Alternative proofs of three theorems of Chetwynd and Hilton

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ABSTRACT. In this paper we give alternative and shorter proofs of three theorems of Chetwynd and Hilton. All these three theorems have been widely used in many research papers.

1 Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let G be a graph. We denote its vertex set, edge set, order, size, minimum degree and maximum degree by $V(G)$, $E(G)$, $|G|$, $e(G)$, $\delta(G)$ and $\Delta(G)$, respectively. We use rG to denote vertex-disjoint union of r copies of a graph G. If $x \in V(G)$, we use $N_G(x)$ (or simply N(x)) to denote the neighbourhood of x and $d_G(x)$ (or simply $d(x)$) the degree of x. If $A \subseteq V(G)$ we use $N(A)$ to denote the neighbourhood of A and use $G - A$ (or simply $G - x$ if $A = \{x\}$) to denote the graph obtained by deleting the set of vertices A and its incident edges from G, and if A and B are disjoint subsets of $V(G)$ we use $e_G(A, B)$ (or simply $e_G(x, B)$ if $A = \{x\}$) to denote the number of edges joining A with B. If $F \subseteq E(G)$ we use $G - F$ to denote the graph obtained by deleting F from G. For $x, y \in V(G)$, we write $xy \in E(G)$ if x and y are adjacent in G. We use K_n and O_n to denote the complete graph and null graph of order n, respectively. The *join* $G + H$ of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Vertices of maximum degree in G are called *major vertices* and others are called *minor vertices*. We write $G \cong i_1^{n_1} i_2^{n_2} \cdots i_\Delta^{n_\Delta}$ if G has n_j vertices of degree i_j, where $j = 1, \cdots, \Delta$.

An *edge colouring* of a graph G is a map $\phi : E(G) \to C$, where C is a set of colours, such that no two adjacent edges receive the same colour.

The *chromatic index* $\chi'(G)$ of G is the least value of $|C|$ for which an edge-colouring $\phi : E(G) \to C$ exists. A well-known theorem of Vizing [9] states that, for any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. A graph G is called *Class 1* if $\chi'(G) = \Delta(G)$ and is called *Class 2* if $\chi'(G) = \Delta(G) + 1$.

The *core* G_Δ of a graph G is the subgraph of G induced by the major vertices of G. We use $d_\Delta(v)$ to denote $e_G(v, V(G_\Delta)\backslash\{v\})$. If G is a connected Class 2 graph having $\Delta(G) = \Delta$ and $\chi'(G-e) < \chi'(G)$ for each edge $e \in E(G)$, then G is said to be Δ*-critical*. From Vizing's Adjacency Lemma (see Lemma 2 below) we know that if G is Δ-critical, then $|G_\Delta| \geq 3$.

In this paper we give alternative and/or shorter proofs of three theorems of Chetwynd and Hilton ([2], [3]). The original proof of Theorem 1 used a result of Chetwynd and Yap [5], whose proof is very tedious. Our proof given here do not use the result of [5]. The proofs of Theorem 2 and Theorem 3 given here are much shorter than the original proofs given by Chetwynd and Hilton. The proof of Theorem 4 given here is basically Chetwynd and Hilton's original proof. We include it here because it is more widely used than Theorem 2 and Theorem 3.

2 Preliminary results

In this section we give a list of results which we shall apply in the next section. The proofs of Lemma 1 to Lemma 5 can be found in [10] and the proofs of Lemma 6 and Lemma 7 can be found in many textbooks on graph theory.

Lemma 1 [8]. *For any simple graph G,*

$$\chi'(G) \leq \Delta(G) + 1.$$

Lemma 2 [9]. *Let G be a Δ-critical graph and let $vw \in E(G)$ where $d(v) = k$. We have*

(i) if $k < \Delta$, then $d_\Delta(w) \geq \Delta - k + 1$;

(ii) if $k = \Delta$, then $d_\Delta(w) \geq 2$;

(iii) $|G_\Delta| \geq \Delta - \delta(G) + 2$; and

(iv) $|G_\Delta| \geq 3$.

Lemma 3 [8]. *Let G be a Class 2 graph. Then G contains a k-critical subgraph for each k satisfying $2 \leq k \leq \Delta(G)$.*

Lemma 4 [1]. *There are no regular Δ-critical graphs for any $\Delta \geq 3$.*

Lemma 5 [2]. *Let $e = vw$ be an edge of a graph G. Suppose $d_\Delta(w) = 1$. Then $\Delta(G - w) = \Delta(G)$ implies that $\chi'(G - w) = \chi'(G)$.*

Lemma 6 [6]. *If G is a simple graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G has a Hamilton cycle.*

Lemma 7 [7]. *A graph G has a perfect matching if and only if*

$$o(G-S) \leq |S| \text{ for all } S \subset V(G),$$

where $o(G-S)$ denotes the number of odd components of $G-S$.

Let J_s be a graph of order s and let $G_0 = J_s + O_{s+2}$. Let G_0' denote a spanning subgraph of G_0 such that each vertex of O_{s+2} is joined to at least $s-1$ vertices of J_s and at least one vertex of O_{s+2} is joined to exactly $s-1$ vertices of J_s.

Lemma 8. *A connected graph G of even order $2n$ has a 1-factor if*

(i) $\delta(G) \geq n-1$ except when $G = G_0$;

(ii) $\delta(G) = n-2$ except when $G = G_0'$ or $G = 3K_3 + K_1$.

Proof. Suppose G has no 1-factor. Then by Tutte's theorem, there exists $S \subset V(G)$ such that $o(G-S) > |S| = s$. Since $|G|$ is even, $o(G-S)$ and $|S|$ have the same parity. Hence $o(G-S) \geq s+2$ and so $s + (s+2) \leq 2n$. Consequently

$$n \geq s+1 \tag{1}$$

Let G_1 be an odd component of $G-S$ with minimum order among all the odd components of $G-S$. Then $|G_1| \leq \frac{2n-s}{s+2}$. Hence $\delta(G) \leq d(x) \leq \frac{2n-s}{s+2} - 1 + s$ for any $x \in V(G_1)$. Now we consider two cases separately:

Case 1. $\delta(G) \geq n-1$. Then $n-1 \leq \delta(G) \leq \frac{2n-s}{s+2} - 1 + s$ together with (1) implies that $s = n-1$ and thus $G = G_0$.

Case 2. $\delta(G) = n-2$. Suppose there exists $x \in V(G_1)$ such that $n-2 < d(x)$ or $d(x) < \frac{2n-s}{s+2} - 1 + s$. Then

$$n-1 \leq d(x) \leq \frac{2n-s}{s+2} - 1 + s$$

or

$$n-2 \leq d(x) \leq \frac{2n-s}{s+2} - 2 + s.$$

However, each of these two inequalities together with $n \geq s+1$ imply that $n = s+1$ and thus $G = G_0'$.

So we may assume that for any $x \in V(G_1)$,

$$n-2 = d(x) = \frac{2n-s}{s+2} - 1 + s \tag{2}$$

However, from (2) we have $|G_1| = \frac{2n-s}{s+2}$ and

$$ns = s^2 + 2s + 2. \tag{3}$$

Clearly, (3) does not hold for $n = s+2$ and $n = s+1$. If $n \geq s+3$, then (3) implies that $s \leq 2$. If $s = 2$, then from (3) it follows that $n = 5$. Hence $|G_1| = \frac{2n-s}{s+2} = \frac{10-2}{2+2} = 2$, which contradicts the fact that $|G_1|$ is odd. If $s = 1$, then from (3) again, we have $n = 5$ and thus $G = 3K_3 + K_1$. ■

3 Proofs of theorems

Theorem 1 [2]. *Let G be a connected graph of order n with $\Delta = \Delta(G) \geq 3$. Suppose $|G_\Delta| = 3$. Then G is Class 2 if and only if $G \cong (n-2)^{n-3}(n-1)^3$ (and thus n is odd).*

Proof. Sufficiency. We have $2e(G) = 3(n-1)+(n-3)(n-2) = (n-1)^2+2$. Hence $e(G) = \frac{n-1}{2}(n-1)+1 > \lfloor \frac{n}{2} \rfloor \Delta$ and so G is Class 2.

Necessity. Suppose G has three major vertices (a, b and c say) and is Class 2. By Lemma 3, G contains a Δ-critical subgraph H. By Lemma 2(iv), H has the same three major vertices a, b, c. By Lemma 2(iii) and Lemma 4, $\delta(H) = \Delta - 1$. Thus $H = G$. Since $|G_\Delta| = 3$, Δ must be even, and thus n is odd.

We next show that $\Delta = n-1$. By Lemma 2(i), $d_\Delta(v) \geq 2$ for each vertex v of G. Hence by counting the number of edges joining $A = \{a, b, c\}$ with $V(G)\backslash A$ in two different ways, we have $2(n-3) \leq 3(\Delta - 2)$. Hence

$$\Delta \geq \frac{2}{3}n \tag{4}$$

For $n = 5$ and $n = 7$, using (4) and the fact that Δ is even, we have $\Delta = n-1$. Hence we assume that $n \geq 9$. Suppose $\Delta < n-1$. Then G has a vertex $d \notin N(a)$. Let $G' = G - \{a, b, d\}$. Then $|G'| = n-3$. Since $n \geq 9$, we have $\delta(G') \geq (\Delta - 1) - 3 \geq \frac{2}{3}n - 4 \geq \frac{n-3}{2} - 1$. By Lemma 8(i), G' has a 1-factor F except when $G' = G_0$. However, when $G' = G_0$, we have $2s+2 = n-3$ and $s = \delta(G_0) = \delta(G') \geq \Delta - 4 \geq \frac{n-3}{2} - 1$, from which it follows that $s = \frac{n-5}{2}$ and $\Delta = \frac{n-3}{2} + 3$. Since the degree of d is $\Delta - 1$ and d is adjacent to only two major vertices, therefore d is adjacent to $\Delta - 3$ minor vertices in G. Thus G' has at most $\Delta - 3 = \frac{n-3}{2} = s+1$ vertices of degree $\Delta - 4 = s$, which contradicts the fact that G_0 has $s+2$ vertices of degree s.

The above shows that G' has a 1-factor F. Now $G^* = G - (F \cup \{ab\})$ is Class 2. Since a is adjacent to only one major vertex c in G^* and $\Delta(G^* - a) = \Delta(G^*) = \Delta - 1$, by Lemma 5, $\chi'(G^* - a) = \chi'(G^*)$. Finally, since $G^* - a$ has only two major vertices, by Lemma 2(iv), $\chi'(G^* - a) = \Delta - 1$. Hence $\chi'(G) = \chi'(G^* - a) + 1 = \Delta$, which is a contradiction. Consequently $\Delta = n - 1$. ■

Theorem 2 [3]. *There does not exist any Δ-critical graph of even order having four major vertices.*

Proof. Suppose such a Δ-critical graph G exists. Clearly, $\Delta \geq 3$. Assume that $2n = |G|$ is minimum among all graphs G which are Δ-critical and having $|G_\Delta| = 4$, and Δ is minimum among all such graphs of order $2n$. Let a, b, c, d be the four major vertices of G and let $A = \{a, b, c, d\}$. By Lemma 4, G can not be regular. By Lemma 2(iii), $4 \geq \Delta - \delta + 2$, where $\delta = \delta(G)$. Hence

$$\delta \geq \Delta - 2 \tag{5}$$

By Lemma 2(i), $d_\Delta(v) \geq 2$ for any vertex $v \in V(G)$ and so $2(2n - 4) \leq e_G(A, V(G)\backslash A) \leq 4(\Delta - 2)$. Hence $\Delta \geq n$ and by (5),

$$\delta \geq \Delta - 2 \geq n - 2 \tag{6}$$

We first prove that G has a 1-factor F. Suppose $\delta = n - 2$. Then $n - 2 = \delta \geq 2$ and (6) imply that $n \geq 4$ and $\Delta = n$. Let $u \in V(G)$ be such that $d(u) = n - 2$. By Lemma 2(i), each vertex in $N(u)$ is adjacent to at least three major vertices. Now $3(n - 2) + 2(2n - (n - 2)) \leq 4\Delta = 4n$ implies that $n \leq 2$, which is a contradiction. Hence $\delta \geq n - 1$. By Lemma 8(i), G has a 1-factor unless $G = G_0$. However, when $G = G_0$, we have $\Delta - \delta = \Delta(G_0) - \delta(G_0) \geq 4$ (because all the major vertices of G are in J_s), which contradicts (5). Hence G has a 1-factor F. Clearly, $G^* = G - F$ is Class 2 and $N_{G^*}(A) = V(G^*)$. By Lemma 3, G^* has a $(\Delta - 1)$-critical subgraph H, which has at most four major vertices a, b, c, d. Suppose H has four major vertices. Then $N_{G^*}(A) = V(G^*)$ implies that $V(H) = V(G^*)$ and thus H is a $(\Delta - 1)$-critical graph of order $2n$, which contradicts the assumption that Δ is minimum among all graphs G of order $2n$ which are critical and having four major vertices. Hence H has three major vertices. By Theorem 1, $|H| \neq |G^*|$ and thus $N_{G^*}(A) = V(G^*)$ implies that there is only one vertex in $V(G^*)\backslash V(H)$. Hence, by Theorem 1 again, $2n - 1 = |H| = \Delta(H) + 1 = \Delta$. Since K_{2n} is Class 1 and $G \subseteq K_{2n}$ with $\Delta(G) = 2n - 1$, G must also be Class 1, which is a contradiction. ■

Theorem 3 [3]. *Let G be a graph of odd order $2n + 1 \geq 5$ with $\Delta = \Delta(G) \geq 3$. Suppose G is Δ-critical and $|G_\Delta| = 4$. Then $e(G) = n\Delta + 1$.*

Proof. By Lemma 2(ii), $d_\Delta(v) \geq 2$ for any $v \in V(G)$, which implies that $2(2n + 1) \leq 4\Delta$. Hence

$$\Delta \geq n + 1 \tag{7}$$

It is known that there are three critical graphs of order 5. Beineke and Fiorini [1] had determined all the critical graphs of order 7. (for proofs of these results, see also Theorem 6.6 and Theorem 6.9 in [10]) All these graphs are of size $n\Delta + 1$. Hence this theorem is true for $n = 2, 3$.

We shall now prove this theorem by induction on Δ. Thus by (7) this theorem is true for $\Delta = 3, 4$. Now we can assume that $n \geq 4$ and thus $\Delta \geq n + 1 \geq 5$.

Let a, b, c, d be the four major vertices of G, $A = \{a, b, c, d\}$, and $\delta = \delta(G)$. Since $|G_\Delta| = 4$, by Lemma 2(iii),

$$\delta \geq \Delta - 2 \tag{8}$$

By Lemma 4, $\delta \neq \Delta$. Hence we have two cases to consider.

Case 1. $\delta = \Delta - 2$. Let $x \in V(G)$ be of degree $\Delta - 2$. By Lemma 2(i), each of the $\Delta - 2$ neighbours of x is adjacent to at least three major vertices of G. Hence

$$3(\Delta - 2) + 2((2n + 1) - (\Delta - 2)) \leq 4\Delta$$

from which it follows that

$$\Delta \geq \frac{4}{3}n \tag{9}$$

Now by putting $n \geq 4$ into (9) we obtain

$$\Delta \geq n + 2 \tag{10}$$

Applying (8) and (10), we have $\delta(G - x) \geq \Delta - 3 \geq n - 1$. By Lemma 8(i), $G - x$ has a 1-factor F except when $G - x = G_0$. However, when $G - x = G_0$, we have $A \subseteq V(J_s)$ and thus $\Delta(G_0) - \delta(G_0) \geq 4$, which contradicts the fact that $3 = \Delta - (\Delta - 3) \geq \Delta(G - x) - \delta(G - x)$. Clearly $G^* = G - F$ is Class 2 and $N_{G^*}(A) = V(G^*)$. By Lemma 3, G^* contains a $(\Delta - 1)$-critical subgraph H, which has at most four major vertices a, b, c and d. Suppose H has three major vertices. Since $d_\Delta(v) \geq 2$ for any $v \in A$, we have $A \subseteq V(H)$. By Theorem 1, $\delta(H) = \Delta(H) - 1$. Also by Theorem 2, $|G^*| - |H| \neq 1$. Now $N_{G^*}(A) = V(G^*)$ implies that $V(H) = V(G^*)$ and thus $2n + 1 = |G^*| = |H| = \Delta(H) + 1 = (\Delta - 1) + 1 = \Delta$, which is false. Hence H has four major vertices. Now $N_{G^*}(A) = V(G^*)$ also implies that $V(H) = V(G^*)$. By the induction hypothesis on Δ, $e(H) = n(\Delta - 1) + 1$. Consequently $e(G) \geq e(H) + n = (n(\Delta - 1) + 1) + n = n\Delta + 1$. Since G is Δ-critical, $e(G) \leq n\Delta + 1$. Therefore $e(G) = n\Delta + 1$.

Case 2. $\delta = \Delta - 1$. We shall prove this case by contradiction also. Suppose $e(G) \leq n\Delta$. Then $4\Delta + (2n - 3)(\Delta - 1) = 2e(G) \leq 2n\Delta$, from which it follows that

$$\Delta \text{ is odd and } \Delta \leq 2n - 3 \tag{11}$$

We consider two subcases separately.

Subcase 2.1. G has a minor vertex w such that $d_\Delta(w) = 2$.

Suppose $\Delta \geq n+2$. Let wc, $wd \in E(G)$. By (11), $d(w) = \Delta - 1 \leq 2n-4$. Hence G has a minor vertex x such that $wx \notin E(G)$. Let $G' = G - \{w, x, d\}$. Then $\Delta(G') \geq \Delta - 2$ and $\delta(G') \geq (\Delta - 1) - 3 = \Delta - 4 \geq (n-1) - 1$, by Lemma 8(i), G' has a 1-factor F except when $G' = G_0$. However, when $G' = G_0$, we have $s = n - 2$ and $s = \delta(G_0) = \delta(G') \geq \Delta - 4$, from which it follows that $\Delta = n + 2$. Since w is adjacent to $\Delta - 3$ minor vertices in G, therefore G' has at most $\Delta - 3 = n - 1 = s + 1$ vertices of degree $\Delta - 4 = n - 2 = s$, contradicting the fact that G_0 has $s + 2$ vertices of degree s. Hence $G - x$ has a 1-factor $F \cup \{wd\}$. Now $G^* = G - (F \cup \{wd\})$ is Class 2 and $N_{G^*}(A) = V(G^*)$. Observe that the major vertices of G^* are a, b, c, d, x. Since w is adjacent to only one major vertex c in G^*, we have $\Delta(G^* - w) = \Delta(G^*) = \Delta - 1$. By Lemma 5, $\chi'(G^* - w) = \chi'(G^*)$. Hence by Lemma 3, $G^* - w$ contains a $(\Delta - 1)$-critical subgraph H, which has at most four major vertices a, b, d, x. Let $S = N_{G^*}(w) \backslash A$. Since $d_\Delta(w) = 2$, $|S| = \Delta - 3$.

Suppose $S \cap V(H) \neq \phi$. Then $\delta(H) \leq (\Delta - 2) - 1 = \Delta(H) - 2$. By Theorem 1, H can not have only three major vertices. Hence H has four major vertices. By Lemma 2(iii), we have $\delta(H) = \Delta(H) - 2$. Hence, by the induction hypothesis on Δ, $2e(H) = (|H| - 1)\Delta(H) + 2$. Suppose H has at least two vertices of degree $\Delta(H) - 2$. Then $(|H| - 1)\Delta(H) + 2 = 2e(H) \leq 2(\Delta(H) - 2) + (|H| - 6)(\Delta(H) - 1) + 4\Delta(H)$, from which it follows that $|H| \leq \Delta(H)$, which is false. Hence H has only one vertex of degree $\Delta(H) - 2$. Since $d_\Delta(c) \geq 2$, we have $c \in V(H)$ and thus $A \subset V(H)$. Next, since every vertex in $S \cap V(H)$ is of degree $\Delta(H) - 2$ in $G^* - w$, the only vertex of degree $\Delta(H) - 2$ in H must be a vertex in $S \cap V(H)$. Hence $d_H(c) = \Delta(H) - 1$. Finally $N_{G^*}(A) = V(G^*)$ implies that $N_{G^*-w}(A) = V(G^* - w)$. Since $A \subset V(H)$, we have $V(H) = V(G^* - w)$. Consequently, $|H| = |G^* - w| = 2n$, which contradicts Theorem 2.

Suppose $S \cap V(H) = \phi$. Then $\Delta = \Delta(H) + 1 \leq |H| \leq |G^* - w| - |S| = 2n - (\Delta - 3)$, from which it follows that $\Delta \leq n + 1$, which contradicts the assumption that $\Delta \geq n + 2$.

By (7), it remains to consider the case that $\Delta = n + 1$. Suppose G has t vertices v such that $d_\Delta(v) \geq 3$. Then $3t + 2((2n + 1) - t) \leq 4\Delta = 4(n + 1)$ implies that $t \leq 2$. From this, it also follows that $\delta(G_\Delta) = 2$. Let $a, b, c \in A$ be such that $d_\Delta(a) = 2$ and ab, $ac \in E(G)$. By (11), $|V(G) \backslash (N(a) \cup A)| \geq (2n + 1) - (n + 2) = n - 1 \geq 3$. Now $t \leq 2$ implies that there exists $x \in (V(G) \backslash A)$ satisfying $xa \notin E(G)$ and $d_\Delta(x) = 2$. Let $G' = G - \{x, a, b\}$. Clearly, $\delta(G') \geq \Delta - 4 = (n - 1) - 2$, by Lemma 8, G'

has a 1-factor F except when $G' = G_0$, G'_0 or $3K_3 + K_1$. If $G' = 3K_3 + K_1$, then $(2n+1)-3 = 10$ implies that $n = 6$ and thus $\Delta = 7$, which contradicts the fact that $\Delta(3K_3 + K_1) = 9$. If $G' = G_0$ or G'_0, then $s = n - 2$. Since $ad \notin E(G)$, we have $d_{G'}(d) \geq \Delta - 2 = s + 1$, and so $d \in V(J_s)$. Let $Y = G - V(O_{s+2})$. Suppose $c \in V(O_{s+2})$. Observe that $e_G(v, A\backslash\{c\}) \geq 1$ for any $v \in V(Y)$. Thus $e(Y) \geq s + 2$. Now $(s+1)(\Delta - 1) + \Delta \leq e(V(O_{s+2}), V(Y)) \leq 3\Delta + s(\Delta - 1) - 2e(Y)$ implies that $\Delta \geq 2n - 1$, which contradicts (11). Hence $c \in V(O_{s+2})$. Since $d_\Delta(v) \geq 2$ for any $v \in V(G)$, we have $e(Y) \geq 2s + 2$. Again, $(s+2)(\Delta - 1) \leq e(V(O_{s+2}), V(Y)) \leq 4\Delta + (s-1)(\Delta - 1) - 2e(Y)$ implies that $\Delta \geq 2n + 1$, which is impossible. Consequently, $G - x$ has a 1-factor $F \cup \{ab\}$.

Clearly, $G^* = G - (F \cup \{ab\})$ is Class 2. Since a is adjacent to only one major vertex c in G^*, we have $\Delta(G^* - a) = \Delta(G^*) = \Delta - 1$. By Lemma 5, $\chi'(G^* - a) = \chi'(G^*)$. Hence by Lemma 3, $G^* - a$ contains a $(\Delta - 1)$-critical subgraph H, which has at most three major vertices b, d and x. Since $d_\Delta(c) \geq 2$, $c \in V(H)$. By Theorem 1, x is adjacent to every vertex in H and in particular xb, xc, $xd \in E(H)$. Thus $d_\Delta(x) \geq 3$ in G, which contradicts the fact that $d_\Delta(x) = 2$.

Subcase 2.2. For any minor vertex v of G, $d_\Delta(v) \geq 3$.

By Lemma 2(i), $3((2n+1) - 4) \leq 4(\Delta - 2)$. Hence $4\Delta \geq 6n - 1$. This together with (11) implies that $n \geq 6$ and $\Delta \geq n + 3$. By (11), G has a minor vertex y which is not adjacent to d. Let db, $dc \in E(G)$ and let $G' = G - \{y, d, b\}$. Then $\delta(G') \geq (\Delta - 1) - 3 \geq n - 1$, and thus by Lemma 6, G' has a 1-factor F_1. Now $G'' = G - (F_1 \cup \{db\})$ is Class 2 and having five major vertices a, b, c, d, y of degree $\Delta - 1$. Clearly, $N_{G''}(A) = V(G'')$.

Suppose $d_\Delta(d) = 2$. Then d is adjacent to only one major vertex c in G'' and $\Delta(G'' - d) = \Delta(G'')$. By Lemma 5, $\chi'(G'' - d) = \chi'(G'')$. Hence, by Lemma 3, $G'' - d$ contains a $(\Delta - 1)$-critical subgraph H, which has three major vertices a, b, y. However, since $d_\Delta(y) \geq 3$ and $dy \notin E(G)$, y must be adjacent to a, b, c. Hence, $c \in V(H)$. Again, since $d_\Delta(v) \geq 3$ for every minor vertex v of G, we have $V(H) = V(G'' - d)$ and thus $|H| = |G''| - 1 = 2n$, which contradicts Theorem 1.

Suppose $d_\Delta(d) = 3$. Since $\Delta \geq n + 3 \geq 7$, G has at least another minor vertex z not adjacent to d. Thus $\delta(G'' - \{a, d, z\}) \geq (\Delta - 2) - 3 \geq (n-1) - 1$. By Lemma 8(i), $G'' - \{a, d, z\}$ has a 1-factor F_2 except when $G'' - \{a, d, z\} = G_0$. However, when $G'' - \{a, d, z\} = G_0$, we have $s = n - 2$ and $s = \delta(G_0) = \delta(G'' - \{a, d, z\}) \geq (\Delta - 2) - 3 \geq n - 2$, from which it follows that $\Delta = n + 3$. As z is adjacent to $\Delta - 4$ minor vertices in G, $G'' - \{a, d, z\}$ has at most $\Delta - 4 = n - 1 = s + 1$ vertices of degree $\Delta - 5 = n - 2 = s$, contradicting the fact that G_0 has $s + 2$ vertices of degree s.

Clearly $G^* = G'' - (F_2 \cup \{da\})$ is Class 2 and having six major vertices a, b, c, d, y and z. Morever, since $d_\Delta(v) \geq 3$ for any minor vertex v of G, we have $N_{G^*}(A) = V(G^*)$. As d is adjacent to only one major vertex c in G^* and $\Delta(G^* - d) = \Delta(G^*)$, by Lemma 5, $\chi'(G^* - d) = \chi'(G^*)$. Hence, by Lemma 3, $G^* - d$ contains a $(\Delta - 2)$-critical subgraph H, which has at most four major vertices a, b, y, z. From (11), we know that Δ is odd. Hence by Theorem 1 and Theorem 2, $|H| \neq \Delta(H) + 1 = \Delta - 1$. Thus $|H| \geq \Delta(H)+2$ and H has four major vertices. By the induction hypothesis on Δ, $2e(H) = (|H| - 1)\Delta(H) + 2$. Suppose $\delta(H) \leq \Delta(H) - 2$. Then $(|H| - 1)\Delta(H) + 2 = 2e(H) \leq (\Delta(H) - 2) + (|H| - 5)(\Delta(H) - 1) + 4\Delta(H)$, from which it follows that $|H| \leq \Delta(H) + 1$, which contradicts the fact that $|H| \geq \Delta(H) + 2$. Thus $\delta(H) = \Delta(H) - 1$. Now $d_{G^*-d}(v) \leq \Delta - 4 = \Delta(H) - 2$ for any $v \in (N(d)\backslash A)$ implies that $(N(d)\backslash A) \cap V(H) = \phi$. Thus $\Delta = \Delta(H) + 2 \leq |H| \leq |G^* - d| - |N(d)\backslash A|$, from which it follows that $\Delta \leq n + 1$, contradicting the fact that $\Delta \geq n + 3$. ■

Corollary 4. *Let G be a Δ-critical graph of order $2n+1$ with $|G_\Delta| = 4$. Then either (i) $G \cong (2n-2)^{2n-3}(2n-1)^4$ or (ii) $G \cong (2n-2)(2n-1)^{2n-4}(2n)^4$.*

Proof. Since G is Δ-critical, by Lemma 2(iii), we have $\delta \geq \Delta - 2$. Now we want to show that G has at most one vertex of degree $\Delta - 2$. Suppose G has at least two vertices of degree $\Delta - 2$. Then by Theorem 3, $2(n\Delta + 1) = 2e(G) = \sum_{v \in V(G)} d_G(v) \leq 2(\Delta - 2) + ((2n+1) - 6)(\Delta - 1) + 4\Delta = 2n\Delta + \Delta - 2n + 1$, from which it follows that $\Delta \geq 2n + 1$, which is false. Hence

$$\text{(i) } G \cong (\Delta - 1)^{2n-3}\Delta^4 \text{ or (ii) } G \cong (\Delta - 2)(\Delta - 1)^{2n-4}\Delta^4$$

By Theorem 3 again, we have

$$\text{(i) } G \cong (2n-2)^{2n-3}(2n-1)^4 \text{ or (ii) } G \cong (2n-2)(2n-1)^{2n-4}(2n)^4.$$ ■

Theorem 5 [4]. *Let G be a connected graph and $\Delta = \Delta(G)$. Suppose $|G_\Delta| = 4$. Then G is Class 2 if and only if, for some n, either (i) $G \cong (2n-2)^{2n-3}(2n-1)^4$, or (ii) $G \cong (2n-2)(2n-1)^{2n-4}(2n)^4$, or (iii) G contains a cut-edge e such that $G - e$ is the union of two disjoint graphs G_1 and G_2, where G_1 is Δ-critical and satisfies $G_1 \cong (2m-1)^{2m-2}(2m)^3$ or $G_1 \cong (2m-2)(2m-1)^{2m-4}(2m)^4$.*

Proof. Sufficiency. If (i) or (ii) holds, then $e(G) = n\Delta + 1 > \lfloor\frac{|G|}{2}\rfloor\Delta$. If (iii) holds, then $e(G_1) = 2m^2 + 1 > \lfloor\frac{|G_2|}{2}\rfloor\Delta$. In either case G is Class 2.

Necessity. Suppose G is Class 2. If G is Δ-critical, then by Corollary 4, (i) or (ii) holds. Suppose G is not critical. Then by Lemma 3, G contains a Δ-critical subgraph G_1, which has at most four major vertices. If G_1 has three major vertices, then by Theorem 1, $G_1 \cong (2m-1)^{2m-2}(2m)^3$

for some m. Since $\Delta(G) = \Delta(G_1) = 2m$, $\delta(G_1) = 2m - 1$ and G has four major vertices, G has exactly one edge e joining G_1 with $G - V(G_1)$. Since G is connected, $G_2 = G - V(G_1)$ must also be connected. Thus e is a cut-edge of G and the end vertex of e in G_1 is a major vertex of G.

Suppose G_1 has four major vertices. Then by Corollary 4, $G_1 \cong (2m-2)(2m-1)^{2m-4}(2m)^4$ for some m. Thus $G_2 = G - V(G_1)$ is joined to G_2 by exactly one edge (e say). ∎

Final Remarks. We are writing a paper on Δ-critical graphs G having $|G_\Delta| = 5$.

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