A sufficient condition for pancyclic graphs

Ferenc Balogh

School of Mathematical Sciences, Australian National University

Abstract. In this paper we prove that, except for the 4-cycle and the 5-cycle, every 2-connected $K(1,3)$-free graph of diameter at most two is pancyclic.

Introduction

All graphs considered in this paper are undirected and finite, without loops or multiple edges. A graph is called $K(1,3)$-free if it has no induced subgraph isomorphic to $K(1,3)$, where $K(m,n)$ denotes the complete bipartite graph on m and n vertices. Let C denote an oriented cycle in a graph $G = (V(G), E(G))$, and C^{op} the same cycle with the opposite orientation. We use x^+ for the successor of x on C and x^- for its predecessor; we also write $x^{++} = (x^+)^+$, $x^{--} = (x^-)^-$, and so on. If $x, y \in V(C)$, then (xCy) denotes the path on C from x to y (excluding x and y, so this path may be empty) according to the orientation of C. Also, (yCx) denotes the path on C from y to x (excluding x and y) according to the orientation of C, $(yC^{op}x)$ is (xCy) in reverse orientation, and similarly $(xC^{op}y)$ is (yCx) in reverse orientation. If P is a path and $x, y \in V(P)$, then $x < y$ indicates that x precedes y in P. The neighborhood $N(x)$ of a vertex x is the set of all vertices adjacent to x. For notation and terminology not defined here, see Bondy and Murty [1].

Recently, there have been results dealing with sufficient conditions for $K(1,3)$-free graphs to be pancyclic. For example, see Faudree *et al.* [4] and [5]. In this paper we consider a well-known sufficient condition for $K(1,3)$-free graphs to be hamiltonian. In [2] Gould showed that every 2-connected $K(1,3)$-free graph of diameter at most 2 is hamiltonian. There have been several generalizations of this result (for example see Gould [6]), but none of them considered pancyclicity. The aim of this paper is to fill the gap: by using a different approach, we prove that this condition is sufficient for a graph (except the 4-cycle and the 5-cycle) to be pancyclic.

This result provides further support for Bondy's 'Metaconjecture' which asserts that almost every nontrivial condition which implies that a graph is hamiltonian also implies that the graph is pancyclic (see [3]).

Postal address: Ferenc Balogh, 4 Habgood Place, Kambah, Canberra, ACT 2902, Australia.

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Main result

Theorem. *Every* 2*-connected* $K(1,3)$*-free graph of diameter at most* 2*, except the* 4*-cycle and the* 5*-cycle, is pancyclic.*

Proof. We assume throughout that

$$G = (V(G), E(G)) \text{ is a } 2\text{-connected } K(1,3)\text{-free graph with } \operatorname{diam} G \leq 2,$$

and that

$$G \text{ is neither a } 4\text{-cycle nor a } 5\text{-cycle.}$$

The first thing to note is that G must contain a triangle. (Given that G is $K(1,3)$-free, the alternative is that G is a cycle, but if a cycle has more than five vertices then its diameter is greater than 2.) This easy observation provides the initial step for a proof by induction. Our inductive hypothesis will be that

$$G \text{ contains a cycle of length } k, \; 3 \leq k < |V(G)|.$$

We prove that then G must contain a cycle of length $k+1$ as well. To this end, we assume the inductive hypothesis and that

$$G \text{ has no cycle of length } k+1,$$

and show that this leads to a contradiction.

Let C be an oriented cycle of length k in G, and let H be a component of $G - V(C)$. Since G is 2-connected, there necessarily exist two distinct vertices a and b on C, having the following properties: (i) there exist vertices $u, v \in H$ (possibly $u = v$) such that a is adjacent to u and b is adjacent to v; (ii) no vertex $x \in V(aCb)$ (this set may be empty) is adjacent to a vertex in H. Let P be a path in H with initial vertex u and end vertex v. Choose H, (aCb), (bCa) and P so that

$$|V(P)| \text{ is minimal.}$$

Before proceeding, we prove that $|V(P)| \in \{1,2,3\}$. Suppose that $|V(P)| > 3$. Then $u \neq v$ and $uv \notin E(G)$, but since $\operatorname{diam} G \leq 2$, there exists a vertex $d \in N(u) \cap N(v)$. If d were in $V(H)$, we would have $|V(P)| = 3$, so this is not the case. As d cannot be in any other component of $G - V(C)$, we must have $d \in V(C)$. Then $d^+u \notin E(G)$ and $d^+v \notin E(G)$, so $\{u, v, d^+, d\}$ induces a $K(1,3)$. This contradiction proves that $|V(P)| \leq 3$.

Now we distinguish two main cases and several subcases. In each, we derive a conclusion that contradicts one or another of the assumptions displayed above.

Case 1. $|V(aCb)||V(bCa)|[|V(aCb)|-1][|V(bCa)|-1] \neq 0$.

Case 1.1. There exist vertices $p, p^+ \in V(aCb)$ such that

$$|[N(u) \cap N(p)] \cup [N(u) \cap N(p^+)]| > 1.$$

Let $m \neq n$, $m \in N(u) \cap N(p)$ and $n \in N(u) \cap N(p^+)$, then by property (ii) $m, n \in V(C)$ and $m, n \notin V(aCb)$. If $n < m$ in $b(bCa)a$, then $|V(nCm)| > 2$ (otherwise a cycle of length $k+1$ can be obtained), and, since $m^-m^+ \in E(G)$ and $n^-n^+ \in E(G)$ (otherwise $\{m, u, m^+, m^-\}$ or $\{n, u, n^+, n^-\}$ induces a $K(1,3)$, or a cycle of length $k+1$ can be obtained), then

$$m^-m^+(m^+Cp)pmunp^+(p^+Cn^-)n^-n^+(n^+Cm^-)m^-$$

is a cycle of length $k+1$. If $m < n$ in $b(bCa)a$, then also $|V(mCn)| > 2$, and

$$m^-m^+(m^+Cn^-)n^-n^+(n^+Cp)pmunp^+(p^+Cm^-)m^-$$

is a cycle of length $k+1$.

Case 1.2. For every pair of vertices $p, p^+ \in V(aCb)$,

$$|[N(u) \cap N(p)] \cup [N(u) \cap N(p^+)]| \leq 1.$$

Using the diameter condition, property (ii) for all $p \in V(aCb)$, and the fact that $|V(aCb)| \geq 2$, we conclude that $|N(u) \cap N(p)| > 0$ for all $p \in V(aCb)$. Obviously,

$$|[N(u) \cap N(p)] \cup [N(u) \cap N(p^+)]| > 0$$

for every pair of vertices $p, p^+ \in V(aCb)$, so, without loss of generality,

$$|[N(u) \cap N(p)] \cup [N(u) \cap N(p^+)]| = 1$$

for every pair of vertices $p, p^+ \in V(aCb)$. Clearly $aa^+ \in E(G)$, and assume that $ap \in E(G)$ for $p \in V(aCb)$, where $p < b^-$ on aCb. We can see that $a = N(u) \cap N(p) = N(u) \cap N(p^+)$ (otherwise the set $[N(u) \cap N(p)] \cup [N(u) \cap N(p^+)]$ would contain at least two vertices, since both $N(u) \cap N(p)$ and $N(u) \cap N(p^+)$ are nonempty) and, consequently, ap^+ is an edge of G. By induction on the vertices of the interval $V(aCb)$, we derive that $ap \in E(G)$ for all $p \in V(aCb)$. If $|V(P)| = 1$ (that is, if $u = v$), then

$$a^-a^+(a^+Cb^-)b^-aub(bCa^-)a^-$$

is a cycle of length $k+1$. If $|V(P)| = 2$ (that is, if $uv \in E(G)$), then

$$a^-a^+(a^+Cb^{--})b^{--}auvb(bCa^-)a^-$$

is a cycle of length $k+1$. It remains to consider the possibility that $|V(P)| = 3$, that is, $V(P) = \{u, y, v\}$ where $uy \in E(G)$, $yv \in E(G)$ and $uv \notin E(G)$. If now $|V(aCb)| = 2$ then $auyvb(bCa)a$ is a cycle of length $k+1$, while if $|V(aCb)| > 2$ then

$$a^-a^+(a^+Cb^{---})b^{---}auyvb(bCa^-)a^-$$

is a cycle of length $k+1$.

Case 2. $|V(aCb)||V(bCa)|[|V(aCb)|-1][|V(bCa)|-1] = 0$.

Here we distinguish three main subcases.

Case 2.1. $|V(aCb)||V(bCa)| = 0$ and $[|V(aCb)|-1][|V(bCa)|-1] \neq 0$. Without loss of generality $|V(aCb)| = 0$ and $|V(bCa)| \geq 2$, where $b = a^+$.

Case 2.1.1. $|V(P)| = 1$. In this case $aub(bCa)a$ is a cycle of length $k+1$.

Case 2.1.2. $|V(P)| = 2$. If there is vertex $x \in V(bCa)$ such that $x^-x^+ \in E(G)$, then

$$auvb(bCx^-)x^-x^+(x^+Ca)a$$

is a cycle of length $k+1$. Suppose that $x^-x^+ \notin E(G)$ for every vertex $x \in V(bCa)$. Clearly $dx \notin E(G)$ for every vertex $d \in V(H)$, otherwise, since G is $K(1,3)$-free, a cycle of length $k+1$ can be obtained. Since $ux \notin E(G)$ for all $x \in V(bCa)$, $ua^+ \notin E(G)$, and by the diameter condition $ax \in E(G)$ for all $x \in V(bCa)$, so $\{u, x^-, x^+, a\}$ induces a $K(1,3)$.

Case 2.1.3. $|V(P)| = 3$. If $|V(bCa)| = 2$, we have a 5-cycle. If $|V(bCa)| > 2$, then $ya^{++} \notin E(G)$ and by the diameter condition there necessarily exists a vertex $d \in N(y) \cap N(a^{++})$. Obviously $d \notin V(P)$ (otherwise a cycle of length $k+1$ can be obtained) and $d \notin \{a^-, a, a^+\}$ (by the minimality of $|V(P)|$). If $d \notin V(P) \cup V(C)$, then again either a cycle of length $k+1$ can be obtained, or we violate the minimality of

$|V(P)|$, or $\{u, v, d, y\}$ is a $K(1,3)$. If $d \in V(C) - \{a^-, a, a^+, a^{++}\}$, then $a^{++}d^+ \in E(G)$ (otherwise $\{y, d^+, a^{++}, d\}$ induces a $K(1,3)$), and

$$a^{++}d^+(d^+Ca)auyd(dC^{op}a^{++})a^{++}$$

is a cycle of length $k+1$.

Case 2.2. $|V(aCb)||V(bCa)| \neq 0$ and $[|V(aCb)| - 1][|V(bCa)| - 1] = 0$.

Now we distinguish two subcases.

Case 2.2.1. Without loss of generality $|V(aCb)| = 1$ and $|V(bCa)| \geq 2$ (where $V(aCb) = \{a^+\}$ and $b = a^{++}$). If now $|V(P)| = 1$ then, using $a^-a^+ \in E(G)$, a cycle of length $k+1$ can be easily obtained. If $|V(P)| = 2$, then $auva^{++}(a^{++}Ca)a$ is again a cycle of length $k+1$. This leaves us with $|V(P)| = 3$. If there exists an $x \in V(bCa)$ such that $x^-x^+ \in E(G)$, then

$$auyva^{++}(a^{++}Cx^-)x^-x^+(x^+Ca)a$$

is a cycle of length $k+1$. Suppose that $x^-x^+ \notin E(G)$ for every vertex $x \in V(bCa)$. Clearly $dx \notin E(G)$ for every vertex $d \in V(H)$, otherwise, since G is $K(1,3)$-free, a cycle of length $k+1$ can be obtained. Since $ux \notin E(G)$ for all $x \in V(bCa)$, $ua^+ \notin E(G)$, $ub \notin E(G)$ (otherwise a cycle of length $k+1$ can be obtained), and by the diameter condition $ax \in E(G)$ for all $x \in V(bCa)$, so $\{u, x^-, x^+, a\}$ induces a $K(1,3)$.

Case 2.2.2. $|V(aCb)| = |V(bCa)| = 1$ (where $V(aCb) = \{a^+\}$, $V(bCa) = \{b^+\}$ and $b = a^{++}$). If $|V(P)| = 1$ then $a^+b^+ \in E(G)$ because G is $K(1,3)$-free, and so a 5-cycle can be easily obtained. If $|V(P)| = 2$, obviously a 5-cycle can be obtained.We are left with $|V(P)| = 3$. Clearly $a^+b^+ \in E(G)$, therefore $yb^+ \notin E(G)$, otherwise a 5-cycle can be obtained. Using the diameter condition, let $d \in N(y) \cap N(b^+)$. If $d \in V(P) \cup V(C)$, then in all possible cases a 5-cycle can be obtained. If $d \notin V(P) \cup V(C)$, then $\{y, v, b, b^+, d\}$ induces a 5-cycle.

Case 2.3. $|V(aCb)||V(bCa)| = 0$ and $[|V(aCb)| - 1][|V(bCa)| - 1] = 0$. If $|V(P)| = 1$ or $|V(P)| = 2$, we have got a 4-cycle, so consider $|V(P)| = 3$. Without loss of generality, let $|V(aCb)| = 0$ and $|V(bCa)| = 1$ (where $b = a^+$, $V(C) = \{a, b, a^-\}$ and $V(P) = \{u, y, v\}$). Since $a^-y \notin E(G)$ (otherwise we would already have a 4-cycle) and $ua^- \notin E(G)$, $va^- \notin E(G)$, $ya^+ \notin E(G)$, $ya \notin E(G)$ and $\operatorname{diam} G \leq 2$, there exists a vertex $d \notin V(C) \cup V(P)$, such that $dy \in E(G)$ and $da^- \in E(G)$, and then either $\{d, u, v, y\}$ induces a $K(1,3)$, or a 4-cycle can be easily obtained.

So in every possible case we get a contradiction, and the proof of the theorem is complete.

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