Isomorphisms Involving Reversing Arcs of Digraphs

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Abstract

A digraph D is reversible if it is isomorphic to the digraph obtained by reversing all arcs of D. A digraph is subreversible if adding any arc between two non-adjacent vertices results in a reversible digraph. We characterize all subreversible digraphs which do not contain cycles of length 3 or 4.

1 Preliminaries

In this paper we consider only connected digraphs. A digraph D is called a simple digraph if for each pair of vertices there is at most one arc between them and there are no loops. A digraph D is called a semi-simple digraph if for each pair of vertices there is at most one arc in each direction, and there are no loops. Let $D = (V(D), E(D))$ be a digraph, where $V(D)$ is the set of vertices of D and $E(D)$ is the set of arcs of D. An arc from a vertex u to a vertex v is denoted by $\overrightarrow{uv}$. If there is an arc between a vertex u and a vertex v, but the direction is not specified, we use the notation $\{u, v\}$ for convenience. A digraph D is called a *double digraph* if $\overrightarrow{uv} \in E(D)$ implies $\overrightarrow{vu} \in E(D)$ for each pair of vertices u and v.

A directed path $P = v_1v_2 \dots v_k$ is a sequence of distinct vertices with arcs $\overrightarrow{v_iv_{i+1}} \in E(D)$ for each $i = 1, 2, \dots, k-1$. A path $P = v_1v_2 \dots v_k$ is a sequence of distinct vertices with arcs $\{v_i, v_{i+1}\} \in E(D)$ for each $i = 1, 2, \dots, k-1$. A directed cycle $C = v_1v_2 \dots v_kv_{k+1}$ is a sequence with arcs $\overrightarrow{v_iv_{i+1}} \in E(D)$ for each i, where $v_{k+1} = v_1$ and no other vertex is repeated. A cycle $C = v_1v_2 \dots v_kv_{k+1}$ is a sequence with arcs $\{v_i, v_{i+1}\} \in E(D)$ for each i, where $v_{k+1} = v_1$ and no other vertex is repeated. The girth (always

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greater than 2) of a digraph is the length of a shortest cycle (except for cycles of length 1 or 2) of this digraph. If there is no cycle of length greater than 2, then the girth of the digraph is defined to be infinity.

The *reverse* of a digraph $D = (V, E)$, denoted by $\widetilde{D}$, is the digraph obtained from D by reversing the direction of every arc of D. A digraph D is *reversible* if D is isomorphic to $\widetilde{D}$. Reversible digraphs arise in a number of situations, see for example [3] and [4]. A digraph D is said to be *subreversible* (SR for short) if for every pair of distinct vertices u and v, where neither $\overrightarrow{uv}$ nor $\overrightarrow{vu}$ is an arc, $D \cup \{\overrightarrow{uv}\}$ is reversible. In this paper, we characterize all SR digraphs whose girths are at least 5.

A digraph (graph) is *vertex-transitive*, transitive for short, if its automorphism group acts transitively on the set of vertices. A digraph (graph) is called *swap-transitive* if, for each pair of vertices u and v, there exists an automorphism which exchanges u and v. A digraph (graph) is called *weak-swap-transitive* if for each pair of vertices u and v not adjacent to each other there exists an automorphism which exchanges u and v. It is easy to see that swap-transitivity is stronger than transitivity. Vertex-transitive digraphs (graphs) have been studied to some extent, see for example [1] and [2].

The *underlying graph* of a digraph D is the undirected graph obtained from D by changing each arc to an edge and each directed loop to a loop. The *primitive* of a graph G (possibly with multiple edges and loops) is the simple graph obtained from G by truncating the loops and replacing each set of parallel edges by a single edge. The primitive of a digraph D is the primitive of its underlying graph. It is easy to see that a semi-simple double digraph whose primitive is a tree is weak-swap-transitive if and only if its primitive is a star. Further, any digraph whose primitive is a complete graph is clearly SR because we cannot add any arc.

The SR digraphs with less than four vertices are easily determined. We concentrate on digraphs with more than three vertices.

Theorem 1 *Let D be a digraph with girth at least* 5 *and more than three vertices. Then D is SR if and only if D is one of the following:*

(1) a directed cycle,

(2) a directed cycle with a loop on each vertex,

(3) a double and weak-swap-transitive digraph without loops,

(4) a double and weak-swap-transitive digraph with loops in the following way: if the primitive of D is not a star (thus must contain a cycle), then D has a loop on each vertex; and if the primitive of D is a star, then D can have loops on each end-vertex (the center vertex may or may not have a loop).

Corollary 1 *Let D be a semi-simple digraph with girth at least* 5 *and more* *than three vertices. Then D is SR if and only if D is either a directed cycle or a weak-swap-transitive double digraph.*

Corollary 2 *Let D be a simple digraph with girth at least 5 and more than three vertices. Then D is SR if and only if D is a directed cycle.*

2 Proof of the Theorem

The sufficiency in theorem 1 is easy to check. The rest of the paper is devoted to the proof of the necessity. Throughout this section, we only consider digraphs with more than three vertices.

Lemma 1 *If a digraph D with girth at least five is SR, then D is either simple (possibly with loops) or double (possibly with loops).*

Proof: Suppose to the contrary. Then there are three vertices x, y, z such that both $\overrightarrow{xy} \in D$ and $\overrightarrow{yx} \in D$, and either $\overrightarrow{yz} \in D$ or $\overrightarrow{zy} \in D$ but not both. Without loss of generality, we assume that $\overrightarrow{yz} \in D$ and $\overrightarrow{zy} \notin D$. Add the arc $\overrightarrow{xz}$ to D and let Φ be an isomorphism between $D \cup \{\overrightarrow{xz}\}$ and its reverse. Since xyz is the unique triangle in the primitive of $D \cup \{\overrightarrow{xz}\}$ (because the girth of D is at least 5), Φ maps the triangle xyz to itself. However, the subgraph of $D \cup \{\overrightarrow{xz}\}$ induced by $\{x, y, z\}$ is not isomorphic to its reverse, a contradiction. □

Lemma 2 *If a digraph D with girth at least five is SR and its primitive is a tree, then D is double and the primitive of D is a star, and either every end-vertex has a loop or no end-vertex has a loop.*

Proof: First we claim that the primitive graph of D is not a path. Otherwise, say $D = v_1v_2v_3....$. After adding one arc $\overrightarrow{v_1v_3}$, vertex v_3 is the only vertex of degree three if D is simple, and the only vertex of degree 5 if D is double. Then vertex v_3 can only be mapped to itself. But it is impossible because the indegree of v_3 is not equal to the outdegree of v_3. Thus the primitive of D is not a path.

If we can prove that D is not simple, then D must be double by Lemma 4. Assume that D is simple. Let E be the set of vertices of degree 1. Since D is not a path, we know that $|E| > 2$ (and no two vertices of E are adjacent). Let E^+ be the vertices of E that have outdegree 1, and let E^- be the vertices of E that have indegree 1. We may assume $|E^+| \geq |E^-|$. If $E^- \neq \phi$, choose a vertex $x \in E^-$; otherwise, let $x \in E^+$. If $E^- - \{x\} \neq \phi$, choose a vertex $y \in E^- - \{x\}$; otherwise, let $y \in E^+ - \{x\}$. It is clear from the choice of x and y that $|E^+ - \{x, y\}| > |E^- - \{x, y\}|$, so that $D \cup \{\overrightarrow{xy}\}$ is not reversible, a contradiction. Thus, D must be a double digraph.

Because D is double and SR, Proposition 7 below implies that that D is weak-swap-transitive. But if the primitive of D is not a star, then there is a vertex x of degree 1 and a vertex y of that is not of degree 1, such that x is not adjacent to y. Contradiction. Again using the hypothesis that D is SR, we can show that either all v_i have a loop or none of them have a loop. □

Lemma 3 *If a digraph D with girth at least five is SR and its primitive is not a tree, then either no vertex of D has a loop or every vertex of D has a loop.*

Proof: Suppose, to the contrary, that D has some vertices with loops and others without loops. Since the girth of D is at least 5, for any path xyz in D, there is no arc between x and z. $D \cup \{\overrightarrow{xz}\}$ or $D \cup \{\overrightarrow{zx}\}$ has a unique cycle of length 3. This unique cycle should be mapped to itself in any isomorphism between $D \cup \{x, z\}$ and its reversal. By properly choosing the direction of the new arc between x and z, we can show that either both x and z have a loop, or neither x nor z have a loop. Since D is connected, the vertex set of D can be partitioned into two nonempty parts X and Y such that every vertex in X has a loop and no vertex in Y has a loop. The above observation implies that if $x \in X$, then $z \in X$ iff there is an even path from x to z, and $z \in Y$ iff there is an odd path from x to z. In particular, there is no arc within X (or Y), so that the primitive of D is a bipartite graph. Since the primitive of D is not a star, each of X and Y has more than one vertex. Then, because the girth of D is at least 5, there must be some $x \in X$ and $y \in Y$, such that x is not adjacent to y. We may assume that there are at least as many directed arcs from X to Y as there are from Y to X. Then $D \cup \{\overrightarrow{xy}\}$ has strictly more directed arcs from X to Y than it does from Y to X, so it is not reversible, contradicting subreversibility. □

In the following propositions, we let D be a SR digraph with girth at least five. By Lemma 1, we only need to consider either simple digraphs or double digraphs. If the primitive of D is a tree, then its primitive must be a star by Lemma 2, and either every end-vertex has a loop, or every end-vertex has no loop. In light of Lemma 2, we only need to consider digraphs whose primitives contain cycles. By Lemma 3, we only need to consider digraphs without loops.

Proposition 1 *If a digraph D with girth at least five is SR and double, then D is weak-swap-transitive.*

Proof: Pick two nonadjacent vertices u and v in D. Since $D \cup \{\overrightarrow{uv}\}$ is reversible, there is an isomorphism Φ from $D \cup \{\overrightarrow{uv}\}$ to its reversal. Since u and v are the only two vertices incident to a single arc, we have

$\Phi(u) = v, \Phi(v) = u$. Hence, Φ is also an automorphism of D, which exchanges u with v. Therefore, D is a weak-swap-transitive digraph. □

Proposition 2 *If C is a cycle of shortest length in a SR simple digraph D with girth at least five, then $V(C)$ induces a directed cycle.*

Proof: Let C be a shortest cycle in D, and let c be the length of C. If C is not a directed cycle, then there exist three vertices x, y, z on C with arcs $\overrightarrow{yx}$ and $\overrightarrow{yz}$. Let the part of the cycle C from x to z (the part not including y) be denoted by $C[x, z]$. Add arc $\overrightarrow{xz}$ to D. Then xyz is the unique cycle of length 3 in $D \cup \{\overrightarrow{xz}\}$. Therefore, the isomorphism Φ from $D \cup \{\overrightarrow{xz}\}$ to its reverse must keep x fixed and exchange y with z. For clarity, we denote the vertex v of the reverse by v^*. Φ should map $C[x, z]$ to some path P from x^* to y^* with the same length $(c-2)$ that does not use vertex z^* (as $z^* = \Phi(y)$). Thus, $P \cup \{x^*, y^*\}$ is a cycle of length $c-1$ in the reverse of $D \cup \{\overrightarrow{xz}\}$ that does not use arc $\overrightarrow{z^*x^*}$. Now the reverse of $D \cup \{\overrightarrow{xz}\}$ is equal to $\tilde{D})\cup\{\overrightarrow{z^*x^*}\}$. Hence, $P \cup \{x^*, y^*\}$ is a cycle of length $c-1$. But this means that the image of $P \cup \{x^*, y^*\}$ under Φ^{-1} is a cycle of length $c-1$ in D, a contradiction. □

Proposition 3 *The intersection of two shortest cycles in a SR simple digraph with girth at least five is either empty or a path (one single vertex will be treated as a path).*

Proof: Let B and C be two shortest cycles in D. Then both B and C are directed cycles by Proposition 2. Suppose, to the contrary, that the intersection of B and C contains two disjoint maximal (directed) paths, say $P_1 = x_1 \ldots y_1$ and $P_2 = y_2 \ldots x_2$. Then, $E(B) \cup E(C) - E(P_1 \cup P_2)$ induces at least two cycles, say C_1 and C_2, neither of which is a directed cycle. We may assume that C_1 is in $B[x_2, x_1] \cup C[x_2, x_1]$, while C_2 is in $B[y_1, y_2] \cup C[y_1, y_2]$, (where $F[u, v]$ denotes a directed subpath of F from u to v). Since a common arc of C_1 and C_2 is necessarily a common arc of B and C, one of the cycles C_1 and C_2 is a shortest cycle in D. However, this shortest cycle does not induce a directed cycle in D, contradicting Proposition 8. □

Proposition 4 *A SR simple digraph with girth at least five must be a directed cycle.*

Proof: Let C be a shortest cycle of a *SR* simple digraph D with girth at least five. By Proposition 8, C is a directed cycle. Suppose to the contrary that D has more vertices than $V(C)$. Then we have at least one more vertex, say z, adjacent to a vertex, say x, on C. (Recall that D is connected.) Without loss of generality, we assume $\overrightarrow{zx} \in D$. Let y be the vertex on C such that $\overrightarrow{yx} \in E(C)$. We add arc $\overrightarrow{zy}$ to D. Let Φ be the isomorphism from $D \cup \{\overrightarrow{zy}\}$ to its reverse. Then $\Phi(x) = z^*$, $\Phi(y) = y^*$, and $\Phi(z) = x^*$, since xzy is the unique cycle of length 3 in D and only the vertex y has both indegree and outdegree 1 in this cycle (recall that D has girth at least 5). Let the directed path on C from the vertex x to the vertex y be denoted by $C[x, y]$ and let $\Phi(C) = B^*$. Then $\Phi(C[x, y]) = B^*[z^*, y^*]$ where $B^*[z^*, y^*]$ is a directed path from z^* to y^* in the reverse of $D \cup \{\overrightarrow{zy}\}$ with only one common vertex y^* with C^* by Proposition 3. Then $B[y, z]$ is a directed path from y to z in $D \cup \{\overrightarrow{zy}\}$ with only one common vertex y with C.

We next add arc $\overrightarrow{yz}$ to D. Let Θ be the isomorphism from $D \cup \{\overrightarrow{yz}\}$ to its reverse. Then

$$\Theta(x) = y^*, \ \Theta(y) = x^*, \text{ and } \Theta(z) = z^*$$

for the same reasons stated above. Let $\Theta(B[y, z]) = T^*[x^*, z^*]$. Then $T[z, x]$ is a directed path from z to x in D. Notice that the lengths of both cycles and paths are graph invariants under isomorphism. Now we have a cycle $T[z, x] \cup \{\overrightarrow{zx}\}$ of shortest length which is not a directed cycle, a contradiction. □

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