meaning that each $v \in V(H)$ is in an even number of E_i's). The circuit space is isomorphic to the $\mathbb{Z}_2$-vector space of circuits in which the sum of circuits is their symmetric difference. A *cycle* is a minimal nonempty dependent set of edges, and so circuits are the edge-disjoint unions of cycles. *Homologous circuits* are circuits such that one is the sum (symmetric difference) of the other and any number of triangles. If H is the octahedron $K_{2,2,2}$ for instance, then every two circuits are homologous to each other; in the cube, no two circuits are homologous. Taking the circuit space modulo the equivalence relation of being homologous produces a subspace whose dimension is denoted $\beta_1(H)$ (so the octahedron has $\beta_1 = 0$ and the cube has $\beta_1 = 5$).

Let $\mathcal{C}(H)$ be the set of all cycles of H. The power set of $\mathcal{C}(H)$ can be made into a $\mathbb{Z}_2$-vector space with basis $\mathcal{C}(H)$ by identifying each $\{C_1, \ldots, C_\ell\}$ with the formal sum $C_1 + \cdots + C_\ell$. For each $C_i = \{E_1, \ldots, E_\ell\} \in \mathcal{C}(H)$, the function f_2 defined by $f_2(C_i) = E_1 + \cdots + E_\ell$ extends uniquely to a linear transformation F_2 from the vector space on the power set of $\mathcal{C}(H)$ to the vector space on the power set of $E(H)$. Since the set of all $f_2(C_i)$'s where C_i is a fundamental cycle with respect to a particular spanning forest of H is a basis for F_2, the rank of F_2 is $|E(H)| - |V(H)| + \beta_0(H)$. The null space of F_2 is called the *polyhedron space* (or 3-*polyhedron space*) and has dimension $|\mathcal{C}(H)| - |E(H)| + |V(H)| - \beta_0(H)$.

In terms of the graph H, a (3-)*polyhedron* is a set $\{C_1, \ldots, C_\ell\} \subseteq \mathcal{C}(H)$ that, viewing each cycle as a set of edges, the symmetric difference $C_1 \oplus \cdots \oplus C_\ell = \emptyset$. The polyhedron space is isomorphic to the $\mathbb{Z}_2$-vector space of polyhedra in which the sum of polyhedra is their symmetric difference. *Homologous polyhedra* are polyhedra such that one is the sum (symmetric difference) of the other and any number of tetrahedra. Taking the polyhedron space modulo the equivalence relation of being homologous produces a subspace whose dimension is denoted $\beta_2(H)$.

This continues. For instance, if $\mathcal{P}(H)$ is the set of all minimal nonempty dependent sets of cycles, then the power set of $\mathcal{P}(H)$ can similarly be made into a $\mathbb{Z}_2$-vector space with basis $\mathcal{P}(H)$. This space will contain (as a null space) a 4-*polyhedron space* with dimension $|\mathcal{P}(H)| - |\mathcal{C}(H)| + |E(H)| - |V(H)| + \beta_0(H)$, with the dimension of the subspace modulo being homologous denoted $\beta_3(H)$.

2 Clique representations

For any graph G, a *clique representation* H *of* G has the form

$$H = \langle V(H), E(H), C(H), P(H), P^4(H), \ldots \rangle$$

where $\langle V(H), E(H) \rangle$ is a graph with distinguished sets $C(H)$ of selected cycles, $P(H)$ of selected polyhedra, $P^4(H)$ of selected 4-polyhedra, and so on. Writing $P(H) = P^3(H)$, $C(H) = P^2(H)$, $E(H) = P^1(H)$, and $V(H) = P^0(H)$, the maximum i with $P^i(H) \neq \emptyset$ is called the *dimension of* H. Theorem 2 below will bound the dimension of H in terms of G. In the dimension-1 case, H is the well-studied *clique tree*, existing if and only if G is chordal [7]. Higher dimensions are intended as generalizations of chordal, with the dimension-2 case detailed in [6].

The actual construction of H from G is given below. (For the readers' benefit, each step will be illustrated parenthetically for G the octahedron; consult Figure 1.)

Step 0. *Take the vertices of* H *to be the maxcliques of* G. For each $V \in V(H)$ with corresponding maxclique Q_i of G, define $\cap V = V(Q_i)$. For each edge $E = Q_iQ_j$ of the complete graph on $V(H)$, define $\cap E = \cap Q_i \bigcap \cap Q_j$.

(For the octahedron, $V(H)$ consists of the eight K_3 subgraphs. If V is the subgraph of H induced by vertices a, b and c, then $\cap V = \{a, b, c\}$.)

Step 1. *Choose the edges for* $E(H)$ *from the edges of the complete graph on* $V(H)$ *by, for each* $i \geq 0$ *in decreasing order, sequentially choosing edges* E *with* $|\cap E| = i$ *such that* E *does not form a cycle with previously-chosen edges* $E_1, E_2, \ldots$ *where each* $\cap E_j$ *contains* $\cap E$. For each $C = \{E_1, \ldots, E_\ell\} \in C(H)$, define $\cap C = \cap E_1 \bigcap \cdots \bigcap \cap E_k$.

(For the octahedron, suppose the 'top' edge E_t joining vertices $\{a, b, c\}$ and $\{a, b, e\}$ with $\cap E_t = \{a, b\}$ is chosen first, followed by the 'left' and 'right' edges E_ℓ and E_r with $\cap E_\ell = \{b, c\}$ and $\cap E_r = \{b, e\}$. The bottom edge E_b could be chosen next, since $\cap E_b = \{b, d\}$ is not contained in each of $\cap E_\ell$, $\cap E_t$, and $\cap E_r$—equivalently, $\cap E_b = \{b, d\} \not\subseteq \cap E_\ell \bigcap \cap E_t \bigcap \cap E_r$. Each of the twelve edges E_i with $|\cap E_i| = 2$ is chosen for $E(H)$, but none of those with $|\cap E_i| = 1$ is chosen; for instance the edge E joining $\{a, b, c\}$ and $\{b, d, e\}$ would not be chosen since $\cap E$ equals $\{b\}$ and would form a cycle with edges E_t and E_r where both $\cap E_t$ and $\cap E_r$ contain $\cap E$. If C is the cycle $\{E_\ell, E_t, E_r, E_b\}$, then $\cap C = \{b\}$. Step 1 ensures that each $C \in C(H)$ will have $\cap C$ properly contained in $\cap E_i$ for each $E_i \in C$.)

Step 2. *Choose the cycles for $C(H)$ from the cycles in $\mathcal{C}(H)$ by, for each $i \geq 0$ in decreasing order, sequentially choosing cycles C with $|\cap C| = i$ such that C is not the sum of previously-chosen cycles $C_1, C_2, \ldots$ where each $\cap C_j$ contains $\cap C$.* For each polyhedron $P = \{C_1, \ldots, C_\ell\}$ of the graph $\langle V(H), E(H)\rangle$, define $\cap P = \cap C_1 \bigcap \cdots \bigcap \cap C_k$.

(For the octahedron, each of the six 4-cycles of H is chosen for $C(H)$, with each $\cap C = \{x\}$ where x runs through $\{a, b, c, d, e, f\}$ as shown in Figure 1. These six cycles form the polyhedron P that is the entire cube, with $\cap P = \emptyset$. Step 2 ensures that each $P \in P(H)$ will have $\cap P$ properly contained in $\cap C_i$ for each boundary cycle $C_i \in P$.)

Step 3. *Choose the polyhedra for $P(H)$ from the polyhedra in $\mathcal{P}(H)$ formed from cycles in $C(H)$ by, for each $i \geq 0$ in decreasing order, sequentially choosing polyhedra P with $|\cap P| = i$ such that P is not the sum of previously-chosen polyhedra $P_1, P_2, \ldots$ where each $\cap P_j$ contains $\cap P$.*

(For the octahedron, the one polyhedron present—the cube itself—is chosen for $P(H)$.)

Step k. *Choose k-polyhedra for $P^k(H)$ that are the sum of members of $P^{k-1}(H)$ by, for each $i \geq 0$ in decreasing order, sequentially choosing P with $|\cap P| = i$ such that P is not the sum of previously-chosen k-polyhedra $P_1, P_2, \ldots$ where each $\cap P_j$ contains $\cap P$; stop if $P^k(H) = \emptyset$.*

(For the octahedron, $P^4(H) = \emptyset$, so H has dimension three and the construction terminates.)

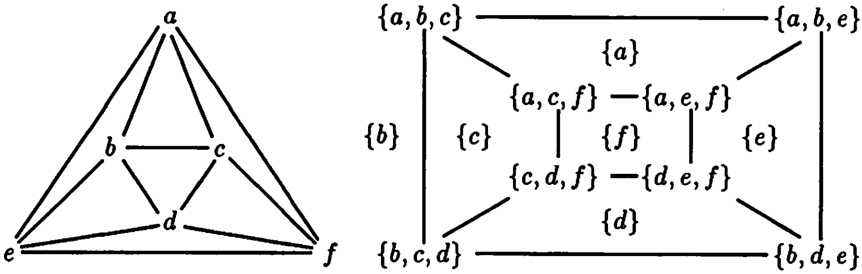

Figure 1: A graph with its dimension 3 clique representation.

Generalizing the dimension-3 clique representation for the octahedron, the d-dimensional (d-partite) octahedron $K_{2,\ldots,2}$ has the d-dimensional cube as its unique clique representation, with each $P^i(H)$ consisting of all the complete subgraphs of order $d - i$.

Figure 2 shows another example of a graph that has a clique representation H of dimension three. One choice for $C(H)$ is shown that consists of the 13 plane faces, each having $|\cap C| = 1$. Each $V \in V(H)$ has $|\cap V| = 3$ with the exact $\cap V$'s inferable from the face labels in the figure; for instance, the top vertex V has $\cap V = \{1, 2, 5\}$ and the vertex V' below that has $\cap V' = \{1, 2, 9\}$. The edge set $E(H)$ is uniquely determined, again with the exact $\cap E$'s inferable from the face labels; for instance the edge $E = VV'$ has $\cap E = \{1, 2\}$. (Notice there are repetitions among the $\cap E$'s and among the $\cap C$'s.) Once again, $|P(H)| = 1$.

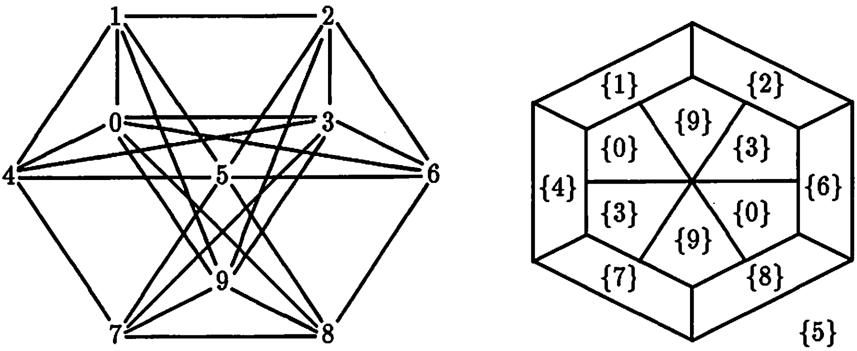

Figure 2: Another graph with a dimension 3 clique representation.

Let R always denote a complete, possibly null (i.e., K_0) subgraph of G, and let H_R denote the subgraph of H induced by those vertices $V \in V(H)$ for which $R \subseteq \cap V$ (making $H_R = H$ when $R = K_0$). Allowing $i = 0$ in the construction of H ensures that each H_R is connected, has its circuit space $\mathcal{C}(H)$ spanned by $\{C \in C(H) : R \subseteq \cap C\}$, and has the vector space $\mathcal{P}^k(H)$ of all k-polyhedra of H spanned by $\{P \in P^k(H) : R \subseteq \cap P\}$. An *$R$-cycle* is a cycle $C : Q_1, \ldots, Q_t, Q_1$ in H for which $\cap C = \cap_i V(Q_i) = R$; *$R$-edges, R-paths, R-polyhedra,* and so on are defined similarly in terms of $\cap_i V(Q_i) = R$. The proof of Lemma 1 below will show that no R-cycle contains an R-edge, and it is similarly true that no R-polyhedron contains an R-cycle, and so on. The interested reader can verify that H will always be triangle-free. Notice that, since $i = 0$ is allowed in Step 1 of the construction, H is always connected.

Let $N(R)$ denote the *common neighborhood of R*, the subgraph of G induced by those vertices that are adjacent to every vertex in R (making $N(R) = G$ when $R = K_0$). Let $a \ominus b$ denote the *proper difference* $\max\{a - b, 0\}$. The following result shows that, although H is not uniquely determined from G, its key features are.

Lemma 1 *Every clique representation H of any graph G satisfies the following conditions:*
H0: *The set $\{\cap V : V \in V(H)\}$ equals the set $\{R : \beta_0 N(R) = 0\}$ (i.e., the set of maxcliques of G; this can also be viewed as a multiset in which each R has multiplicity $1 \ominus \beta_0 N(R)$).*
H1: *The multiset $\{\cap E : E \in E(H)\}$ equals the multiset $\{R : \beta_0 N(R) > 1\}$ with each R having multiplicity $\beta_0 N(R) \ominus 1$.*
H2: *The multiset $\{\cap C : C \in C(H)\}$ equals the multiset $\{R : \beta_1 N(R) > 0\}$ with each R having multiplicity $\beta_1 N(R)$.*
Hk: *The multiset $\{\cap P : P \in P^k(H)\}$ equals the multiset $\{R : \beta_{k-1} N(R) > 0\}$ with each R having multiplicity $\beta_{k-1} N(R)$.*

Proof: Condition H0 follows immediately from Step 0.

To show condition H1, choose any R (complete or null subgraph) in G. Suppose $v_1, \ldots, v_{\beta_0 N(R)}$ are chosen from different components of $N(R)$. For each $i \leq \beta_0 N(R)$, choose a maxclique Q_i of G such that $R \cup \{v_i\} \subseteq Q_i$. Suppose $i \neq j$. If $w \in Q_i \cap Q_j$, then w is adjacent to both v_i and v_j and so $w \in R$ (since v_i and v_j are in different components of $N(R)$). Therefore $Q_i \cap Q_j \subseteq R$ and so $Q_i \cap Q_j = R$. Since H_R is connected, the vertices Q_i and Q_j of H_R are connected by an R-path that, since v_i and v_j are in different components of $N(R)$, contains an R-edge.

In Step 1, taking i in decreasing order ensures that, for every cycle C chosen, each edge E of C has $|\cap E| \geq |\cap E_f|$, where E_f is the final edge chosen for C. Also, $\cap E_f$ properly contains $\cap C$, so each $\cap E \neq \cap C$. Thus no edge E of H is in an $\cap E$-cycle, and so no R-cycle contains an R-edge. Therefore H_R consists of $\beta_0 N(R)$ connected subgraphs without R-edges, interconnected by $\beta_0 N(R) - 1$ many R-edges. Thus the multiplicity of R in the multiset $\{R : \beta_0 N(R) > 1\}$ equals the number of R-edges in $E(H)$.

To show condition H2, choose any R in G. Call a circuit C in $N(R)$ *null-homologous* if it is the sum of triangles from $N(R)$ (i.e., if it is homologous to the empty circuit). We first show that each chordless cycle $C : v_1, v_2, \ldots, v_\ell, v_1$ in $N(R)$ that is not null-homologous can be mapped to an R-cycle C^+ in H_R as follows: For each $v_i v_{i+1} \in E(C)$ (using arithmetic modulo ℓ in subscripts) choose a maxclique $Q(i, i+1)$ of G that contains R such that $V(C) \cap Q(i, i+1) = \{v_i, v_{i+1}\}$; then $Q(i-1, i)$ and $Q(i, i+1)$ are vertices of H_R that are connected by a path—call it Π_i—in $H_{\{v_i\} \cup R}$; and these Π_i's piece together in H to form an R-cycle C^+.

Call two R-cycles *$\supset R$-equivalent* if either is the sum of the other and any number of $C_i \in C(H)$ where each $\cap C_i$ properly contains R. Call an R-cycle *$\supset R$-generated* if it is the sum of any number of $C_i \in C(H)$ where each $\cap C_i$ properly contains R. Step 2 of the definition of clique representation ensures that no R-cycle $C \in C(H)$ is the sum of several $C_i \in C(H)$ where each $\cap C_i \supseteq R$ and some $\cap C_i = R$. None of the C^+'s we constructed can be $\supset R$-generated, since if $C^+ = \bigoplus_i C_i$ where each C_i contains an $x_i \notin R$, then $C : v_1, v_2, \ldots, v_\ell, v_1$ would be the sum of various triangles of the forms $v_i v_{i+1} x_j$, $v_i x_j x_k$ and $x_i x_j x_k$, contrary to C not being null-homologous. Taking into account the choices possible for the $Q(i, i+1)$'s and the Π_i's, each chordless C in $N(R)$ that is not null-homologous can be mapped to a family of non-$\supset R$-equivalent, non-$\supset R$-generated C^+'s in H_R.

We next show that each non-$\supset R$-generated R-cycle $C : Q_1, Q_2, \ldots, Q_\ell$, Q_1 of H_R can be mapped to a cycle C^- of $N(R)$: Choose distinct $v_1, \ldots, v_h$ from $V(G)$ where for each $i \leq h$ there is a subpath $\Lambda_i : Q_{s(i,1)}, \ldots, Q_{s(i,n_i)}$ of C with $v_i \in \cap \Lambda_i$. It is not hard to see that $h \geq 4$ and C^- is chordless. None of these C^- can be null-homologous, since if $C^- = \bigoplus_i T_i$ with each T_i a triangle, then for each $x \in \bigcup_i T_i \setminus V(C^-)$, $N(x)$ would be a circuit C_x and C would be the sum of the $R \cup \{x\}$-cycles C_x^+, contrary to C being non-$\supset R$-generated. Taking into account the choices possible for the v_i's, we have actually mapped each non-$\supset R$-generated R-cycle C of H_R to a family of chordless, null-homologous cycles C^- of $N(R)$.

Finally, observe that each chordless cycle C of $N(R)$ that is not null-homologous can be a $(C^+)^-$, and that each non-$\supset R$-generated R-cycle C of H_R can be a $(C^-)^+$. Thus the homology classes in $N(R)$—equivalence classes of cycles with respect to being homologous—are in one-to-one correspondence with the $\supset R$-equivalence classes of H_R, and this shows that the multiplicity of R in the multiset $\{R : \beta_1 N(R) > 0\}$ equals the number of R-cycles in $C(H)$.

Condition Hk follows by a similar, dimensionally-altered argument. □

3 Dimension-k chordal graphs

A *k-circuit* of G is any set $\{T_1, \ldots, T_\ell\}$ of complete subgraphs, each of order $k+1$, such that each order-k complete subgraph of G is in an even number of the T_i's. So 1-circuits are the same as circuits, and every *deltahedron*, meaning a polyhedron formed from triangles, is a 2-circuit.

Define a graph G to be *dimension-k chordal* if every vertex-minimal subgraph of G that contains a k-circuit is isomorphic to K_{k+2}. For instance, a graph is dimension-1 chordal if and only if every vertex-minimal subgraph that contains a cycle is a triangle, which makes being dimension1-chordal equivalent to being chordal. Dimension-2 chordal graphs can contain no deltahedra bigger than tetrahedra, but there are other graphs they cannot contain, such as the G in Figure 2: the 18 triangles other than triangle 039 form a 2-circuit, so G is a vertex-minimal subgraph that contains yet is not itself a 2-circuit. The graphs G in Figures 1 and 2 are examples of dimension-3 chordal graphs, while deleting vertex 5 from Figure 2 would leave a dimension-2 chordal graph.

Theorem 1 *For every graph G and every clique representation H of G, the following are equivalent:*

(1) *G is dimension-k chordal.*

(2) $|V(H)|-|E(H)|+|C(H)|-|P(H)|+\cdots+(-1)^k|P^k(H)|=1.$

(3) *H has dimension less than or equal to k.*

Proof: To show (1) $\Longrightarrow$ (2), suppose G is dimension-k chordal and define the *characteristic* of G by

$$\text{char } G=|V(G)|-|E(G)|+\alpha_2(G)-\alpha_3(G)+\cdots,$$

where each $\alpha_i(G)$ counts the number of complete subgraphs dimension i (order $i+1$) in G. Since G is dimension-k chordal, [8, Corollary 12] implies that, for each complete (possibly null) subgraph R of G,

$$\text{char } N(R)=\beta_0N(R)-\beta_1N(R)+\cdots+(-1)^{k-1}\beta_{k-1}N(R).$$

From that and [4, Lemma 1] (see also [5]),

$$\begin{aligned}1 &= \sum_R[1-\text{char } N(R)]\\ &= \sum_R[1-\beta_0N(R)+\beta_1N(R)-\cdots+(-1)^k\beta_{k-1}N(R)]\\ &= \sum_R[1\ominus\beta_0N(R)]-\sum_R[\beta_0N(R)\ominus 1]+\sum_R\beta_1N(R)+\\ &\qquad\cdots+(-1)^k\sum_R\beta_{k-1}N(R)].\end{aligned}$$

Lemma 1 then shows that every clique representation of G satisfies condition (2) of the theorem.

To show that (2) $\Longrightarrow$ (3), suppose H is any clique representation of G satisfying condition (2). Thus

$$|P^k(H)|=|P^{k-1}(H)|-\cdots+(-1)^k|E(H)|-(-1)^k|V(H)|+(-1)^k.$$

Since $\beta_0(H)=1$ by Step 1 of the construction, this equality shows that $|P^k(H)|$ equals the dimension of the k-polyhedron space of H. By Step k, $P^k(H)$ spans that space, and so $P^k(H)$ must be independent. Therefore $P^{k+1}(H)=\emptyset$ in Step $(k+1)$, and so H has dimension at most k.

To show that (3) $\Longrightarrow$ (1), suppose G is not dimension-k chordal, so there is a vertex-minimal induced subgraph G' of G such that $G'\not\cong K_{k+2}$ and G' contains a k-circuit $\{T_1,\ldots,T_\ell\}$. As in [8, Theorem 9], each $v\in$

$V(T_1) \cup \cdots \cup V(T_\ell)$ determines a $(k-1)$-circuit C_v in $N(v)$ that is not the sum of K_{k+1}'s in $N(v)$ and that has, for every complete subgraph Q of order k in G',

$$|\{C_v : Q \text{ in } C_v\}| = |\{T_i : Q \text{ in } T_i\}|.$$

So each such Q is in an even number of C_v's, and so $\{C_v : v \in V(G')\}$ is a dependent set of $(k-1)$-circuits. Suppose H is any clique representation of G. By the proof of condition Hk of Lemma 1, each such C_v corresponds to a $(k-1)$-circuit of H that is in the sum of $(k-1)$-circuits of H_v that are in $P^k(H)$. Since all these $(k-1)$-circuits of $P^k(H)$ form a dependent subset of $P^k(H)$, Step k of the construction shows that $P^{k+1}(H) \neq \emptyset$ and so the dimension of H would be greater than k. □

When k is less than the dimension of H, the equality in condition (2) becomes an inequality: ≥ 1 when k is even and ≤ 1 when k is odd.

Theorem 2 *For every graph G and clique representation H of G, the dimension of H is less than or equal to the boxicity, chromatic number, and tree-width of G (as well as the clique number of G and $\frac{1}{2}|V(G)|$). In particular, every bipartite or series-parallel graph is dimension-2 chordal.*

Proof: Suppose H is a dimension-d clique representation of a graph G.

If G has clique number ω (meaning that the largest complete subgraph in G has order ω), then G is $K_{\omega+1}$-free, so G contains no ω-circuit and so is dimension-ω chordal; then $d \leq \omega$ by Theorem 1.

The other inequalities will follow by results in [9] after showing that $d \leq \text{Chord}(G)$, where this is the least k such that $E(G) = E_1 \cap \cdots \cap E_k$ where each graph $(V(G), E_i)$ is chordal. This follows by induction on d, making free use of the results and terminology of [8] and [9]. The $d \leq 2$ cases are simple: $d = 1$ implies G is chordal, so $1 = \text{Chord}(G)$; $d = 2$ implies G is not chordal, so $2 \leq \text{Chord}(G)$. Suppose $d \geq 3$. By Theorem 1, G is dimension-d chordal but not dimension-$(d-1)$ chordal. As in [8], this implies that some induced subgraph G' of G has $\beta_{d-1}(G') > 0$. Suppose G' is a vertex-minimal induced subgraph of this sort. By [8, Theorem 9], every $v \in V(G')$ has $\beta_{d-2}(N(v)) > 0$ and so $N(v)$ is at best dimension-$(d-1)$ chordal. By Theorem 1, clique representations for $N(v)$ will have dimension at least $d-1$. Thus the inductive hypothesis on $N(v)$ implies that $d - 1 \leq \text{Chord}(N(v))$ for each $v \in V(G')$. The contrapositive of [9, Theorem 6] (strengthened as described in the paragraph following its proof) then implies that $d \leq \text{Chord}(G)$. □

The generalized octahedron $K_{2,\ldots,2}$ shows that the upper bounds in Theorem 2 are sharp. Figure 1 of [9] shows that the dimension of H can be less than $\text{Chord}(G)$.

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