The Equitable Edge-colouring of Outerplanar Graphs

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Abstract. An edge-colouring of a graph G is *equitable* if, for each vertex v of G, the number of edges of any one colour incident with v differs from the number of edges of any other colour incident with v by at most one. In the paper, we prove that any outerplanar graph has an equitable edge-colouring with k colours for any integer $k \geq 3$.

1 Introduction

Throughout the paper, all graphs are finite, simple and undirected. Let G be a graph. $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. $N_G(v)$ denotes the set of vertices adjacent to a vertex v. A vertex v is called a k-vertex if $|N_G(v)| = k$. An *odd cycle* is a cycle in which the number of edges is odd.

An *edge-colouring* of G is an assignment of colours to the edges of G. An edge-colouring with two colours will be called a *bicolouring*. Given an edge-colouring of G with k colours $1, 2, \dots, k$, for $v \in V(G)$, let $c_i(v)$ denote the number of edges incident with v coloured i. An edge-colouring of G with k colours $1, 2, \dots, k$ is called *equitable* if for each $v \in V(G)$,

$$|c_i(v) - c_j(v)| \leq 1 \qquad (1 \leq i < j \leq k).$$

A graph G is called *equitable* if G has an equitable edge-colouring with k colours for any integer $k \geq 1$. It is an interesting problem to determine whether a graph is equitable or not. Hilton and Werra [1] have proved that if k does not divide $d(v)$ for all vertex $v \in V(G)$, then G has an equitable edge-colouring with k colours. Werra [3] has proved that all bipartite graphs are equitable. In the paper, we consider the equitable edge-colouring of outerplanar graphs and obtain the following result.

Theorem. *A connected outerplanar graph is equitable if and only if it is not an odd cycle.*

The theorem implies that the edge chromatic number of any connected outerplanar graph G is $\Delta(G)+1$ if and only if G is an odd cycle.

2 Proof of Theorem

Lemma 2.1. *Let G be a 2-connected outerplanar graph of order at least 5. Then one of the following conditions holds:*

(1) *G has two adjacent 2-vertices u and v;*

(2) *G has a 2-vertex u such that $N(u) = \{v, w\}$, $N(v) = \{u, w, x\}$, $N(w) = \{u, w, y\}$ and $xy \notin E(G)$;*

(3) *G has a 2-vertex u such that $N(u) = \{v, w\}$, $N(w) = \{u, v, x\}$, $vx \notin E(G)$ and $d(x) = 2$;*

(4) *G has two nonadjacent 2-vertices u and v such that $N(u) = \{x_1, x_2\}$, $N(v) = \{y_1, y_2\}$, $N(x_1) = \{u, x_2, y_1\}$, $N(y_1) = \{v, x_1, y_2\}$ and $x_2y_2 \in E(G)$;*

(5) *G has a 2-vertex u such that $N(u) = \{v, w\}$, $vw \in E(G)$ and $4 \leq d(v) \not\equiv 0(mod\,3)$;*

(6) *G has a 2-vertex u such that $N(u) = \{v, w\}$, $N(w) = \{u, v, x\}$, $xv \in E(G)$, $6 \leq d(v) \equiv 0(mod 3)$ and $3 \leq d(x) \leq 4$;*

(7) *G has a 6-vertex w such that $N(w) = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $d(w_1) = d(w_4) = 2$, $d(w_2) = d(w_5) = 3$ and $w_1w_2, w_2w_3, w_3w_6, w_4w_5, w_5w_6 \in E(G)$.*

Proof: At first, let us give a few definitions. Let $G = (V, E, F)$ be a plane graph. A face which is not the unbounded face is called an *interior face*. An *interior edge* is an edge incident with two interior faces. An *exterior edge* is an edge incident with the unbounded face of G. The *degree* $d(f)$ of a face f is the number of vertices incident with f. A *k-face* is the face of degree k. $N_f(e)$ denotes the set of faces incident with an edge e. An interior face f is called *pendant* if f is adjacent to at most one interior face. A sequence $f_1f_2\ldots f_n$ is an *interior face sequence* if $f_1, f_2, \ldots, f_n$ are interior faces of G and f_i is adjacent to f_{i+1} for $i = 1, 2, \ldots n-1$.

Let G be a 2-connected outerplane graph of order at least 5 which is a counter-example. If $\Delta(G) = 2$, G is a cycle. (1) holds. If G has a pendant interior face f such that $d(f) \geq 4$, then there must be an exterior edge

$uv \in E(G)$ having $d(u) = d(v) = 2$, which implies (1). So $\Delta(G) \geq 3$ and any pendant face is a 3-face.

Let $f_1 f_2 \ldots f_l$ be a longest interior face sequence of G. Then f_1 and f_l are pendant interior faces, that is, 3-faces. Since $\Delta(G) \geq 3$, $l \geq 2$. If $l = 2$, then $|G| = 4 < 5$. So $l \geq 3$. Let the vertices incident with f_2 be $v_1, v_2, \ldots, v_m$, where $v_1v_2, \ldots, v_{m-1}v_m, v_mv_1 \in E(G)$, $N_f(v_1v_m) = \{f_2, f_3\}$ and $N_f(v_av_{a+1}) = \{f_1, f_2\}$ for some $a (1 < a < m)$. Let $u \in N(v_a) \cap N(v_{a+1})$ and u be incident with f_1. Then $d(u) = 2$. If $v_{a-1}v_a$ is an interior edge of G, then $d(v_a) = 4$, (5) holds. So $v_{a-1}v_a$ is an exterior edge of G. Suppose that $a + 1 < m$. If $v_{a+1}v_{a+2}$ is an interior edge of G, then (5) holds; otherwise, $d(v_a) = d(v_{a+1}) = 3$, (2) holds. So $a + 1 = m$, that is to say, $v_{a+1} = v_m$. Suppose that $m \geq 5$. If $x_{a-2}x_{a-1}$ is an exterior edge of G, then $d(v_{a-1}) = 2$, (3) holds; otherwise, (2) or (5) holds. Suppose that $m = 4$. If $x_{a-2}x_{a-1}$ is an exterior edge of G, then (3) holds; otherwise, (4) holds. So $m = 3$. Note that $d(v_{a+1}) \geq 4$, for otherwise, $d(v_{a+1}) = 3$ and $m \geq 4$, (2) holds. Hence, Combining these results, we have that $m = 3$, $d(v_{a+1}) \geq 4$, f_1 and f_2 are 3-faces, and v_1v_2 is an exterior edge of$G$.

If $4 \leq d(v_3) \not\equiv 0 (mod 3)$, (5) holds. So we have $6 \leq d(v_3) \equiv 0 (mod 3)$. Since $d(v_3) \geq 6$, $l \geq 5$. Let the vertices incident with the face f_3 be $x_1, x_2, \ldots, x_n$, where $x_1x_2, \ldots, x_{n-1}x_n, x_nx_1 \in E(G)$, $N_f(x_1x_n) = \{f_3, f_4\}$, $x_b = v_1$ and $x_{b+1} = v_3$ for some $b (1 \leq b < n)$. Suppose that $b > 1$. Then $3 \leq d(v_1) \leq 5$. If $3 \leq d(v_1) \leq 4$, (6) holds. If $d(v_1) = 5$, let $\{f'_2, f_3\} = N_f(x_{b-1}x_b)$ and $\{f'_1, f'_2\} = N_f(v_1v'_2)$, where $v'_2 \in N(v_1) \backslash \{x_{b-1}, v_2, v_3\}$, then $f'_1 f'_2 f_3 \ldots f_l$ is also a longest interior sequence. So f'_1 and f'_2 are the 3-faces, (5) holds. Suppose that $b = 1$. It is easy to prove that $d(v_3) = 6$. Let $\{f''_2, f_3\} = N_f(x_{b+1}x_{b+2})$ and $\{f''_1, f''_2\} = N_f(v_3v''_2)$, where $v''_2 \in N(v_3) \backslash \{x_{b+2}, u, v_2, v_1\}$. Then $f''_1 f''_2 f_3 \ldots f_l$ is also a longest interior sequence. So f''_1 and f''_2 are the 3-faces. If $n = 3$, (7) holds; otherwise, by the same discussion as the case $b > 1$, (5) or (6) holds.

Hence, this contradicts the hypothesis and we complete the proof of the lemma. □

Lemma 2.2. *Let G be a 2-connected outerplanar graph of order at least 3. Then one of the following conditions holds:*

(1) *G has two adjacent 2-vertices u and v;*

(2) *G has a 2-vertex u adjacent to a 3-vertex v such that $N(u) \subset N(v)$;*

(3) *G has two nonadjacent 2-vertices u and v adjacent to a common 4-vertex w such that $(N(u) \cup N(v)) \backslash \{w\} = N(w) \backslash \{u, v\}$.*

The lemma is similar with lemma 2.2. Its proof omit here.

Proof of Theorem: In [1] and [3], it was proved that any connected graph G has an equitable bicolouring if and only if G is not odd cycle. It is easy to prove that if any 2-connected maximal subgraph of a graph G is equitable, G is equitable. So in the following, it is only to prove that any 2-connected outerplanar graph G has an equitable edge-colouring with k colours $1, 2, \ldots, k$ for any integer $k \geq 3$. We shall prove the result by induction on $|G|$, the number of vertices of G.

When $|G| \leq 4$, the result is obvious. Now assume that $p \geq 5$ is an integer and that the theorem holds for all 2-connected outerplanar graphs with less than p vertices, and let G be a 2-connected outerplanar graph of order p. In the following, we shall obtain a 2-connected outerplanar graph G^* of order less than p. By the induction hypothesis, G^* has an equitable edge-colouring φ with k colours. On the basis of φ, we shall construct an equitable edge-colouring σ of G using the same set of k colours. To save space, we only give in the following the construction of G^* and the colouring of some edges of G. Any uncoloured edge of G is coloured the same colour as in φ of G^*. For each vertex $v \in V(G^*)$, let $c_i(\varphi, v) = |\{vv' \in E(G^*) \mid \varphi(vv') = i\}|$ and $C_\varphi(v) = \{i \mid c_i(\varphi, v) = \min\limits_{1\leq j\leq k} c_j(\varphi, v)\}$. Then $|C_\varphi(v)| \geq 1$.

Case 1 $k = 3$.

According to Lemma 2.1, the proof can be divided into the following seven subcases.

Subcase 1.1 G has two adjacent 2-vertices u and v.

Let $w_1 \in N(u)\backslash v$ and $w_2 \in N(v)\backslash u$. Let $G^* = G - u + vw_1$ and let

$$\sigma(uw_1) = \varphi(vw_1), \quad \sigma(uv) \in \{1,2,3\}\backslash\{\sigma(uw_1), \varphi(vw_2)\}.$$

Subcase 1.2 G has a 2-vertex u such that $N(u) = \{v, w\}$, $N(v) = \{u, w, x\}$, $N(w) = \{u, w, y\}$ and $xy \notin E(G)$.

Let $G^* = G - \{u, v\} + wx$. Then $\varphi(wx) \neq \varphi(wy)$. Let

$$\begin{aligned}\sigma(wy) &= \sigma(uv) = \varphi(wy),\\ \sigma(uw) &= \sigma(vx) = \varphi(wx),\\ \sigma(vw) &\in \{1,2,3\}\backslash\{\sigma(uv), \sigma(uw)\}.\end{aligned}$$

Subcase 1.3 G has a 2-vertex u such that $N(u) = \{v, w\}$, $N(w) = \{u, v, x\}$, $vx \notin E(G)$ and $d(x) = 2$.

Let $G^* = G - \{u, w\} + vx$. Then $d_{G^*}(x) = 2$, that is to say, $|C_\varphi(x)| = 1$. Let $\{\alpha_1\} = C_\varphi(x)$. Then $\alpha_1 \neq \varphi(vx)$. Let $\alpha_2 \in C_\varphi(v)$.

Subcase 1.3.1 $\alpha_2 \neq \varphi(vx)$. Let

$$\sigma(uv) = \sigma(wx) = \varphi(vx),\ \sigma(vw) = \alpha_2,\ \sigma(uw) \in \{1,2,3\}\backslash\{\sigma(uv), \sigma(vw)\}.$$

Subcase 1.3.2 $\alpha_2 = \varphi(vx)$. Let

$$\sigma(uv) = \sigma(vw) = \alpha_2,\ \sigma(wx) = \alpha_1,\ \sigma(uw) \in \{1,2,3\}\backslash\{\sigma(uv), \sigma(vw)\}.$$

Subcase 1.4 G has two nonadjacent 2-vertices u and v such that $N(u) = \{x_1, x_2\}$, $N(v) = \{y_1, y_2\}$, $N(x_1) = \{u, x_2, y_1\}$, $N(y_1) = \{v, x_1, y_2\}$ and $x_2y_2 \in E(G)$.

Let $G^* = G - \{u, v, x_1, y_1\}$. Let $\alpha_1 \in C_\varphi(x_2)$, $\alpha_2 \in C_\varphi(y_2)$,

$$\alpha_3 \in \begin{cases} C_\varphi(x_2)\backslash\alpha_1, & \text{if } |C_\varphi(x_2)| \geq 2, \\ \{1,2,3\}\backslash C_\varphi(x_2), & \text{otherwise.} \end{cases}$$

and

$$\alpha_4 = \begin{cases} C_\varphi(y_2)\backslash\alpha_2, & \text{if } |C_\varphi(y_2)| \geq 2, \\ \{1,2,3\}\backslash C_\varphi(y_2), & \text{otherwise.} \end{cases}$$

Then $\{\alpha_1, \alpha_3\} \cap \{\alpha_2, \alpha_4\} \neq \emptyset$. Without loss of generality, assume that $\alpha_1 = \alpha_2$. Let

$$\sigma(ux_2) = \sigma(x_1y_1) = \sigma(vy_2) \in \alpha_1,\ \sigma(x_1x_2) = \alpha_3,\ \sigma(y_1y_2) = \alpha_4,$$
$$\sigma(ux_1) \in \{1,2,3\}\backslash\{\alpha_1, \alpha_3\},\ \sigma(vy_1) \in \{1,2,3\}\backslash\{\alpha_2, \alpha_4\}.$$

Subcase 1.5 G has a 2-vertex u such that $N(u) = \{v, w\}$, $vw \in E(G)$ and $4 \leq d(v) \not\equiv 0(mod\,3)$.

Let $G^* = G - u$. Then $d_{G^*}(v) \not\equiv 2(mod\,3)$, that is to say, $|C_\varphi(v)| \geq 2$. So let

$$\sigma(uw) \in C_\varphi(w),\ \sigma(uv) \in C_\varphi(v)\backslash\sigma(uv).$$

Subcase 1.6 G has a 2-vertex u such that $N(u) = \{v, w\}$, $N(w) = \{u, v, x\}$, $xv \in E(G)$, $6 \leq d(v) \equiv 0(mod\,3)$, and $3 \leq d(x) \leq 4$.

Subcase 1.6.1 $d(x) = 3$. Let $y \in N(x)\backslash\{w, v\}$.

Subcase 1.6.1.1 $vy \in E(G)$.

Let $G^* = G - \{u, w, x\}$. Since $d_{G^*}(v) \equiv 0(mod\,3)$, $|C_\varphi(v)| = 3$. Let

$$\sigma(xy) = \sigma(wv) \in C_\varphi(y),\ \sigma(uw) = \sigma(vx) \in \{1,2,3\}\backslash\sigma(xy),$$
$$\sigma(wx) = \sigma(uv) \in \{1,2,3\}\backslash\{\sigma(uw), \sigma(wv)\}.$$

Subcase 1.6.1.2 $vy \notin E(G)$.

Let $G^* = G - \{u, x, w\} + vy$. Then $d_{G^*}(v) \equiv 1(mod\,3)$. So $|C_\varphi(v)| = 2$. Let

$$\sigma(xy) = \sigma(wv) = \varphi(vy),\ \sigma(uw) = \sigma(vx) \in C_\varphi(v)\backslash\sigma(xy),$$
$$\sigma(uv) \in C_\varphi(v)\backslash\sigma(vx),\ \sigma(wx) \in \{1,2,3\}\backslash\{\sigma(uw), \sigma(wv)\}.$$

Subcase 1.6.2 $d(x) = 4$.

Let $G^* = G - \{u, w\}$. Then $d_{G^*}(v) \equiv 1(mod 3)$ and $d_{G^*}(x) = 3$. So $|C_\varphi(v)| = 2$ and $|C_\varphi(x)| = 3$. Let

$$\sigma(uv) = \sigma(wx) \in C_\varphi(v),\ \sigma(vw) \in C_\varphi(v)\backslash\sigma(uv),$$
$$\sigma(uw) \in \{1,2,3\}\backslash\{\sigma(uv), \sigma(wv)\}.$$

Subcase 1.7 G has a 6-vertex w such that $N(w) = \{w_1, w_2, w_3, w_4, w_5, w_6\}$, $d(w_1) = d(w_4) = 2$, $d(w_2) = d(w_5) = 3$, and $w_1w_2, w_2w_3, w_3w_6, w_4w_5, w_5w_6 \in E(G)$.

Let $G^* = G - \{w, w_1, w_2, w_4, w_5\}$. Let $\alpha_1 \in C_\varphi(w_3)$, $\alpha_2 \in C_\varphi(w_6)$,

$$\alpha_3 \in \begin{cases} C_\varphi(w_3)\backslash\alpha_1, & \text{if } |C_\varphi(w_3)| \geq 2, \\ \{1,2,3\}\backslash C_\varphi(w_3), & \text{otherwise.} \end{cases}$$

and

$$\alpha_4 = \begin{cases} C_\varphi(w_6)\backslash\alpha_2, & \text{if } |C_\varphi(w_6)| \geq 2, \\ \{1,2,3\}\backslash C_\varphi(w_6), & \text{otherwise.} \end{cases}$$

Then $\alpha_1 \neq \alpha_3$ and $\alpha_2 \neq \alpha_4$. Let $\sigma(ww_3) = \sigma(w_1w_2) = \alpha_1$, $\sigma(ww_1) = \sigma(w_2w_3) = \alpha_2$, $\sigma(ww_6) = \sigma(w_4w_5) = \alpha_3$, $\sigma(ww_4) = \sigma(w_5w_6) = \alpha_4$, $\sigma(ww_2) \in \{1,2,3\}\backslash\{\alpha_1, \alpha_2)\}$, $\sigma(ww_5) \in \{1,2,3\}\backslash\{\alpha_3, \alpha_4)\}$.

Case 2 $k \geq 4$.

According to Lemma 2.2, the proof can be divided into the following three subcases.

Subcase 2.1 G has two adjacent 2-vertices u and v.

The subcase is similar to Subcase 1.1.

subcase 2.2 G has a 2-vertex u adjacent to a 3-vertex v such that $N(u) \subset N(v)$.

Let $\{w\} = N(u)\backslash v$ and $\{x\} = N(v)\backslash\{u, w\}$. Let $G^* = G - u$. Then $d_{G^*}(v) = 2$. So $|C_\varphi(v)| \geq 2$. Let

$$\sigma(uw) \in C_\varphi(w),\ \sigma(uv) \in C_\varphi(v)\backslash\sigma(uw).$$

Subcase 2.3 G has two nonadjacent 2-vertices u and v adjacent to a common 4-vertex w such that $(N(u) \cup N(v))\backslash\{w\} = N(w)\backslash\{u, v\}$.

Let $\{x\} = N(u)\backslash w$ and $\{y\} = N(v)\backslash w$. Let $G^* = G - \{u, v\}$.

Subcase 2.3.1 $C_\varphi(x) \cap C_\varphi(y) \neq \emptyset$. Let

$$\sigma(vy) = \varphi(wy),\ \sigma(ux) = \sigma(wy) \in C_\varphi(x) \cap C_\varphi(y),$$
$$\sigma(vw) \in \{1,2,3,4\}\backslash\{\varphi(wx), \sigma(wy), \sigma(vy)\},$$
$$\sigma(uw) \in \{1,2,3,4\}\backslash\{\varphi(wx), \sigma(wy), \sigma(wv)\}.$$

Subcase 2.3.2 $C_\varphi(x) \cap C_\varphi(y) = \emptyset$. Let

$$\sigma(ux) \in C_\varphi(x),\ \ \sigma(vy) \in C_\varphi(y),$$
$$\sigma(uw) \begin{cases} = \sigma(vy), & \text{if } \sigma(vy) \notin \{\varphi(wx), \varphi(wy)\}, \\ \in \{1,2,3,4\} \backslash \{\sigma(ux), \varphi(wx), \varphi(wy)\}, & \text{otherwise.} \end{cases}$$
$$\sigma(wv) \in \{1,2,3,4\} \backslash \{\varphi(wx), \varphi(wy), \sigma(uw)\}.$$

From all above cases, it is easy to verify that σ is an equitable edge-colouring of G with k colours. Hence the theorem is true. □

Acknowledgment. The author is very grateful to the referee for his detailed suggestions and corrections which have greatly contributed to this final version. This work was partially supported by National Natural Science Foundation of China.

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