Forbidden Graphs and Irredundant Perfect Graphs

Joël Puech
Mathématique, Bât. 425, Université Paris-Sud, 91405 Orsay cedex, France
e-mail : Joel.PUECH@math.u-psud.fr

Abstract

The domination number $\gamma(G)$ and the irredundance number $ir(G)$ of a graph G have been considered by many authors. It is well known that $ir(G) \leq \gamma(G)$ holds for all graphs G. In this paper we determine all pairs of connected graphs (X, Y) such that every graph G containing neither X nor Y as an induced subgraph satisfies $ir(G) = \gamma(G)$.

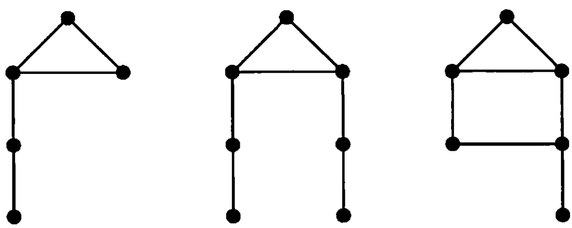
1 Introduction

The graphs $G = (V(G) = V, E(G))$ we consider here are simple and finite of order $|V(G)| = n(G)$. The degree, neighborhood, closed neighborhood of a vertex x of G are respectively denoted by $d_G(x)$, $N_G(x)$, $N_G[x]$ (where $N_G[x] = N_G(x) \cup \{x\}$), or simply by $d(x)$, $N(x)$, $N[x]$ if there is no ambiguity. If $X \subseteq V$, then $N(X) = \bigcup_{x \in X} N(x)$, $N[X] = N(X) \cup X$ and $N_2[X] = N[N[X]]$ ($N_2[X]$ is the set of vertices of G at distance at most 2 from X). We denote by $G[X]$ the subgraph induced by X in G and by Y_X (respectively Z_X) the sets of nonisolated (respectively isolated) vertices of $G[X]$. We say that X is an *independent* set is $Y_X = \emptyset$. If H is an induced subgraph of G, we say that $H \preceq G$.

The X-*private neighborhood* of a vertex x of X is the set $N[x] \setminus N[X \setminus \{x\}]$ and is denoted $\text{pn}(x, X)$. Its elements are the X-private neighbors of x. The X-private neighbors of x which are not contained in X are called *external* and we denote by $B_X(x)$=epn(x, X) the set of external X-private neighbors of x. We observe that the X-private neighborhood of x is $B_X(x)$ if $x \in Y_X$ and $\{x\} \cup B_X(x)$ if $x \in Z_X$. We denote $B_X = \bigcup_{x \in X} B_X(x)$, $Q_X = N(X) \setminus (X \cup B_X)$, $U_X = V \setminus (X \cup B_X \cup Q_X)$, A vertex x of a set

X of vertices in *redundant* in X if $\text{pn}(x, X) = \emptyset$, *irredundant* otherwise. The set X is *irredundant* in G if all its vertices are irredundant. The irredundant set X is maximal if $X \cup \{v\}$ is redundant for all $v \in V \setminus X$. The characterization of maximal irredundant sets was explicitely expressed in [1]: the irredundant set X of G is maximal if and only if for each $v \in N[U_X]$ there exists $x \in X$ such that $\text{pn}(x, X) \subseteq N[v]$. In this case we say that v annihilates x. The set of the vertices of U_X annihilating a vertex $x \in Y_X$ is denoted by $U_X(x)$. The minimum cardinality of a maximal irredundant set is denoted by $ir(G)$. The set X is dominating in G if every vertex of $V \setminus X$ has at least one vertex in X, that is if $N[X] = V$. The minimum cardinality of a dominating set is denoted $\gamma(G)$. It is well known that since every minimal dominating set of G is a maximal irredundant set, $ir(G) \leq \gamma(G)$. We say that a graph G is *irredundance perfect* if for every induced subgraph H of G we have $ir(H) = \gamma(H)$ and we say that a graph is $(H_1, H_2, \ldots, H_k)$*-free* if G contains no induced subgraph isomorphic to any H_i , $i = 1, 2, \ldots, k$. As the property $(H_1, H_2, \ldots, H_k)$-free is hereditary among the induced subgraphs of a graph G, to prove that the property for G to be $(H_1, H_2, \ldots, H_k)$-free implies its irredundance perfection, it is sufficient to prove that this property implies $ir(G) = \gamma(G)$.

For a maximal irredundant set X, we recall the following well known results concerning each vertex u of U_X:
- R_1: there exists at least one vertex in Y_X which is annihilated by u, that is $U_X = \bigcup_{x \in Y_X} U_X(x)$.
- R_2: for every vertex v out of X, which is adjacent to no vertices of Z_X and which is adjacent to u, there exists at least one vertex of Y_X which is annihilated by v.



D_2 $D_{2,2}$ W'

Figure 1:

An induced path of length k, which contains exactly k edges and $k+1$ vertices, is denoted by P_{k+1}. A tripode $T_{i,j,k}$ is constructed by connecting one endvertex of three induced paths which contain respectively i, j, k vertices to a new vertex called the root. Thus $T_{i,j,k}$ contains $i + j + k + 1$ vertices. We consider also the small graphs of Figure 1.

In [2], Faudree, Favaron and Li, studied the relation between a graph G being $(H_1, H_2, \ldots, H_k)$-free and equalities between two of the six parameters concerning domination. In this article, we characterize the pairs (X, Y) for which the property G is (X, Y)-free implies that $ir(G) = \gamma(G)$. Indeed, we prove the following:

Theorem 1.1 *(See Figure 1)*
Let (X, Y) be a pair of connected graphs and let n_0 be a given positive integer.
Then, G is (X, Y)-free implies that G is irredundance perfect for any connected graph G of order at least n_0, if and only if, one of the following statements holds, even if it means exchanging X and Y:

- $X \preceq P_5$,
- $X \preceq P_6$ *and* $Y \preceq W'$,
- $X \preceq D_{2,2}$ *and* $Y \preceq T_{1,1,2}$,
- $X \preceq D_2$ *and* $Y \preceq T_{1,1,3}$.

In Section 2, we consider some facts that will be useful in Sections 3 and 4, where we prove respectively that $(D_{2,2}, T_{1,1,2})$-free graphs and $(D_2, T_{1,1,3})$-free graphs are irredundance perfect. Conversely, in Section 5, we define a family of graphs that are not irredundance perfect that will be used in the proof of Theorem 1.1. In Section 6, results of [2], [3], and of the preceding sections will be used in order to prove Theorem 1.1.

2 Preliminaries

In this section, we consider G to be any $(D_{2,2}, T_{1,1,3})$-free graph and X any maximal irredundant set of G. Since a $(D_{2,2}, T_{1,1,2})$-free graph and a $(D_2, T_{1,1,3})$-free graph are also a $(D_{2,2}, T_{1,1,3})$-free graph, we can apply the following results to Sections 3 and 4.

Lemma 2.1 *If y_1 and y_2 are two vertices of Y_X which are not in the same connected component of Y_X, if y_1' and y_2' are respective X-private neighbors of y_1 and y_2, and if y_1' is adjacent to some vertex u_1 of U_X, then y_1' and y_2' are not adjacent.*

Proof. Suppose on the contrary that $y_1'y_2' \in E$. For $i = 1, 2$ let x_i be a neighbor of y_i in X. Then $G[x_2, y_2, y_2', y_1', u_1, y_1] \not\simeq T_{1,1,3}$ implies that $uy_2' \in E$. Hence $G[x_1, y_1, y_1', u_1, y_2', y_2, x_2] \simeq D_{2,2}$, which gives a contradiction. □

Definition 2.2 *In Sections 3 and 4, we will define a set of special components of Y_X such that every component C which is not special satisfies the following* **Property** $\mathcal{P}$: $\forall y \in C$ *the set* $U_X(y)$ *is a clique. We denote by* Λ *the set of all the vertices of special components.*

We consider S a maximal independent set of the subgraph of G induced by the set $\{u \in U_X \mid u$ *does not annihilate some vertex of* $\Lambda\}$ *so that S dominates* $U_X \setminus \bigcup_{y\in\Lambda} U_X(y)$. *Then, for every u in S we choose one vertex* $y(u)$ *such that* $u \in U_X(y(u))$. *That defines a function y from S to* $Y_X \setminus \Lambda$, *and we denote by* $y(S)$ *the set* $\{y(u) \mid u \in S\}$.

Proposition 2.3 *Every function y as in Definition 2.2 is injective, and therefore we have* $|y(S)| = |S|$.

Proof. Suppose to the contrary that the function y is not injective. Then, there exist u and v, two distinct vertices of the independent set S, such that $y(u) = y(v) = y$. Note that since by definition any vertex y of $y(S)$ is not in Λ, the set $U_X(y)$ is a clique. We obtain a contradiction since both u and v of the independent set S belong to such a $U_X(y)$. □

Definition 2.4 *We say that a subset* $\{y(u_1), y(u_2)\}$ *of* $y(S)$ *induces the* **structure T** *if for* $i = 1,2$ *there exists* $y'_i \in B_X(y(u_i))$ *which is not adjacent to* u_j *where* $j \neq i$.
Recall that u_1u_2 *is not an edge since both* u_1 *and* u_2 *belong to the independent set S and that* u_i *annihilates* $y(u_i)$ *for* $i = 1,2$. *Note that the extra edges of* $G[y(u_1), y'_1, u_1, y(u_2), y'_2, u_2]$ *are possibly among* $y(u_1)y(u_2)$ *and* $y'_1y'_2$.

Proposition 2.5
Every subset $\{y(u_1), y(u_2)\}$ *of* $y(S)$ *induces the structure T.*

Proof. Otherwise, without loss of generality we can say that u_2 dominates $B_X(y(u_1))$. That is both u_1 and u_2 are located in the clique $U_X(y(u_1))$. Thus, we obtain a contradiction since both u_1 and u_2 belong to the independent set S. □

3 The class $(D_{2,2}, T_{1,1,2})$

In this section we consider a $(D_{2,2}, T_{1,1,2})$-free graph G and a maximal irredundant set X. The aim is to prove that G is irredundance perfect.

Proposition 3.1 $\forall y \in Y_X$, $U_X(y)$ *is a clique.*

Proof. Suppose on the contrary that there exist u and v two nonadjacent vertices of $U_X(y)$ where $y \in Y_X$. Let y' be any vertex in $B_X(y)$, and let x be any neighbor of y in X. Then $G[x, y, y', u, v] \simeq T_{1,1,2}$, which gives a contradiction. □

Definition 3.2 *A connected component C of Y_X is said to be a* **special component**

- **of type 1** *if C is reduced to $\{y_1, y_2\}$ and for $i = 1, 2$*
  there exists $(b_i, u_i) \in B_X(y_i) \times U_X(y_i)$ such that u_i is adjacent to neither u_j nor b_j where $j \neq i$;
  and every vertex of $B_X(y_i)$ annihilates either $y(u_1)$ or $y(u_2)$.
- **of type 2** *if there exists $y \in C$ such that every $b \in B_X(y)$ dominates* $\bigcup_{x \in C}[B_X(x) \cup U_X(x)]$.

Note that by Proposition 3.1, every component C which not special satisfies Property $\mathcal{P}$.

Proposition 3.3 *The set $y(S)$ is independent.*

Proof. Otherwise there exist two vertices $y(u_1)$ and $y(u_2)$ in $y(S)$ such that $y(u_1)y(u_2)$ is not an edge. Observe that by Proposition 2.5, the subset $\{y(u_1), y(u_2)\}$ of $y(S)$ induces the structure T. Moreover we have $y'_1 y'_2 \notin E$ since otherwise $G[u_1, y'_1, y'_2, u_2, y(u_1)] \simeq T_{1,1,2}$. Let C be the connected component of Y_X including both $y(u_1)$ and $y(u_2)$. We will show that C is a special component of type 1, which gives a contradiction.

Claim 1 The component C is reduced to $\{y(u_1), y(u_2)\}$.
Otherwise let x be another vertex of C. Without loss of generality we can suppose that $xy(u_1)$ is an edge. Then $G[u_1, y'_1, y(u_1), x, y(u_2)] \not\simeq T_{1,1,2}$ implies that $xy(u_2)$ is an edge. Therefore $G[u_1, y'_1, y(u_1), x, y(u_2), y'_2, u_2] \simeq D_{2,2}$, a contradiction.

Claim 2 Every vertex of $B_X(y(u_i))$ annihilates either $y(u_1)$ or $y(u_2)$.
Let t and w be two vertices of $B_X(y(u_1))$. Since t is adjacent to the vertex u_1 of U_X, by Result R_2, t annihilates some vertex x of Y_X. By Lemma 2.1, x lies in the component C, and by Claim 1, x is either $y(u_1)$ or $y(u_2)$.

By Claims 1,2 and by the structure T, C is a special component of type 1, where for $i = 1, 2$ $y_i = y(u_i)$ and $b_i = y'_i$. □

Theorem 3.4 *Every $(D_{2,2}, T_{1,1,2})$-free graph is irredundance perfect.*

Proof.
Procedure: Let X be a maximal irredundant set of a $(D_{2,2}, T_{1,1,2})$-free graph which is not dominating (otherwise $ir = \gamma$). We will construct D, a dominating set with the same cardinality as X. Then, take X such that $|X| = ir$. Hence $|D| = ir \leq \gamma$. But by definition of γ, $|D| \geq \gamma$. Thus, $|D| = \gamma$ and therefore $ir = \gamma$.

Construction of D: First we put $Z_X \cup [Y_X \setminus (\Lambda \cup y(S))] \cup S$ in D. Moreover if C is a special component
of type 1 we put b_1 and b_2 in D,
of type 2 we put b and $C \setminus \{y\}$ in D.

Assume that there is a vertex t undominated by D. By the construction of D, we can say that t is neither in X since $y(S)$ is an independent set included in Y_X (see Proposition 3.3) and therefore is dominated by at least one vertex of $Y_X \setminus (\Lambda \cup y(S))$, nor in $B_X \cup U_X$ since S dominates $U_X \setminus \bigcup_{y \in \Lambda} U_X(y)$ (for special components it is clear by the definition of each type). Thus $t \in Q_X$.

Claim 1 The vertex t cannot dominate a special component of type 1.
We consider a special component C of type 1. First note that $b_1 b_2 \notin E$, for otherwise $G[u_2, b_2, b_1, u_1, y_1] \simeq T_{1,1,2}$. Suppose on the contrary that t dominates the component C. Since both the vertices b_1 and b_2 are in D, $G[u_1, b_1, y_1, t, y_2, b_2, u_2] \not\simeq D_{2,2}$ implies that t is adjacent to u_1 or to u_2. Suppose for instance that t is adjacent to u_1. Then t is not adjacent to u_2 for otherwise $G[b_2, u_2, t, y_1, u_1] \simeq T_{1,1,2}$. By Result R_2, since t is adjacent to the vertex u_1 of U_X, the vertex t annihilates some vertex x of Y_X. Moreover since t is undominated by D, the vertex x is in $Y_X \setminus C$. Then $tx \notin E$, since otherwise $G[b_2, y_2, t, x, u_1] \simeq T_{1,1,2}$. If $x' \in B_X(x)$, then $x'u_2 \in E$, for otherwise by Lemma 2.1 we have $G[x, x', t, y_1, y_2, b_2, u_2] \simeq D_{2,2}$. We obtain a contradiction, since then $G[y_1, t, x', u_2, x] \simeq T_{1,1,2}$.

Claim 2 The vertex t is adjacent to only one component C_t of Y_X.
Note that if t is adjacent to l_i in the component C_i, then by Claim 1 if C_i is special and since $y(S)$ is an independent set otherwise, there exists a neighbor k_i of l_i located in the component C_i such that k_i is not adjacent to t. Then suppose that t is adjacent to two components C_1 and C_2. Let l'_1 be any vertex in $B_X(l_1)$. Then $G[k_2, l_2, t, l_1, k_1, l'_1] \not\simeq T_{1,1,2}$ implies that $l'_1 t \in E$. Hence t dominates $B_X(l_1)$. Therefore by the construction of D and since t is undominated by D, we assert that C_1 is not a special component.

Let y be l_1, and let b be any vertex in $B_X(y)$. Recall that t dominates $B_X(y)$. First we prove that every vertex of $C_1 \setminus \{y\}$ is adjacent to y. Indeed, otherwise there exists an induced path yxz in C_1. Note that $tx \notin E$ since $y(S)$ is an independent set and therefore $x \in D$. Observe that $tz \notin E$ for otherwise $G[k_2, l_2, t, y, z] \simeq T_{1,1,2}$. Then $G[k_2, l_2, t, b, y, x, z] \simeq D_{2,2}$, a contradiction.

In the following let $x \in C_1 \setminus \{y\}$ and $x' \in B_X(x)$. Recall that x is adjacent to y and note that since $x \notin y(S)$ ($y(S)$ is an independent set) we have $tx \notin E$. Then $G[x', x, y, b, t, l_2, k_2] \not\simeq D_{2,2}$ implies that $bx' \in E$. Moreover, if y' is any vertex of $B_X(y) \setminus \{b\}$, $G[b, y', t, l_2, k_2] \not\simeq T_{1,1,2}$ implies that $by' \in E$ (recall that $ty' \in E$ since t dominates $B_X(y)$). Thus b dominates $\bigcup_{x \in C_1} B_X(x)$. If furthermore x satisfies $U_X \neq \emptyset$, let u be any vertex in $U_X(x)$. Then $G[u, x', b, y, t, l_2, k_2] \not\simeq D_{2,2}$ implies that $ub \in E$, and therefore b dominates $\bigcup_{x \in C_1} U_X(x)$.

Thus C_1 is a special component of type 2, which gives a contradiction.

Since $t \in Q_X$, the vertex t is adjacent to at least two vertices of the component C_t. By Claim 1, clearly C_t is not a special component. Therefore the vertex t is adjacent to at least two vertices $y(u_1)$ and $y(u_2)$ of $C_t \cap y(S)$. Note that for $i = 1, 2$ we have $tu_i \notin E$ since $u_i \in D$. Let $w_1 w_2 \cdots w_k$ where $k \geq 1$ be a path in C_t of minimal length linking $y(u_1)$ to $y(u_2)$. Without loss of generality we can suppose that $\forall i \in \{1, 2, \cdots, k\}$ $tw_i \notin E$. Note that by Proposition 2.5, the subset $\{y(u_1), y(u_2)\}$ of $y(S)$ induces the structure T. Then $G[u_1, y'_1, y(u_1), w_1, t] \not\simeq T_{1,1,2}$ implies that $ty'_1 \in E$, and that $y'_1 y'_2 \notin E$ for otherwise $G[u_1, y'_1, y'_2, u_2, y(u_2)] \simeq T_{1,1,2}$. By symmetry we also have $ty'_2 \in E$. If $k \geq 2$ then $G[u_2, y'_2, t, y'_1, y(u_1), w_1, w_2] \simeq D_{2,2}$, a contradiction. Hence $k = 1$. Let $w = w_1$ and $w' \in B_X(w)$. Then $G[y'_1, y(u_1), w, w', y(u_2)] \not\simeq T_{1,1,2}$ implies that $y'_1 w' \in E$, and by symmetry, $y'_2 w' \in E$. Then $G[u_1, y(u_1), y'_1, w', y'_2] \not\simeq T_{1,1,2}$ implies that $w' u_1 \in E$. We obtain a contradiction since then $G[u_1, y'_2, w', w, y(u_1)] \simeq T_{1,1,2}$. Thus t cannot exist, D is a dominating set, and the theorem holds. □

4 The class $(D_2, T_{1,1,3})$

In this section we consider a $(D_2, T_{1,1,3})$-free graph G and a maximal irredundant set X. The aim is to prove that G is irredundance perfect.

Definition 4.1 *A connected component C of Y_X is said to be a* **special component**

- **of type 1** *if there exists $y \in C$ such that $U_X(y)$ is not a clique.*
- **of type 2** *if C is an induced path $y_1 x y_2$ of length 2, and if for $i = 1, 2$ there exists $(b_i, u_i) \in B_X(y_i) \times U_X(y_i)$ such that $b_1 b_2 \in E$, $u_1 u_2 \notin E$, and $u_i b_j \notin E$ when $j \neq i$.*

• **of type 3** *if for $i = 1,2$ there exist $x_i \in C$ and $(p_i, w_i) \in B_X(x_i) \times U_X(x_i)$ such that $x_1x_2 \in E$, $w_1w_2 \notin E$, and $w_ip_j \notin E$ when $j \neq i$.*

Note that every component C which is not special (of type 1) satisfies Property $\mathcal{P}$.

Proposition 4.2 *If C is a special component*

• *of type 1, then C is a star centered at y.*

• *of type 3 such that $|C| \geq 3$, then every vertex of $C \setminus \{x_1, x_2\}$ is adjacent to exactly one vertex among x_1 and x_2, and moreover $p_1p_2 \in E$.*

Proof. Let C be a special component of type 1, let u and v be two nonadjacent vertices of $U_X(y)$, and let y' be any vertex in $B_X(y)$. First, there is no induced path zxy of length 2 where z and x are in C, for otherwise $G[z, x, y, y', u, v] \simeq T_{1,1,3}$. Thus every vertex of $C \setminus \{y\}$ is adjacent to y. Now, suppose to the contrary that there exist two adjacent vertices x and z of $C \setminus \{y\}$. Then $G[x, z, y, y', u] \simeq D_2$, a contradiction. Thus C is a star centered at y.

Let C be a special component of type 3 such that $|C| \geq 3$. First, note that there is no vertex y of $C \setminus \{x_1, x_2\}$ which is adjacent to both x_1 and x_2, for otherwise $G[y, x_1, x_2, p_2, w_2] \simeq D_2$. Since $|C| \geq 3$, without loss of generality, we can suppose that there exists $y_1 \in C \setminus \{x_1, x_2\}$ which is adjacent to x_1 but not to x_2. Then $p_1p_2 \in E$, for otherwise $G[w_2, p_2, x_2, x_1, y_1, p_1] \simeq T_{1,1,3}$. Finaly, there is no induced path $z_1y_1x_1$ of length 2 where z_1 and y_1 are in $C \setminus \{x_1, x_2\}$, for otherwise $G[z_1, y_1, x_1, p_1, p_2, w_1] \simeq T_{1,1,3}$. By symmetry the result holds. □

Proposition 4.3

1. *The set $y(S)$ is independent.*
2. *Every subset $\{y(u_1), y(u_2)\}$ of $y(S)$, such that $y(u_1)$ and $y(u_2)$ are in the same connected component of Y_X, induces the structure T with $y'_1y'_2 \notin E$.*

Proof.

1. Otherwise there exist $y(u_1)$ and $y(u_2)$ in $y(S)$ such that $y(u_1)y(u_2) \in E$. Note that by Proposition 2.5, the subset $\{y(u_1), y(u_2)\}$ of $y(S)$ induces the structure T. Then the connected component of Y_X including both $y(u_1)$ and $y(u_2)$ is a special component of type 3, which gives a contradiction.
2. By Proposition 2.5, the subset $\{y(u_1), y(u_2)\}$ of $y(S)$ induces the structure T. Suppose to the contrary that $y'_1y'_2 \in E$. By 1. we have $y(u_1)y(u_2) \notin E$. Let x be a neighbor of $y(u_1)$ in X. Then $xy(u_2) \in E$ for otherwise $G[x, y(u_1), y'_1, y'_2, y(u_2), u_2] \simeq T_{1,1,3}$. We will show that the connected component C including both $y(u_1)$ and $y(u_2)$ is reduced to $\{y(u_1), x, y(u_2)\}$ so

that C is a special component of type 2, which gives a contradiction. Indeed suppose otherwise that z is a vertex of $C\backslash\{y(u_1), x, y(u_2)\}$. Note that z cannot be adjacent to both x and $y(u_1)$ since otherwise $G[x, z, y(u_1), y'_1, u_1] \simeq D_2$. By symmetry, without loss of generality, we can suppose that, either z is adjacent to x but neither to $y(u_1)$ nor to $y(u_2)$, or z is adjacent to $y(u_1)$ but not to x. In the first case, $G[u_1, y'_1, y(u_1), x, z, y(u_2)] \simeq T_{1,1,3}$ gives a contradiction. In the second case, $G[u_2, y'_2, y'_1, y(u_1), x, z] \simeq T_{1,1,3}$ gives a contradiction. □

Definition 4.4
Let C be a connected component of Y_X and let $\mathcal{F}$ be a family $\{(y_i, b_i, u_i)\}_{i\in I}$ where $y_i \in C$ such that $U_X(y_i) \neq \emptyset$, $b_i \in B_X(y_i)$, and $u_i \in U_X(y_i)$. We suppose henceforth that every vertex x of $C^* = C \setminus \{y_i\}_{i\in I}$ is adjacent to at least one of the y_i's. We choose one of them, say y_{i_x}, which is called the **mate** *of x. Then we choose one vertex x' of $B_X(x)$ such that if possible*

1. *x' is not adjacent to the set U_X*
2. *x' is not adjacent to b_{i_x}.*

The set $C' = \{b_i\}_{i\in I} \cup \{x'\}_{x\in C^*}$ is a collection of X-private neighbors of vertices of C, and is said to be a **private sample** *of C induced by the family $\mathcal{F}$.*

Proposition 4.5
Let C' be a private sample of C induced by the family $\{(y_i, b_i, u_i)\}_{i\in I}$. Then:
1. *The X-private neighborhood of vertices of C is dominated by the set C'.*
2. *If $t \in Q_X$ is adjacent to at least one vertex of C and to no vertices of C', then t cannot be adjacent to the set U_X.*

Proof.
1. Let x be a vertex in C and x' be the unique vertex of $B_X(x)$ which is in C'. Suppose to the contrary that there exists $b \in B_X(x)$ which is not dominated by C'. Then b is not adjacent to U_X, for otherwise, by Result R_2 the vertex b annihilates some vertex s of Y_X, and by Lemma 2.1 the vertex s is in C, which gives a contradiction with the hypothesis b is not dominated by C'. Then by restriction 1 completed in order to make the choice of x', we assert that x' is not adjacent to U_X and therefore x is not an y_i for some $i \in I$. Let y_{i_x} be the mate of x. Then b is adjacent neither to x' nor to b_{i_x}, since both x' and b_{i_x} belong to C'. Therefore by restriction 2 completed in order to make the choice of x', we assert that $x'b_{i_x} \notin E$. We obtain a contradiction since $G[u_{i_x}, b_{i_x}, y_{i_x}, x, x', b] \simeq T_{1,1,3}$.
2. Suppose to the contrary that t is adjacent to U_X. By Result R_2, the vertex t annihilates some vertex s in Y_X. Since t is adjacent to no vertices of C', the vertex s is located in a different component from C. Let s' be any

vertex in $B_X(s)$, y be any neighbor of x in C, and x' be the unique vertex of $B_X(x)$ which is in C'. Note that by Lemma 2.1 we have $s'x' \notin E$, and since $x' \in C'$ we have $tx' \notin E$. Then $G[s', s, t, x, x'] \not\simeq D_2$ implies that $ts \notin E$, and $G[y, x, t, s', s] \not\simeq D_2$ implies that $ty \notin E$. We obtain a contradiction since $G[s, s', t, x, x', y] \simeq T_{1,1,3}$. □

Theorem 4.6 *Every $(D_2, T_{1,1,3})$-free graph is irredundance perfect.*

Proof.

Procedure: This procedure is similar to Theorem 3.4.

Construction of D: First we put $Z_X \cup [Y_X \setminus (\Lambda \cup y(S))] \cup S$ in D. Moreover if C is a special component

of type 1 we put in D a private sample C' of C induced by the family $\{(y, b, u)\}$ where (b, u) is any couple in $B_X(y) \times U_X(y)$.

of type 2 we put in D a private sample C' of C induced by the family $\{(y_i, b_i, u_i)\}_{i=1,2}$.

of type 3 we put in D a private sample C' of C induced by the family $\{(x_i, p_i, w_i)\}_{i=1,2}$.

Assume that there is a vertex t undominated by D. By construction of D, we can say that t is not in X since $y(S)$ is an independent set included in Y_X (see Proposition 4.3_1) and therefore is dominated by at least one vertex of $Y_X \setminus (\Lambda \cup y(S))$. The vertex t is not in $B_X \cup U_X$ since S dominates $U_X \setminus \bigcup_{y \in \Lambda} U_X(y)$ (for special components it is clear by the definition of each type and by Proposition 4.5_1). Thus $t \in Q_X$.

Claim 1 The vertex t is adjacent to only one connected component C_t of Y_X.

If t is adjacent to the vertex l_i in the connected component C_i, let k_i be any neighbor of l_i in C_i, and let l'_i be the unique vertex of $B_X(l_i)$ which is in C'_i if C_i is special, and any vertex of $B_X(l_i)$ otherwise. Suppose that t is adjacent to two components C_1 and C_2. Either C_1 is special or not. In the first case, since $l'_1 \in D$ we have $tl'_1 \notin E$. Then $tk_2 \notin E$ for otherwise $G[k_2, l_2, t, l_1, l'_1] \simeq D_2$, and $tk_1 \notin E$ for otherwise $G[k_1, l_1, t, l_2, k_2] \simeq D_2$. In the second case, since $y(S)$ is an independent set and by the construction of D, we have $tk_1 \notin E$. Then $tk_2 \notin E$ for otherwise $G[k_2, l_2, t, l_1, k_1] \simeq D_2$, and $tl'_1 \notin E$ for otherwise $G[l'_1, l_1, t, l_2, k_2] \simeq D_2$. Therefore in either cases we obtain a contradiction since $G[k_2, l_2, t, l_1, k_1, l'_1] \simeq T_{1,1,3}$.

For the following claims, note that since $t \in Q_X$, we have $N(t) \cap C_t \geq 2$. Remark also that by the construction of D and by Proposition 4.5_2, if C_t is a special component, then t cannot be adjacent to the set U_X (see Claims 2, 3 and 4).

Claim 2 The component C_t is not a special component of type 1.
Suppose the contrary. Let u and v be two nonadjacent vertices of $B_X(y)$, and let y' be the unique vertex of $B_X(y)$ which is in C'_t. Note that since $y' \in D$ we have $ty' \notin E$, and recall that by Proposition 4.2, C_t is a star centered at y. Since $N(t) \cap C_t \geq 2$, we can suppose that there exists a neighbor x of y in C_t which is adjacent to t. We assert that t is not adjacent to y for otherwise $G[t, x, y, y', u] \simeq D_2$. Then $G[t, x, y, y', u, v] \simeq T_{1,1,3}$, a contradiction.

Claim 3 The component C_t is not a special component of type 2.
Suppose the contrary. Recall that C is an induced path $y_1 x y_2$ of length 2. Since $N(t) \cap C_t \geq 2$, we can suppose that t is adjacent to y_1. Then, $tx \notin E$ for otherwise $G[t, x, y_1, b_1, u_1] \simeq D_2$ ($tb_1 \notin E$, because $b_1 \in D$). We obtain a contradiction since then $G[u_2, b_2, b_1, y_1, t, x] \simeq T_{1,1,3}$.

Claim 4 The component C_t is not a special component of type 3.
Suppose the contrary. Recall that every vertex of $C_t \setminus \{x_1, x_2\}$ is adjacent to exactly one vertex among x_1 and x_2. Note that for $i = 1, 2$, we have $tp_i \notin E$ since $p_i \in D$ and since t is undominated by D. Since $N(t) \cap C_t \geq 2$, we can suppose that either t is adjacent to both x_1 and x_2, or without loss of generality we can suppose that there exists a neighbor y_1 of x_1 in C_t which is adjacent to t. The first case cannot happen for otherwise $G[t, x_2, x_1, p_1, w_1] \simeq D_2$. In the second case, $tx_1 \notin E$ for otherwise $G[t, y_1, x_1, p_1, w_1] \simeq D_2$. We obtain a contradiction since then $G[t, y_1, x_1, p_1, p_2, w_1] \simeq T_{1,1,3}$.

Claim 5 The component C_t must be special.
Suppose the contrary. Since $t \in Q_X$ and by the definition of D, the vertex t is adjacent to at least two vertices $y(u_1)$ and $y(u_2)$ of $C_t \cap y(S)$. Note that by Proposition 4.3, the subset $\{y(u_1), y(u_2)\}$ of $y(S)$ induces the structure T with $y(u_1)y(u_2) \notin E$ and $y'_1 y'_2 \notin E$. Moreover since u_1 and u_2 are in D, they are not adjacent to t. We assert that t is adjacent neither to y'_1 nor to y'_2. Indeed if for instance $ty'_1 \in E$ then $G[y'_1, y(u_1), t, y(u_2), y'_2] \not\simeq D_2$ implies that $ty'_2 \in E$. Therefore $G[y'_1, y(u_1), t, y'_2, u_2] \simeq D_2$, which gives a contradiction. Let x be a neighbor of $y(u_1)$ in C_t. By construction of D and since $y(S)$ is an independent set, we have $tx \notin E$. Then $ty(u_2) \in E$ for otherwise $G[y'_2, y(u_2), t, y(u_1), y'_1, x] \simeq T_{1,1,3}$. Let x' be any vertex in $B_X(x)$. Note that both $G[u_1, y'_1, y(u_1), x, x', y(u_2)] \not\simeq T_{1,1,3}$ and $G[u_1, y'_1, x', x, y(u_2)] \not\simeq D_2$ imply that x' is adjacent to exactly one vertex among y'_1 and u_1. By symmetry x' is adjacent to exactly one vertex among y'_2 and u_2. If x' is adjacent to y'_1 and to y'_2 then $G[u_1, y'_1, x', y'_2, u_2, y(u_2)] \simeq T_{1,1,3}$. If x' is adjacent to y'_1 and to u_2 then $G[y'_2, u_2, x', y'_1, u_1, y(u_1)] \simeq T_{1,1,3}$. Thus x' is adjacent to both u_1 and u_2 where x' is any vertex in $B_X(x)$. Therefore u_1 and u_2 are two nonadjacent vertices of $U_X(x)$ and C_t is a special component

of type 1, which gives a contradiction.

Thus, we obtain a contradiction since the component C_t cannot exist. □

5 A family of graphs

Proposition 5.1

Let k be an integer such that $k \geq 2$ and G_k be the graph of order $3k+6$ in Figure 2.
Then we have $ir(G_k) < \gamma(G_k)$, and $T_{2,2,l} \not\leq G_k$ for every integer $l \geq 1$.

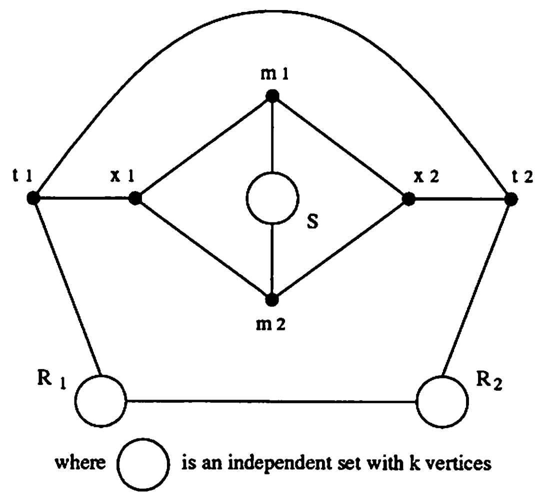


Figure 2: the family of graphs G_k

Proof. Note that G_k is such that R_1, R_2, S are all independent sets with k vertices, for $i = 1, 2$, the vertex t_i dominates the set R_i, the vertex m_i dominates the set S, and for every $r_i \in R_i$, we have $r_1 r_2 \in E$.

First we assert that $ir(G_k) \leq 3$. Indeed the set $X = \{x_1, x_2, m_1\}$ is irredundant since the X-private neighborhoods of x_1, x_2, m_1 are respectively $\{t_1\}$, $\{t_2\}$, S and therefore are nonempty. Now it is sufficient to prove that X is maximal irredundant and we use the characterization mentionned in the introduction. We have $U_X = R_1 \cup R_2$ and therefore $N[R_1 \cup R_2] = R_1 \cup R_2 \cup \{t_1, t_2\}$. But, for $i = 1, 2$, t_i annihilates x_i and any $r_i \in R_i$ also annihilates x_i. Hence the assertion holds.

On the other hand we have $\gamma(G_k) \geq 4$. Indeed, we consider any dominating set D and we will show that $|D| \geq 4$. Without loss of generality, since D must dominate $R_1 \cup R_2$, we can suppose that either $|R_1 \cap D| \geq 1$ or both t_1 and t_2 are in D. In the first case, since D dominates R_1, we must have $R_1 \subseteq D$, $t_1 \in D$, or $|R_2 \cap D| \geq 1$. In either cases we obtain that the set $[R_1 \cup R_2 \cup \{t_1\}] \cap D$ contains at least two vertices and does not dominate the set $S \cup \{m_1, m_2\}$. In the second case, t_1 and t_2 of D does not dominate the set $S \cup \{m_1, m_2\}$. Thus, in both cases, since $S \cup \{m_1, m_2\}$ does not contain any dominating vertex, we have $|D| \geq 4$. Hence $ir(G_k) < \gamma(G_k)$.

Suppose that there exist an integer l ($l \geq 1$) and an induced subgraph H of G_k isomorphic to $T_{2,2,l}$. Therefore H contains no induced cycles. We denote by a the root of H, by $b_i c_i$ for $i = 1, 2$ the two induced paths isomorphic to P_2, and by d the neighbor of a which is included in the induced path isomorphic to P_l. First note that *two of* b_1, b_2, d *cannot be in the same independent set* R_1, R_2 *or* S, for otherwise, without loss of generality we can suppose that one is b_1 and the other, denoted by v, is either b_2 or d, and then the graph $H[a, b_1, c_1, v]$ is an induced cycle, a contradiction. Since $d_{G_k}(a) = 3$ and by symmetry, the root a is among the vertices m_1, x_1, t_1. If the root is m_1, then $\{b_1, b_2, d\} = \{x_1, x_2, s\}$ where s is any vertex of S. Note that for $i = 1, 2$ the vertex c_i cannot be m_2, for otherwise the graph $H[m_1, x_1, m_2, x_2]$ is an induced cycle. Hence we can suppose that for $i = 1, 2$ we have $b_i = x_i$ and $c_i = t_i$. Then the graph $H[a, b_1, c_1, c_2, b_2]$ is an induced cycle, a contradiction. If the root is x_1, then $\{b_1, b_2, d\} = \{m_1, m_2, t_1\}$. Without loss of generality we can suppose that $b_1 = m_1$ and therefore c_1 is x_2 or any vertex of S. Then we obtain a contradiction since $H[x_1, m_1, c_1, m_2]$ is an induced cycle. If the root is t_1, then $\{b_1, b_2, d\} = \{x_1, t_2, r_1\}$ where r_1 is any vertex of R_1. Note that for $i = 1, 2$ the vertex c_i cannot be any vertex of R_2, for otherwise $H[r_1, r_2, t_2, t_1]$ is an induced cycle, and therefore $d = r_1$. We can suppose that $b_1 = x_1$, $b_2 = t_2$, so that $c_2 = x_2$. Then we obtain a contradiction, since c_1 can only be m_1 or m_2 and therefore $H[t_1, t_2, x_2, c_1, x_1]$ is an induced cycle, a contradiction. Thus the graph H cannot exist. □

6 The proof of the theorem

First, recall that we proved in Sections 3 and 4 that respectively $(D_{2,2}, T_{1,1,2})$-free graphs and $(D_2, T_{1,1,3})$-free graphs are irredundance perfect. Moreover note that in [3], it is proved that P_5-free graphs and (P_6, W')-free graphs are irredundance perfect.

Conversely, let (X, Y) be a pair of connected graphs, neither of which is a subgraph of P_5 or a subgraph of each other, and let n_0 be a given integer. Suppose that the condition G is (X, Y)-free implies G is irredundance perfect for any graph G of order at least n_0. Then, by the Theorem 5.4 of