Concerning the Number of Edges in a Graph with Diameter Constraints

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Abstract

We study problems related to the number of edges of a graph with diameter constraints. We show that the problem of finding, in a graph of diameter $k \geq 2$, a spanning subgraph of diameter k with the minimum number of edges is NP-hard. In addition, we propose some efficient heuristic algorithms for solving this problem. We also investigate the number of edges in a critical graph of diameter 2. We collect some evidence which supports our conjecture that the number of edges in a critical graph of diameter 2 is at most $\Delta(n-\Delta)$ where Δ is the maximum degree. In particular, we show that our conjecture is true for $\Delta \leq \frac{1}{2}n$ or $\Delta \geq n-5$.

1 Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *size* of G, denoted by $\varepsilon(G)$, is the number of edges contained in G. In general, each edge e of G is associated with a non-negative integer $l(e)$ called the *length* of e. For a path P of G joining two vertices x and y, the length $l(P)$ of P is the sum of the lengths of the edges of P. Thus the *distance* between two vertices x and y of G is defined as:

$$d(x,y) = min\{l(P) : \ P \ is \ a \ path \ joining \ x \ and \ y\}$$

The *diameter* of G is defined as:

$$d(G) = max\{d(x,y) : \ x,y \in V(G)\}.$$

JCMCC 36 (2001), pp. 161-173

When $l(e) = 1$ for each $e \in E(G)$, the distance between two vertices x and y is just the number of edges contained in a shortest path connecting x and y. Hence the diameter is the maximum distance between all pairs of vertices.

The diameter of a graph is an important graph theoretic parameter with considerable applications, for instance, in communic4)network which satisfies certain requirements and which is optimal according to some criterion such as cost, output, or performance. The review by Caccetta [2] contains an excellent account of such work. The problem of characterizing graphs with a prescribed diameter is very much unresolved. Indeed, not even graphs of diameter 2 have been completely characterized. In characterizing graphs with a prescribed diameter, it is fruitful to consider a subclass of graphs, the so called critical graphs which are defined below.

Suppose that G is a graph of diameter k, i.e., $d(G) = k$. An edge e of G is *critical* if $d(G - e) > k$; otherwise e is *non-critical.* If every edge of G is critical, then G is called a *critical* graph of diameter k. Critical graphs of diameter k with $l(e) = 1$ can be recognized in time $O(n^2m)$, where $|V(G)| = n$ and $|E(G)| = m$, see [1].

Suppose that each edge e of G is associated with another non-negative integer $c(e)$ called the *cost* of e. For a subgraph G' of G, the cost $c(G')$ of G' is defined to be the sum of all the costs of edges of G'. When $c(e) = 1$ for each $e \in E(G)$, the cost of a subgraph G' of G is simply the size of G' and thus the problem of finding a minimum cost network corresponds to finding a subgraph of minimum size (and having the prescribed properties).

In this paper, we shall mainly consider graphs with $l(e) = c(e) = 1$ for each edge e. We study some problems related to the number of edges in a graph with diameter constraints. These constraints include prescribed diameter and criticality of edges. We show that the problem of finding, in a graph of diameter $k \geq 2$, a spanning subgraph of diameter k with minimum number of edges is NP-hard. In addition, we detail several efficient heuristic algorithms for solving this problem. We test these heuristics on 100 random graphs. Our study of critical graphs of diameter 2 leads us to conjecture that the number of edges in such a graph is bounded above by $\Delta(n - \Delta)$ where Δ is the maximum degree. We prove several results which verify this conjecture for all $\Delta \leq \frac{1}{2}n$ or $\Delta \geq n - 5$.

2 Some NP-hard Problems

The following practical problem considered by Plesnik [10] arises from network designs.

Problem 2.1 *Given a graph G and an integer B, find a spanning subgraph H of G with cost $c(H) \leq B$ and with minimum diameter $d(H)$.*

Plesnik [10] showed that the above problem is NP-hard. Given integers D and B, we can check in polynomial time whether a graph G satisfies $d(G) \leq D$ and $c(G) \leq B$. Hence the following problem is NP-complete.

Problem 2.2 *Given a graph G and integers D and B, does there exist a spanning subgraph H of G with $d(H) \leq D$ and $c(H) \leq B$.*

In the case when $l(e) = c(e) = 1$ for each edge e of G, Problem 2.2 in fact asks whether a given graph contains a spanning subgraph which has diameter $\leq D$ and at most B edges. Clearly, a spanning subgraph H of G with diameter $\leq D$ and with minimum number of edges must be critical. Suppose that G has diameter k. A spanning critical subgraph of G of diameter k can be obtained through the following procedure: While G has diameter k and is not critical, delete non-critical edges. Hence every graph G contains a spanning critical subgraph H of diameter $d(G)$.

However deleting non-critical edges in different orders may result in graphs of different sizes. This is due to the fact that a graph can contain two spanning critical subgraphs of different sizes. For instance, it is easily seen that $K_4 - e$ contains spanning critical subgraphs of diameter 2 of sizes 3 and 4. Figure 1 shows a graph G of diameter 2 which contains two spanning critical subgraphs H_1 and H_2 of sizes $\frac{2}{9}(n^2 + n - 2)$ and $2n - 4$ respectively. Thus it is natural to propose the following problem:

Problem 2.3 *Let $k \geq 2$ be an integer. Given a graph G of diameter k, find a spanning graph H of G of diameter k of minimum size.*

Chung and Garey [4] proposed the following problem.

Problem 2.4 *Given a graph G and integers t and k. Determine whether there exists a subgraph of G obtained by deleting t edges that has diameter no more than k, and, if so, find such a subgraph.*

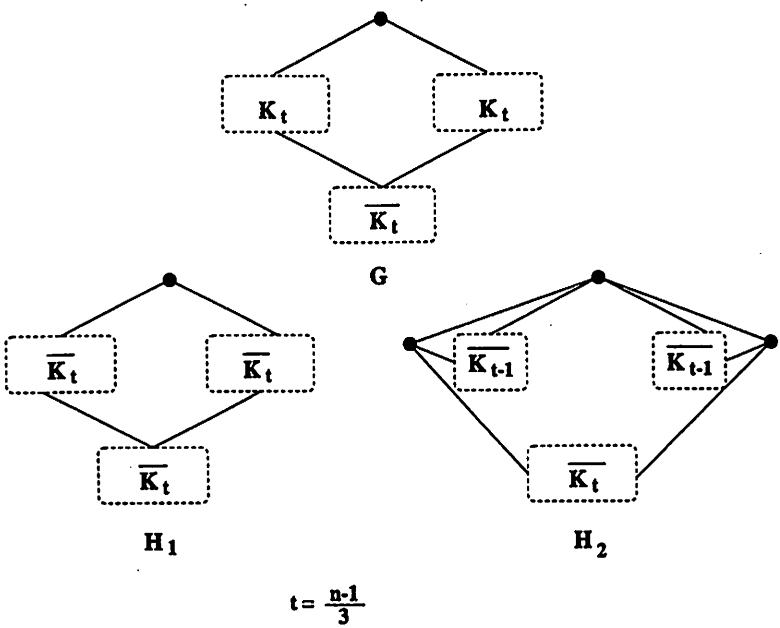

Figure 1:

If Problem 2.4 is polynomial time solvable, then so is Problem 2.3. Our next result shows both of these problems are difficult.

Theorem 2.5 *Problem 2.3 is NP-hard.*

Proof: We reduce the minimum dominating set problem (which is NP-hard, cf. [7]) to our problem. Let H be any graph. We construct a graph of diameter k as shown in Figure 2. Note that the vertices a_{k-1}, b_{k-1} and c_{k-1} are connected to all the vertices in H.

We shall show that every minimum dominating set C of H gives rise to a spanning critical subgraph S_C of G of diameter k of minimum size and conversely, every spanning critical subgraph of G of diameter k of minimum size gives rise to a minimum dominating set C of H.

Given a dominating set C ($\neq V(H)$) of H, we define a spanning subgraph S_C of G as follows: The subgraph S_C contains all edges of G incident with u, v, a_i, b_i, c_i, d_i ($i = 1, 2, \ldots, k-2$), and c_{k-1}, all edges of G between C and $\{a_{k-1}, b_{k-1}\}$, exactly one edge from each $x \in V(H) - C$ to C, and no other edge of G. It is easy to check that the spanning subgraph S_C is critical and the number of edges in S_C is

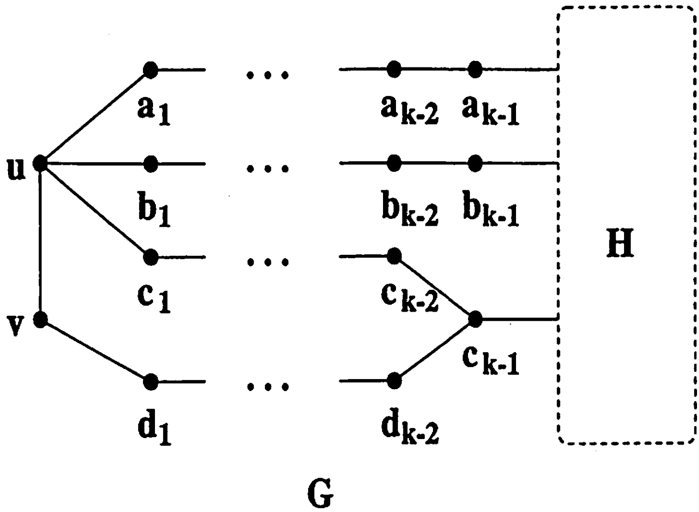

Figure 2:

$$\varepsilon(S_C) = |C| + 2|V(H)| + 4(k-1) + 1. \qquad (1)$$

Consider a spanning critical subgraph S of G. We show that there exists a dominating set C' with $\varepsilon(S) \geq \varepsilon(S_{C'})$. Clearly S must contain all edges of G incident with u, v, a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, k-2)$, and c_{k-1}. Let

$$X = \{v \in V(H) : va_{k-1} \in E(S), vb_{k-1} \notin E(S)\},$$

$$Y = \{v \in V(H) : va_{k-1} \notin E(S), vb_{k-1} \in E(S)\},$$

and

$$Z = \{v \in V(H) : va_{k-1} \in E(S), vb_{k-1} \in E(S)\}.$$

In the subgraph of S induced by $V(H)$, each vertex of $V(H) - X - Y - Z$ is incident with either exactly one edge to Z or exactly two edges to $X \cup Y$. In the latter case one edge is to X and the other is to Y. Let A denote the set of all vertices of $V(H) - X - Y - Z$ which are incident with exactly one edge to Z and B the set of all vertices which are incident with exactly two edges to $X \cup Y$. Then A, B, X, Y, Z form a partition of $V(H)$. Without loss of generality, assume that $|X| \leq |Y|$. The number of edges in S is

$$\varepsilon(S) \geq |V(H)| + 2|X| + |Y| + 2|Z| + |A| + 2|B| + 4(k-1) + 1.$$

Note that each vertex in Y (X) must have at least one neighbour in $X \cup Z$ ($Y \cup Z$). It is easy to see that $X \cup Z = C'$ is a dominating set of H. Hence, by (1), the number of edges in $S_{C'}$ is

$$\begin{aligned}\varepsilon(S_{C'}) &= |C'| + 2|V(H)| + 4(k-1) + 1 \\ &= |V(H)| + 2|X| + |Y| + 2|Z| + |A| + |B| + 4(k-1) + 1 \\ &\leq \varepsilon(S).\end{aligned}$$

Therefore if C_{min} is a minimum dominating set of H, then $S_{C_{min}}$ (which can be constructed in polynomial time) is a spanning critical graph of G of minimum size. This completes the proof. □

Corollary 2.6 *Problems 2.2 and 2.4 are NP-complete.* □

Note that the graph G constructed in the proof of Theorem 2.5 has maximum degree $n - 4k + 5$. So we have in fact proved that Problem 2.3 is NP-hard even if we assume that the input graph has maximum degree $n - 4k + 5$. When $k = 2$, Problem 2.3 is polynomially solvable assuming that the input graph has maximum degree $> n - 4k + 5 = n - 3$.

3 Critical Graphs of Diameter 2

In this section, we consider only unweighted graphs G of diameter 2.

Plesnik [9] observed that all known critical graphs of diameter 2 have no more than $\lfloor \frac{1}{4}n^2 \rfloor$ edges, and that the complete bipartite graph $K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$ is a critical graph of diameter 2 which has exactly $\lfloor \frac{1}{4}n^2 \rfloor$ edges. Simon and Murty stated the following (cf. [3]):

Conjecture 3.1 *Let G be a critical graph of diameter 2 . Then*

$$\varepsilon(G) \leq \lfloor \frac{1}{4}n^2 \rfloor,$$

and equality holds if and only if $G \cong K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$.

Caccetta and Häggkvist [3] proved that every critical graph contains at most $.27n^2$ edges. With more careful analysis and computation, Fan [5] showed that such a graph contains less than $.2532n^2$ edges. From a completely different approach, Füredi [6] proved that there exists an integer n_0 such that the conjecture is true when G contains at least n_0 vertices. The integer n_0 provided by Füredi is huge (a tower of 2's of height about 10^{14}).

Conjecture 3.1 is believed to be correct. But the complete establishment of it seems to be difficult. We believe the following stronger conjecture holds.

Conjecture 3.2 *Let G be a graph of diameter 2 and maximum degree Δ and let v be a vertex of degree Δ. Suppose that each edge of G not incident with v is critical. Then*

$$\varepsilon(G) \leq \Delta(n-\Delta).$$

Moreover, if $\Delta \neq n-2$, then $\varepsilon(G) = \Delta(n-\Delta)$ if and only if $G \cong K_{\Delta,n-\Delta}$.

Note that in the above conjecture the edges incident with v are not assumed to be critical. Also note that there are critical graphs of diameter 2 and maximum degree $n-2$ which have $2(n-2)$ edges, yet they are not isomorphic to $K_{2,n-2}$, see Figure 3. So the assumption '$\Delta \neq n-2$' is neccessary.

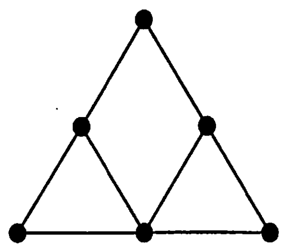

Figure 3:

When $\Delta(G) \leq \frac{1}{2}n$, we have $\varepsilon(G) \leq \frac{1}{2}n\Delta \leq \Delta(n-\Delta)$. Thus the first part of Conjecture 3.2 is true. In the sequel, we shall show that the first part of Conjecture 3.2 is true when $\Delta \geq n-5$.

Let x, y, z be three vertices of G. Then (x, y, z) is called an *extremal triple* if $x \sim y \sim z$ is the only path of length ≤ 2 connecting x and z. Note that if (x, y, z) is an extremal triple then x is not adjacent to z.

For each vertex v of G, let $L_1(v)$ (resp. $L_2(v)$) denote the set of all vertices which have distance 1 (resp. 2) from v. We shall use the following notation.

$$S = \{xy \in E(G) : \ x, \ y \in L_1(v) \text{ or } x, \ y \ \in L_2(v)\}$$

and

$$T = \{ab \notin \ E(G) : \ a \in L_1(v) \text{ and } b \in L_2(v)\}.$$

Lemma 3.3 *Let G be a graph of diameter 2 and let v be a vertex of degree Δ. If $|S| \leq |T|$ (where S and T are defined as above), then*

$$\varepsilon(G) \leq \Delta(n-\Delta).$$

Proof: Clearly, we have

$$\varepsilon(G) = \Delta + |S| + \Delta(n - \Delta - 1) - |T| \leq \Delta(n - \Delta).$$

□

Lemma 3.4 *Let G be a graph of diameter 2 and let v be a vertex of degree Δ. Suppose that each edge not incident with v is critical and suppose that, for each edge xy where $x, y \in L_2(v)$, there exists a vertex $z \in L_1(v)$ such that (x, y, z) is an extremal triple. Then*

$$\varepsilon(G) \leq \Delta(n - \Delta).$$

Proof: Consider an element $xy \in S$. If $x, y \in L_2(v)$, then by assumption there exists a vertex $z \in L_1(v)$ such that (x, y, z) is an extremal triple. If $x, y \in L_1(v)$, then the criticality of xy assures that there exists a vertex $z \in L_2(v)$ such that (x, y, z) is an extremal triple. Thus $f : xy \mapsto xz$ defines a mapping from S to T. It is easy to check it is an injection. Hence we have $|S| \leq |T|$ and, by Lemma 3.3, $\varepsilon(G) \leq \Delta(n - \Delta)$. □

Corollary 3.5 *Let G be a graph of diameter 2 and let v be a vertex of degree Δ. Suppose that each edge not incident with v is critical and that there exists a vertex $u \in L_1(v)$ which is adjacent to all vertices in $L_2(v)$. Then*

$$\varepsilon(G) \leq \Delta(n - \Delta).$$

Proof: It is easy to verify that G satisfies the assumption of Lemma 3.4. □

Lemma 3.6 *Let G be a graph of diameter 2 and let v be a vertex of degree Δ. Suppose that each edge not incident with v is critical and suppose that $L_2(v)$ contains no induced path of length 2, that is, $L_2(v)$ is a union of disjoint (possibly one) cliques. Then*

$$\varepsilon(G) \leq \Delta(n - \Delta).$$

Proof: We establish an injection f from S to T: let f be defined the same as in the proof of Lemma 3.4, except for those $xy \in S$ with $x, y \in L_2(v)$ such that there is no $z \in L_1(v)$ such that (x, y, z) is an extremal triple. By the assumption, xy is in some clique C contained in $L_2(v)$. The criticality of xy assures that x and y are the only vertices in C and $x \sim y$ is the only path ≤ 2 joining x and y. Let $x' \in L_1(v)$ be any vertex adjacent to x. Then $x' \not\sim y$ and we define $f : xy \mapsto x'y$. Now f is an injection from S to T and hence, by Lemma 3.3, $\varepsilon(G) \leq \Delta(n - \Delta)$. □

Lemma 3.7 *Let G be a graph of diameter 2 and let v be a vertex of degree Δ. Suppose that each edge not incident with v is critical and that $L_2(v)$ induces a connected subgraph of G with at most $n - \Delta - 1$ edges. Then*

$$\varepsilon(G) \le \Delta(n - \Delta).$$

Proof: For $i = 0, 1, \ldots, n - \Delta - 1$, let

$$S_i = \{x \in L_1(v) : \textit{ x is adjacent to precisely i vertices in } L_2(v)\},$$

and write $s_i = |S_i|$. Then $L_1(v)$ is partitioned into $S_0, S_1, \ldots, S_{n-\Delta-1}$. If $S_{n-\Delta-1} \ne \emptyset$, then, by Corollary 3.5, we have $\varepsilon(G) \le \Delta(n - \Delta)$. So assume that $S_{n-\Delta-1} = \emptyset$. Consider a vertex $x \in S_i$ where $i \ne n - \Delta - 1$. Let T_x consists of all vertices $z \in L_2(v)$ such that (x, y, z) forms an extremal triple for some vertex $y \in L_1(v)$. Since x is adjacent to precisely i vertices in $L_2(v)$ which induces a connected subgraph, $|T_x| \le n - \Delta - i - 2$ if $i \ge 1$ and $|T_x| \le n - \Delta - 1$ if $i = 0$. Since each edge xy of $L_1(v)$ is associated with at least one extremal triple (x, y, z) with $z \in L_2(v)$, the number of edges contained in $L_1(v)$ is at most $(n - \Delta - 1)s_0 + \sum_{i=1}^{n-\Delta-2}(n - \Delta - i - 2)s_i$. Thus we have

$$\begin{aligned}
\varepsilon(G - v) &\le (n - \Delta - 1)s_0 + (n - \Delta - 2)\textstyle\sum_{i=1}^{n-\Delta-2} s_i + n - \Delta - 1 \\
&= (n - \Delta - 1)\textstyle\sum_{i=0}^{n-\Delta-2} s_i + n - \Delta - 1 - \sum_{i=1}^{n-\Delta-2} s_i \\
&= (n - \Delta - 1)\Delta + n - \Delta - 1 - \textstyle\sum_{i=1}^{n-\Delta-2} s_i.
\end{aligned}$$

If $\sum_{i=1}^{n-\Delta-2} s_i \ge n - \Delta - 1$, then we have $\varepsilon(G) \le \Delta(n - \Delta)$. So assume that $\sum_{i=1}^{n-\Delta-2} s_i < n - \Delta - 1$. We may also assume, according to an earlier remark, that $\Delta > \frac{1}{2}n$. Thus $s_0 \ge 2$. Since no two vertices of S_0 are adjacent, each vertex of S_0 is adjacent to at most $\sum_{i=1}^{n-\Delta-2} s_i$ vertices in $L_1(v)$. Hence the number of edges contained in $L_1(v)$ is at most $s_0 \sum_{i=1}^{n-\Delta-2} s_i + \sum_{i=1}^{n-\Delta-2}(n - \Delta - i - 2)s_i$. Therefore we have

$$\begin{aligned}
\varepsilon(G - v) &\le s_0 \textstyle\sum_{i=1}^{n-\Delta-2} s_i + (n - \Delta - 2)\sum_{i=1}^{n-\Delta-2} s_i + n - \Delta - 1 \\
&= s_0 \textstyle\sum_{i=1}^{n-\Delta-2} s_i + (n - \Delta - 1)\sum_{i=1}^{n-\Delta-2} s_i - \sum_{i=1}^{n-\Delta-2} s_i + n - \Delta - 1 \\
&= \Delta(n - \Delta - 1) + (1 - s_0)(n - \Delta - 1 - \textstyle\sum_{i=1}^{n-\Delta-2} s_i) \\
&< \Delta(n - \Delta - 1).
\end{aligned}$$

□

Lemma 3.8 *Let G be a graph of diameter 2 and let v be a vertex of degree Δ. Let Q_i, $i = 1, 2, \ldots, l$, be the connected components of the subgraph induced by $L_2(v)$ and $n_i = |V(Q_i)|$ for each i. Suppose that each* *edge not incident with v is critical and each Q_i contains at most n_i edges. Then*

$$\varepsilon(G) \leq \Delta(n-\Delta).$$

Proof: For each $i = 1, 2, \ldots, l$, let G_i be the subgraph of G induced by $\{v\} \cup L_1(v) \cup V(Q_i)$ and let H_i be the subgraph obtained from G_i by removing all non-critical edges not incident with v. Clearly $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_l)$. Each H_i has diameter 2 and satisfies the assumptions of Lemma 3.7. Hence $\varepsilon(H_i) \leq \Delta(n_i + 1)$. Therefore

$$\begin{aligned}
\varepsilon(G) &\leq \textstyle\sum_{i=1}^{l} \varepsilon(H_i) - (l-1)\Delta \\
&\leq \textstyle\sum_{i=1}^{l} \Delta(n_i + 1) - (l-1)\Delta \\
&= \Delta(\textstyle\sum_{i=1}^{l} n_i + 1) \\
&= \Delta(n - \Delta).
\end{aligned}$$

□

Let G be a graph of diameter 2 and let v be a vertex of degree Δ. Suppose that each edge not incident with v is critical. When $\Delta \geq n-4$, $L_2(v)$ contains at most three vertices. It is easy to see that G satisfies the assumptions of Lemma 3.8. Hence $\varepsilon(G) \leq \Delta(n-\Delta)$. When $\Delta = n-5$, $L_2(v)$ contains precisely four vertices. By lemmas 3.6 and 3.8, we may assume that $L_2(v)$ induces a subgraph with precisely five edges. In this case, for each edge xy in $L_2(v)$ there exists a vertex z in $L_1(v)$ such that (x, y, z) is an extremal triple. Hence the assumption of Lemma 3.4 is satisfied and G contains at most $\Delta(n-\Delta)$ edges. Combining this and an earlier remark, we have proved the following:

Theorem 3.9 *Let G be a graph of diameter 2 and maximum degree Δ with $\Delta \leq \frac{1}{2}n$ or $\Delta \geq n-5$. Let v be a vertex of degree Δ. Suppose that each edge not incident with v is critical. Then*

$$\varepsilon(G) \leq \Delta(n-\Delta).$$

□

4 Some Heuristic Algorithms

In this section we demonstrate some heuristic solutions to Problem 2.3 for $k = 2$. In other words, we would like to find an efficient algorithm for finding, in a given graph G of diameter 2, a spanning subgraph of G of diameter 2 containing as few edges as possible. We implemented the following heuristics on a Silicon Graphics workstation in C:

Algorithm 4.1 *Let G be a graph of diameter 2.*
while *G is non-critical* **do**

1. *Remove a non-critical edge incident with a vertex of smallest degree.*
2. *Update G.*

Algorithm 4.2 *Let G be a graph of diameter 2.*
For *each vertex v* **do***:*

1. *Remove all non-critical edges with both endpoints in $L_1(v)$.*
2. *Remove all non-critical edges with both endpoints in $L_2(v)$.*
3. *Remove all non-critical edges connecting vertices in $L_1(v)$ and vertices in $L_2(v)$.*
4. *Remove all non-critical edges incident with v.*

Among all the n 2-critical subgraphs obtained, choose the one with the smallest number of edges.

Interchanging steps 1 and 2 in Algorithm 4.2 gives a rise to another heuristic which we refer to as **Algorithm 4.3.**

We implemented other algorithms where stages 1-4 above were executed in other possible permutations. Finally, we compared these algorithms to the 'greedy algorithm'.

Algorithm 4.4 [The 'Greedy Algorithm'] *Let G be a graph of diameter 2.*
Sort the edges in G in a random order.
While *G is non-critical, remove a non-critical edge with the smallest index.*

Checking whether an edge $e = (i, j)$ is critical in G can be easily done in $O(n^2)$, by computing rows i and j in the matrix $(A_e + I)^2$ (where A_e denotes the adjacency matrix of $G - e$). Hence algorithms 4.2 and 4.3 are of order $O(mn^3)$.

Algorithm 4.1, however, can be implemented more efficiently: note that a critical edge in G remains critical in any induced subgraph of G containing that edge. Hence every edge is checked only once for criticality. Furthermore, once an edge (i, j) is removed, the degrees of i and j decrease by exactly one, hence finding the next unobserved edge incident with the vertex of smallest degree can be done in $O(1)$. Thus Algorithm 4.1 is of order $O(mn^2)$. Algorithm 4.4 has the same order.

We have applied the heuristic algorithms above to 100 random graphs of orders ranging between 10 and 70 vertices, and with varying maximum degrees. A random graph of diameter 2 was constructed by first generating a random graph G (with a user defined edge probability p). If G has diameter greater than 2, then we arbitrarily add edges to G joining non-adjacent vertices, until a graph of diameter 2 is obtained.

Our computational analysis shows that if the maximum degree in the graph is relatively large (say, between $\frac{3}{4}n$ and $n-2$), then Algorithm 4.2 gives the best results. Furthermore, it is achieved for a vertex v of relatively large degree. If the maximum degree in the graph is relatively small (say, around $\frac{1}{2}n$), then Algorithm 4.3 gives the best results, which were achieved, in this case, for a vertex of small degree. In both cases, Algorithm 4.1 was the second best.

To be more precise, out of 60 random graphs with large maximum degree, in 51 of them Algorithm 4.2 produced subgraphs which were 2-11% smaller than the subgraphs produced by any of the other algorithms. In 5 of them Algorithms 4.2 and 4.3 produced subgraphs of similar sizes and only in a small portion of them (4 graphs) the results of other algorithms were better than those of algorithm 4.2. When 40 random graphs with maximum degrees around $\frac{1}{2}n$ were checked, Algorithm 4.3 produced subgraphs which were 1-3% smaller than the subgraphs produced by other algorithms in 37 of them.

Finally, the first three heuristics gave better results than the 'Greedy Algorithm' in almost all input graphs.

Algorithm 4.1 is more efficient than the other algorithms, except for the 'greedy algorithm' which gives poor results. It also produces results which are in general, within 4% of the best possible. Thus, we believe that Algorithm 4.1 has an advantage over the others.

The intuition behind Algorithm 4.1 lies in Conjecture 3.1. If the conjecture holds, then the largest critical graph of diameter two is the complete bipartite graph $K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$, where $\Delta - \delta$ is at most one. Thus, you would expect that small critical graphs would have the property that $\Delta - \delta$ would be large. (the star $K_{n-1,1}$ is an extreme example of a small critical graph). Algorithm 4.1 attempts to widen the gap between Δ and δ by removing edges from vertices of small degree. By comparison, when we removed edges from vertices of large degree on a sample of 60 random graphs, in 57 of them the resulting critical graphs were 3-20% larger than those produced by Algorithm 4.1.

Acknowledgement

This work has been supported by Australian Research Coucil Grant No. A49532661.

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