Antimagic Face Labeling of Convex Polytopes Based on Biprisms

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ABSTRACT. The paper defines (a, d)-face antimagic labeling of a certain class of convex polytopes. The possible values of d are determined as $d = 2$, 4 or 6. For $d = 2$ and 4 we produce $(9n + 3, 2)$ and $(6n + 4, 4)$-face antimagic labelings for the polytopes.

1 Introduction

We shall consider non-orientable connected plane graphs without loops, multiple edges or isolated vertices. The vertex (edge) set of a graph G will be denoted by $V(G)$ $(E(G))$, respectively. A graph is said to be *plane* if it is drawn on the Euclidean plane so that none of the edges cross each other except at vertices of the graph. For a plane graph $G = (V, E, F)$, it makes sense to determine its faces, including the unique face of infinite area. Let $F(G)$ be the face set and $|F(G)|$ be the number of faces of G.

The *weight* $w(f)$ of a face $f \in F(G)$ under an edge labeling $\delta: E \to \{1, 2, \dots, |E(G)|\}$ is the sum of the labels of the edges surrounding that face.

A connected plane graph $G = (V, E, F)$ is said to be *(a, d)-face antimagic* if there exist positive integers $a, d \in N$ and a bijection $\delta: E(G) \to \{1, 2, \dots, |E(G)|\}$ such that the induced mapping $\delta^*: F(G) \to W$ is also

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a bijection, where $W = \{w(f): f \in F(G)\} = \{a, a+d, a+2d, \ldots, a+(|F(G)|-1)d\}$ is the set of weights of faces.

If $G = (V, E, F)$ is (a, d)-face antimagic and $\delta\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$ is a corresponding bijective mapping of G then δ is said to be an (a, d)-face antimagic labeling of G.

Hartsfield and Ringel [5] introduced the concept of an antimagic graph. An *antimagic graph* is a graph whose edges can be labeled with the integers $1, 2, \ldots, q$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex, that is, no two vertices have the same sum. Hartsfield and Ringel conjecture that every tree different from K_2 is antimagic, and, more strongly, that every connected graph other than K_2 is antimagic. Bodendiek and Walther [1] defined the concept of an (a, d)-antimagic graph as a special case of an antimagic graph as follows: If $G = (V, E)$ is a connected graph of order $p = |V| \geq 3$ and size $q = |E| \geq 2$, then G is said to be (a, d)-antimagic iff there exists a bijection $f\colon E \to \{1, 2, \ldots, q\}$ and two positive integers $a, d \in N$ such that the induced mapping $g_f\colon V(G) \to N$, defined by $g_f(v) = \Sigma\{f(u, v)\colon (u, v) \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, \ldots, a+(p-1)d\}$ is the set of weights of vertices.

Clearly, (a, d)-face antimagic labeling of a graph of convex polytope G is equivalent to (a, d)-antimagic labeling of a dual graph G^*.

Bodendiek and Walther showed [2] that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected (a, d)-antimagic graphs. For special graphs called parachutes, (a, d)-antimagic labelings are described in [3,4].

In this paper we construct (a, d)-face antimagic labelings of a certain class of convex polytopes.

Let $I = \{1, 2, 3, \ldots, n\}$ and $J = \{1, 2, 3\}$ be index set and B_n be the graph of a biprism. The *biprism* B_n, $n \geq 3$, is a graph which can be defined as the Cartesian product $P_3 \times C_n$ of a path on three vertices with a cycle on n vertices, embedded in the plane. Let us denote the vertex set of B_n by $V(B_n) = \{z_{j,i}\colon j \in J \text{ and } i \in I\}$ and edge set by $E(B_n) = \{z_{j,i}z_{j,i+1}\colon j \in J \text{ and } i \in I\} \cup \{z_{1,i}z_{2,i}\colon i \in I\} \cup \{z_{2,i}z_{3,i}\colon i \in I\}$. We make the convention that $z_{j,n+1} = z_{j,1}$ for $j \in J$.

The face set $F(B_n)$ contains $2n$ 4-sided faces and two n-sided faces (internal and external). We insert exactly one vertex x into the internal n-sided face of B_n and exactly one vertex y into the external n-sided face of B_n. Suppose that n is even, $n \geq 4$, and consider the graph $\mathbb{B}_n$ with the vertex set $V(\mathbb{B}_n) = V(B_n) \cup \{x, y\}$ and the edge set $E(\mathbb{B}_n) = E(B_n) \cup \{z_{1,2k-1}x\colon k = 1, 2, \ldots, \frac{n}{2}\} \cup \{z_{3,2k-1}y\colon k = 1, 2, \ldots, \frac{n}{2}\}$. Then $\mathbb{B}_n$, $n \geq 4$, is the graph of the convex polytope on $|V(\mathbb{B}_n)| = 3n+2$ vertices, $|E(\mathbb{B}_n)| = 6n$ edges and consisting of $|F(\mathbb{B}_n)| = 3n$ 4-sided faces. Let the vertices of $\mathbb{B}_n$ be labeled as in Figure 1.

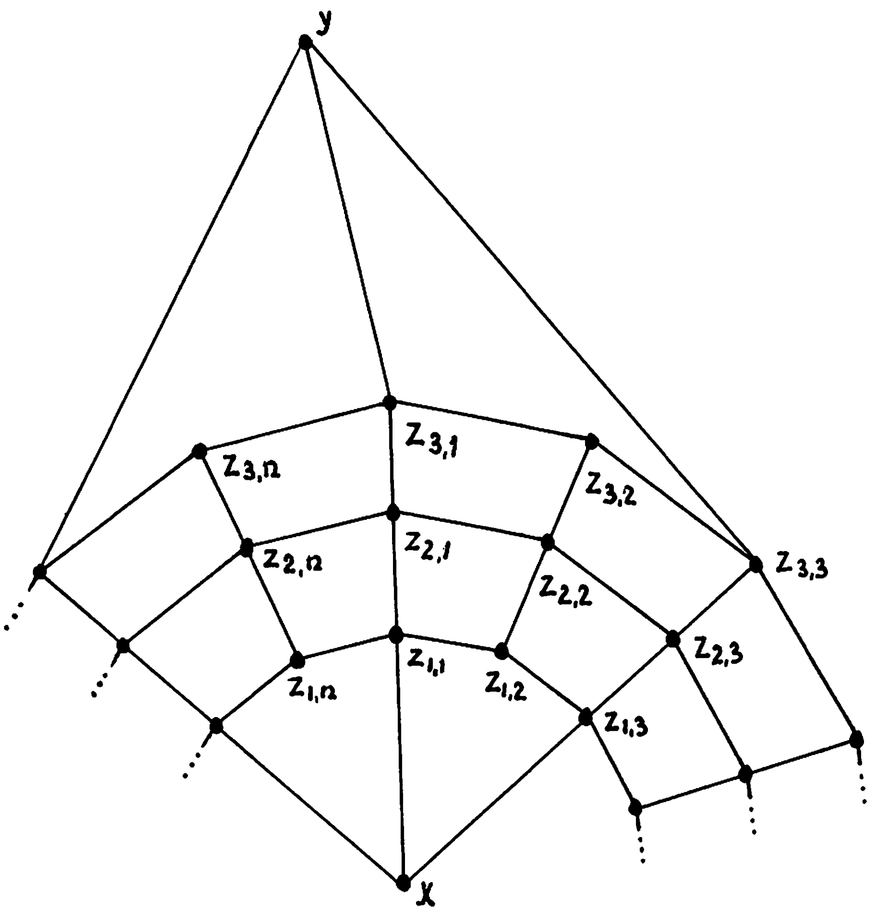

Figure 1

2 Conditions

In this section we deal with the necessary conditions for an (a, d)-face antimagic labeling of $\mathbb{B}_n$. Assume that $\mathbb{B}_n$ is (a, d)-face antimagic on $|V(\mathbb{B}_n)| = 3n + 2$ vertices, $|E(\mathbb{B}_n)| = 6n$ edges and $|F(\mathbb{B}_n)| = 3n$ faces. Clearly, the sum of weights in the set $W = \{w(f)\colon f \in F(\mathbb{B}_n)\} = \{a, a + d, a + 2d, \dots, a + (3n - 1)d\}$ is $3na + \frac{3nd(3n-1)}{2}$. The sum of all the edge labels used to calculate the weights of faces is equal to $6n(1 + 6n)$.

Thus the following equation holds

$$4(1+6n) = 2a + d(3n-1). \tag{1}$$

By putting $a \geq 10$ (the minimal value of weight which can be assigned to a 4-sided face is $a = 10$) we get from (1) that $0 < d < 8$. Since n is even, it follows from (1) that d is even. This implies that equation (1) has exactly the three different solutions $(a, d) = (9n+3, 2)$ or $(6n+4, 4)$ or $(3n+5, 6)$, respectively.

3 Constructions

We shall use the functions

$$\varphi(u) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod 2, \\ 0 & \text{if } u \equiv 0 \pmod 2, \end{cases}$$

$$\psi(u) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod 4, \\ 0 & \text{if } u \equiv 3 \pmod 4, \end{cases}$$

$$\lambda(p, q) = \begin{cases} 1 & \text{if } p \leq q, \\ 0 & \text{if } p > q. \end{cases}$$

to simplify later notations.

If n is even, $n \geq 4$, then we construct the edge labelings δ_1, δ_2, δ_3 and δ_4 of the convex polytope $\mathbb{B}_n$ in the following way:

$$\begin{aligned} \delta_1(z_{j,i}z_{j,i+1}) = & \left[\left(\frac{3-j}{2}n+i\right)\varphi(i) + \left(\frac{4-j}{2}n+i-1\right)\varphi(i+1)\right]\lambda\left(i, \frac{n}{2}\right) \\ & + \left[\left(\frac{2-j}{2}n+i+1\right)\varphi(i) + \left(\frac{3-j}{2}n+i\right)\varphi(i+1)\right] \\ & \lambda\left(\frac{n}{2}+1, i\right), \text{ for } i \in I \text{ and } j \in J \text{ odd.} \end{aligned}$$

$$\begin{aligned} \delta_1(z_{2,i}z_{2,i+1}) = & \left[\frac{5n-i+1}{2}\varphi(i) + \left(3n - \frac{i}{2}\right)\varphi(i+1)\right] \\ & \lambda(i, n-1) + 3n\lambda(n, i), \text{ for } i \in I. \end{aligned}$$

$$\begin{aligned} \delta_1(z_{j,i}z_{j+1,i}) = & \left[(5-j)n + \frac{i+1}{2}\right]\varphi(i) + \left\{\left[(11-2j)\frac{n}{2}+i-1\right]\lambda\left(i, \frac{n}{2}\right)\right. \\ & \left. + [(5-j)n+i]\lambda\left(\frac{n}{2}+2, i\right)\right\}\varphi(i+1), \\ & \text{for } i \in I \text{ and } j \in J - \{3\}. \end{aligned}$$

$$\delta_1(z_{1,i}x) = \left[\left(6n - \frac{i-5}{2}\right)\psi(i) + \left(6n - \frac{i-1}{2}\right)\psi(i+2)\right]\lambda(3,i) + \left(\frac{11}{2}n+2\right)\lambda(i,1), \text{ for } i \in I \text{ odd.}$$

$$\delta_1(z_{3,i}y) = \left[\left(\frac{11n}{2} - \frac{i-5}{2}\right)\psi(i) + \left(\frac{11n}{2} - \frac{i-1}{2}\right)\psi(i+2)\right]\lambda(3,i) + (5n+2)\lambda(i,1), \text{ for } i \in I \text{ odd.}$$

$$\begin{aligned}
&\delta_2(z_{j,i}z_{j,i+1}) \\
&= \left\{\left[\left(\frac{3-j}{2}n+i\right)\varphi(i) + \left(\frac{5-j}{2}n-i+1\right)\varphi(i+1)\right]\lambda\left(i,\frac{n}{2}-1\right)\right. \\
&\left.+\left[\left(\frac{2-j}{2}n+i+2\right)\varphi(i) + \left(\frac{6-j}{2}n-i+1\right)\varphi(i+1)\right]\lambda\left(\frac{n}{2},i\right)\right\}\varphi(j) \\
&+\left\{\left[\left(\frac{3nj}{2}-i\right)\varphi(i) + (nj+i-1)\varphi(i+1)\right]\lambda\left(i,\frac{n}{2}-1\right)\right. \\
&\left.+\left[\left(\frac{7nj}{4}-i\right)\varphi(i) + \left(\frac{3nj}{4}+i+1\right)\varphi(i+1)\right]\lambda\left(\frac{n}{2},i\right)\right\}\varphi(j+1), \\
&\text{for } i \in I - \{n\} \text{ and } j \in J.
\end{aligned}$$

$$\delta_2(z_{j,n}z_{j,1}) = \left(\frac{4-j}{2}n\right)\varphi(j) + \frac{5nj}{4}\varphi(j+1), \text{ for } j \in J.$$

$$\delta_2(z_{j,i}z_{j+1,i}) = (5-j)n+i, \text{ for } i \in I \text{ and } j \in J - \{3\}.$$

$$\delta_2(z_{1,i}x) = \frac{11n+i+1}{2}\varphi(i), \text{ for } i \in I.$$

$$\delta_2(z_{3,i}y) = \left(5n + \frac{i+1}{2}\right)\varphi(i), \text{ for } i \in I.$$

$$\begin{aligned}
\delta_3(z_{j,i}z_{j,i+1}) &= \{[(3-j)n+2i-1]\varphi(i) + [(4-j)n+2i-3]\varphi(i+1)\}\lambda\left(i,\frac{n}{2}\right) \\
&+ \{[(2-j)n+2i+1]\varphi(i) + [(3-j)n+2i-1]\varphi(i+1)\} \\
&\lambda\left(\frac{n}{2}+1,i\right), \text{ for } i \in I \text{ and } j \in J \text{ odd.}
\end{aligned}$$

$$\delta_3(z_{2,i}z_{2,i+1}) = [(5n-i)\varphi(i) + (6n-i-1)\varphi(i+1)]\lambda(i,n-1) + (6n-1)\lambda(n,i), \text{ for } i \in I.$$

$$\delta_3(z_{j,i}z_{j+1,i}) = [(4-2j)n+i+1]\varphi(i) + \left\{[(5-2j)n+2i-2]\lambda\left(i,\frac{n}{2}\right) + [(4-2j)n+2i]\lambda\left(\frac{n}{2}+2,i\right)\right\}\varphi(i+1),$$
for $i \in I$ and $j \in J - \{3\}$.

$$\delta_3(z_{1,i}x) = (5n+2)\lambda(i,1) + (6n-i+3)\lambda(3,i), \text{ for } i \in I,\ i \text{ odd}.$$

$$\delta_3(z_{3,i}y) = (4n+2)\lambda(i,1) + (5n-i+3)\lambda(3,i), \text{ for } i \in I,\ i \text{ odd}.$$

$$\delta_4(z_{j,i}z_{j,i+1}) = \{[(3-j)n+i]\varphi(i) + [(5-j)n-i-1]\varphi(i+1)\}\varphi(j) + \{(3jn-i-2)\varphi(i) + (2jn+i-1)\varphi(i+1)\}\varphi(j+1),$$
for $i \in I - \{n\}$ and $j \in J$.

$$\delta_4(z_{j,n}z_{j,1}) = [(5-j)n-1]\varphi(j) + (3jn-1)\varphi(j+1), \text{ for } j \in J.$$

$$\delta_4(z_{j,i}z_{j+1,i}) = (4-2j)n+2i, \text{ for } i \in I \text{ and } j \in J - \{3\}.$$

$$\delta_4(z_{1,i}x) = (5n+i+1)\varphi(i), \text{ for } i \in I.$$

$$\delta_4(z_{3,i}y) = (4n+i+1)\varphi(i), \text{ for } i \in I.$$

4 Results

Let us denote the weights of 4-sided faces of $\mathbb{B}_n$ (under the edge labeling δ) by

$$\alpha_i(\delta) = \delta(z_{2,i}z_{2,i+1}) + \delta(z_{3,i}z_{3,i+1}) + \delta(z_{2,i}z_{3,i}) + \delta(z_{2,i+1}z_{3,i+1}),$$
$$\beta_i(\delta) = \delta(z_{3,i}z_{3,i+1}) + \delta(z_{3,i+1}z_{3,i+2}) + \delta(z_{3,i}y) + \delta(z_{3,i+2}y),$$
$$\varrho_i(\delta) = \delta(z_{1,i}z_{1,i+1}) + \delta(z_{2,i}z_{2,i+1}) + \delta(z_{1,i}z_{2,i}) + \delta(z_{1,i+1}z_{2,i+1}),$$
and
$$\sigma_i(\delta) = \delta(z_{1,i}z_{1,i+1}) + \delta(z_{1,i+1}z_{1,i+2}) + \delta(z_{1,i}x) + \delta(z_{1,i+2}x), \text{ for } i \in I.$$

Theorem 1. *For $n \geq 4$, $n \equiv 0 \pmod 2$, the convex polytope $\mathbb{B}_n$ has a $(9n+3,2)$-face antimagic labeling.*

Proof: If $n \equiv 0 \pmod 4$ then we label the edges of $\mathbb{B}_n$ by the edge labeling δ_1 and if $n \equiv 2 \pmod 4$ then we label the edges of $\mathbb{B}_n$ by the edge labeling δ_2. It is a matter of routine checking to see that the edge labelings δ_1 and δ_2 use each integer $1, 2, \ldots, 6n$ exactly once and this implies that the labelings δ_1 and δ_2 are bijections from the edge set $E(\mathbb{B}_n)$ to the set $\{1, 2, \ldots, 6n\}$.

Under the edge labelings δ_1 and δ_2 the weights of all 4-sided faces constitute sets W_i, $i = 1, 2, 3, 4$, of consecutive integers:

$$
\begin{aligned}
W_1 &= \{\alpha_i(\delta_1)\colon i \in I\} = \{\alpha_i(\delta_2)\colon i \in I\} = \{9n+3, 9n+5, \dots, 11n+1\},\\
W_2 &= \{\beta_i(\delta_1)\colon i \in I, i \text{ odd}\} = \{\beta_i(\delta_2)\colon i \in I, i \text{ odd}\}\\
&= \{11n+3, 11n+5, \dots, 12n+1\},\\
W_3 &= \{\varrho_i(\delta_1)\colon i \in I\} = \{\varrho_i(\delta_2)\colon i \in I\} = \{12n+3, 12n+5, \dots, 14n+1\},\\
W_4 &= \{\sigma_i(\delta_1)\colon i \in I, i \text{ odd}\} = \{\sigma_i(\delta_2)\colon i \in I, i \text{ odd}\}\\
&= \{14n+3, 14n+5, \dots, 15n+1\}.
\end{aligned}
$$

We can see that each 4-sided face of $\mathbb{B}_n$ receives exactly one label of weight from $\bigcup_{k=1}^4 W_k$ and each number from $\bigcup_{k=1}^4 W_k$ is used exactly once as a label of weight. This proves that δ_1 and δ_2 are $(9n+3, 2)$-face antimagic labelings.

Theorem 2. *For $n \geq 4$, $n \equiv 0 \pmod 2$, the convex polytope $\mathbb{B}_n$ has a $(6n+4, 4)$-face antimagic labeling.*

Proof: Label the edges of $\mathbb{B}_n$ by the edge labeling δ_3 (if $n \equiv 0 \pmod 4$) or the edge labeling δ_4 (if $n \equiv 2 \pmod 4$), respectively. It is not difficult to check that the edge labelings δ_3 and δ_4 are bijections from $E(\mathbb{B}_n)$ to the set $\{1, 2, \dots, 6n\}$. By direct computation we obtain that the weights of 4-sided faces under the edge labeling δ_k $(k = 3, 4)$ constitute the set

$$
\begin{aligned}
W = \{\alpha_i(\delta_k)\colon i \in I\} &\cup \{\beta_i(\delta_k)\colon i \in I, i \text{ odd}\} \cup \{\varrho_i(\delta_k)\colon i \in I\}\\
&\cup \{\sigma_i(\delta_k)\colon i \in I, i \text{ odd}\} = \{6n+4, 6n+8, \dots, 18n\}.
\end{aligned}
$$

This completes the proof.

Concluding this paper, let us pose the following conjecture:

Conjecture 1. *For $n \geq 4$, $n \equiv 0 \pmod 2$, the convex polytope $\mathbb{B}_n$ has a $(3n+5, 6)$-face antimagic labeling.*

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