On the Nordhaus-Gaddum Problem for the Edge Cost of a Graph

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Abstract. Let G be a simple graph with n vertices, and let $\overline{G}$ denote the complement of G. A well-known theorem of Nordhaus and Gaddum [6] bounds the sum $\chi(G) + \chi(\overline{G})$ and product $\chi(G)\chi(\overline{G})$ of the chromatic numbers of G and its complement in terms of n. The *edge cost ec(G)* of a graph G is a parameter connected with node fault tolerance studies in computer science. Here we obtain bounds for the sum and product of the edge cost of a graph and its complement, analogous to the theorem of Nordhaus and Gaddum.

1. Introduction.

In this paper we consider only simple graphs, that is finite, undirected graphs having no loops or multiple edges. In general we follow the notation of Bondy and Murty [2] except that the graph $G = (V, E)$ has n vertices (or nodes) and m edges. In 1956 Nordhaus and Gaddum [6] established that the chromatic numbers of a graph G with n vertices (or nodes) and its complement $\overline{G}$ satisfy the inequalities

$$2\sqrt{n} \leqslant \chi(G) + \chi(\overline{G}) \leqslant n + 1 \text{ and } n \leqslant \chi(G)\chi(\overline{G}) \leqslant \frac{(n+1)^2}{4}.$$

Analogous results have since been obtained for many other graph-theoretic parameters, see Chartrand and Mitchem [3]. Most recently Achuthan, Achuthan and Simanihuruk [1] have obtained a result for the n-path-chromatic number of a graph.

Graph theoretic models are frequently used for node fault tolerance studies in computer networks, see for example [4,5]. Starting with an undirected graph G on n nodes we augment it by introducing an additional node w and extra edges to form an augmented graph G^+. We say that the system is *(one) node fault tolerant* if after removing any node from G^+ the resulting graph contains a graph isomorphic to G. We take $ec(G)$, the *edge cost* of G, to be the minimum number of extra edges that must be added in the augmented graph G^+ in order to ensure node fault tolerance.

Clearly we can ensure node fault tolerance by joining every node of G to w, so $ec(G) \leqslant n$. Let Δ denote the maximum node degree of G. Removing a node of maximal degree removes Δ edges, and so

$$\Delta \leqslant ec(G) \leqslant n. \tag{1}$$

The upper bound is attained when G is the complete graph K_n. We expect $ec(G)$ to be large if G is 'close' to K_n, in the sense that it contains many edges. Then the complementary graph $\overline{G}$ is sparse and so we expect $ec(\overline{G})$ to be small. Thus there are strong *a priori* reasons for expecting that we might obtain a Nordhaus-Gaddum theorem for $ec(G)$.

Theorem 1. *Let G be an undirected graph on n nodes, then*

$$n \leqslant ec(G) + ec(\overline{G}) \leqslant 2n \tag{2}$$

and

$$0 \leqslant ec(G).ec(\overline{G}) \leqslant n^2. \tag{3}$$

Further all these bounds are attained. The lower bounds are attained when G or $\overline{G}$ is the complete graph K_n. The upper bounds are attained by the regular proper subgraphs of K_n.

The lower bound of the multiplicative inequality is trivial and the upper bounds follow easily from (1) so it only remains to prove the lower bound for the additive inequality, which we shall do in Section 2. First however we note that when $G = K_n$, the complete graph, the lower bounds are attained but when $G = C_n$, the circuit graph, the upper bounds are attained. Further, if $G_1 \subset G_2$ then $ec(G_1) \leqslant ec(G_2)$. This suggests that for *most* graphs we might expect the values of the sum and product to attain the upper bound.

2. Proof of Theorem 1.

Let $d = 2m/n$ denote the average node degree of G. We have

$$2n \geqslant ec(G) + ec(\overline{G})$$

$$\geqslant \Delta(G) + \Delta(\overline{G})$$
$$\geqslant d(G) + d(\overline{G}) = n - 1 \quad (4)$$

since G and $\overline{G}$ combine to give K_n. Further $\Delta > d$ unless G is regular of degree d. Since $ec(G) + ec(\overline{G})$ is an integer we have the required inequality

$$ec(G) + ec(\overline{G}) \geqslant n$$

unless G is regular of degree d. Theorem 1 now follows from the following lemma.

Lemma 1. *Let G be a graph regular of degree $d > 0$. Then*

$$ec(G) = n. \quad (5)$$

Proof. We may suppose that $ec(G) < n$, so there is a node x of G which is not adjacent to the additional node w. Choose any neighbour y of x, and remove y from G. In the resulting subgraph x has degree $d - 1$, so we cannot obtain a subgraph isomorphic to G. Therefore $ec(G) = n$.

3. 'r-node' fault tolerance.

The idea of node fault tolerance has been extended to study fault tolerance when r nodes are removed. Starting with the graph G on n nodes we augment it by adding r nodes $w_1, \ldots, w_r$ and extra edges to form an augmented graph G^*. We say that the system is *r-node fault tolerant* if, after removing any r nodes from G^*, the resulting graph contains a graph isomorphic to G. We denote by $ec(G; r)$ the minimum number of edges we must include so that G^* is r—node fault tolerant.

We obtain an upper bound for $ec(G : r)$ by observing that we can always obtain a subgraph isomorphic to G by connecting each of the n nodes of G to each of $w_1, \ldots, w_r$ and constructing the complete graph K_r on $w_1, \ldots, w_r$. Hence

$$ec(G : r) \leqslant rn + r(r - 1)/2. \quad (6)$$

Further, this bound is attained by the complete graph K_n.

Similarly, let $d_1, \ldots, d_n$ be the node degree sequence of G, ordered so that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$. Removing the r nodes of degrees $d_1, \ldots, d_r$ from G removes at least

$$d_1 + d_2 + \ldots + d_r - r(r - 1)/2$$

edges from G, since at most $r(r-1)/2$ of the edges have both ends in the set of r nodes being removed. Thus

$$ec(G;r) \geqslant d_1 + d_2 + \ldots + d_r - r(r-1)/2.$$

The corresponding formula for $\overline{G}$ is

$$ec(\overline{G};r) \geqslant (n-1-d_n) + (n-1-d_{n-1}) + \ldots + (n-1-d_{n-r}) - r(r-1)/2$$
$$= rn - r(r+1)/2 - (d_{n-r} + \ldots + d_n).$$

Hence

$$ec(G:r) + ec(\overline{G};r) \geqslant r(n-r) + (d_1 + \ldots + d_r) - (d_{n-r} + \ldots + d_n). \quad (7)$$

It is now straightforward to obtain a Nordhaus - Gaddum theorem for ec(G;r).

Theorem 2. *For all graphs G,*

$$r(n-r) \leqslant ec(G:r) + ec(\overline{G};r) \leqslant 2rn + r(r-1) \quad (8)$$

and

$$0 \leqslant ec(G;r).ec(\overline{G};r) \leqslant (rn + r(r-1)/2)^2. \quad (9)$$

The upper bounds follow from (6). The additive lower bound follows from (7) whilst the multiplicative lower bound is trivial (and attained by K_n).

Suppose now that r is fixed and we let $n \to \infty$. Since K_n attains the upper bound in (6) the additive lower bound is asymptotically the best possible, specifically

$$rn + r(r-1)/2 \leqslant ec(K_n) + ec(\overline{K_n}).$$

In order to see that the upper bounds are also asymptotically best possible we need to consider regular graphs in more detail.

4. Regular graphs.

We begin by extending Lemma 1 to the more general case.

Lemma 2. *Let G be a graph regular of degree* $d > 0$. *Then*

$$ec(G:r) \geqslant rn. \quad (10)$$

Proof. We may suppose that $ec(G) < rn$, so there is a node x of G which is not adjacent to each of the additional node $w_1, \ldots, w_r$. If $d \geqslant r$ we can choose r neighbours $y_1, \ldots, y_r$ of x, and remove $y_1, \ldots, y_r$ from G. If $d < r$ we remove the neighbours $y_1, \ldots, y_d$ of x from G and $r - d$ of the added nodes w_i, choosing nodes that are neighbours of x where possible. In the resulting subgraph x has degree at most $d - 1$, so we cannot obtain a subgraph isomorphic to G. Therefore $ec(G) \geqslant rn$.

It is now easy to see that the upper bounds in Theorem 2 are asymptotically best possible, that is for fixed r and letting $n \to \infty$. For a graph G regular of degree $d \geqslant r$ we have

$$ec(G) + ec(\overline{G}) \geqslant 2rn \text{ and } ec(G) + ec(\overline{G}) \geqslant (rn)^2,$$

provided that neither G nor $\overline{G}$ is K_n. We can also sharpen Theorem 2 for regular graphs, the improved lower bounds follow easily from Lemma 2.

Theorem 3. *Let G be regular of degree d where* $0 < d < n - 1$. *Then*

$$2rn \leqslant ec(G : r) + ec(\overline{G}; r) \leqslant 2rn + r(r-1) \quad (11)$$

and

$$(rn)^2 \leqslant ec(G; r).ec(\overline{G}; r) \leqslant (rn + r(r-1)/2)^2. \quad (12)$$

References

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