Some friends of Alltop's designs $4-(2^f+1,5,5)$

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Abstract

We construct several families of simple 4-designs, which are closely related to Alltop's series with parameters $4-(2^f+1,5,5)$, f odd. More precisely, for every $q=2^f$, where $gcd(f,6)=1, f\geq 5$ we construct designs with the following parameters:

$$4-(q+1,6,\lambda),\ \text{where } \lambda\in\{60,70,90,100,150,160\},$$

$$4-(q+1,8,35),$$

and

$$4-(q+1,9,\lambda),\ \text{where } \lambda\in\{63,147\}.$$

1 Introduction

We continue with the construction of 4-designs defined on projective lines in characteristic 2. For earlier results see [2, 3, 4, 5]. Alltop's series with parameters $4-(2^f+1,5,5)$ (see [1]) is used as a starting-point. Our new families of 4-designs with $k\in\{6,8,9\}$ and constant λ are closely related to Alltop's family.
We start with a description of Alltop's family of designs, which is different from the one given in [1]):

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1.1 A description of Alltop's family

Let $q = 2^f, gcd(f, 6) = 1, f \geq 5$. Denote by $\mathcal{P}_1(q)$ the projective line and by $G = PGL_2(q)$ the projective group in its sharply 3-transitive action on $\mathcal{P}_1(q)$. For each subset $X \subset \mathcal{P}_1(q)$ denote by $G(X)$ the stabilizer of X in G. For each 4-subset $S \subset \mathcal{P}_1(q)$ we have that $G(S)$ is a four-group (an elementary abelian group of order 4). In fact, it is obvious that $S = \{\infty, 0, 1, a\}$ is stabilized by the four-group V generated by $(\tau \mapsto a/\tau)$ and by $(\tau \mapsto (\tau + a)/(\tau + 1))$. We use [3], Lemma 1 in the following form:

Lemma 1 *Let $q = 2^f$. The stabilizer in G of a k-subset of $\mathcal{P}_1(q)$ divides $k(k-1)(k-2)$.*

As elements of order 3 are fixed-point-free on $\mathcal{P}_1(q)$ and the Sylow-2-subgroups of G are abelian, we conclude that the stabilizer of every 4-subset S is indeed a four-group V. Moreover V operates regularly on S and has precisely one fixed point in $\mathcal{P}_1(q)$.
Let $F \subset \mathcal{P}_1(q)$ be a 5-subset. As $|G|$ is not divisible by 5 and elements of order 3 are fixed-point-free we conclude from Lemma 1 that $|G(F)|$ divides 4. It is easily seen that only two cases occur: either $G(F)$ is trivial or $G(F)$ is a four-group. Define a design $\mathcal{B}_0$ on $\mathcal{P}_1(q)$, whose blocks are the 5-sets with a nontrivial stabilizer. As each such 5-set can be written in a unique way as the union of a 4-set S and the fixed point of $G(S)$ we see that the number of blocks is $b(\mathcal{B}_0) = \binom{q+1}{4}$. This is an equivalent description of Alltops's design from [1]. We claim that the parameters are $4 - (q + 1, 5, 5)$. Indeed, let $S = \{a, b, c, d\}$ be given. We have to count the elements $x \in \mathcal{P}_1(q) - S$ such that the 5-set $F = S \cup \{x\}$ has a nontrivial stabilizer. This stabilizer is then a four-group V. We see that the fixed point of V must be an element of F. If the fixed point is x, then $V = G(S)$ and x is the fixed point of V. So assume the fixed point of V is in S. As $G(S)$ is transitive on S we may assume a is this fixed point. Then V must contain an element σ achieving the following operation: $\sigma(a) = a, \sigma : b \longleftrightarrow c, d \longleftrightarrow x$. Because of the sharp triple transitivity of G we see that an involution σ is uniquely determined by the first three operations. This shows that x is uniquely determined as image of d under σ. We see that the number of blocks containing S is indeed $\lambda = 1 + 4 \cdot 1 = 5$.

1.2 Description of results

Let $q = 2^f, gcd(f, 6) = 1, f \geq 5$, and S an arbitrary 4-subset of $\mathcal{P}_1(q)$. Define $T(S)$ as the 5-set of those points which complement S to a block in Alltop's design. Write

$$T(S) = \{T_*(S)\} \cup T_0(S), \tag{1}$$

where $T_*(S)$ is the fixed-point of $G(S)$. This defines the 4-set $T_0(S)$. We show that the operator T_0 is involutorial, i.e.

$$T_0(T_0(S)) = S \text{ for every 4-set } S \text{ (see Lemma 5).} \tag{2}$$

If we define blocks of a design $\mathcal{B}_1$ to be the 8-sets of the shape

$$S \cup T_0(S) \ (S \text{ an arbitrary 4-set}), \tag{3}$$

we get a 4-design with $k = 8, \lambda = 35$ (see Theorem 1).
If we take as blocks the sets $S \cup T(S)$, we get a 4-design $\mathcal{B}_2$ with $k = 9, \lambda = 63$ (Theorem 2). This last design is blockwise disjoint from the design $4-(q+1, 9, 84)$ as defined in [3], so the union of the two designs yields a 4-design with $k = 9, \lambda = 147$.
Another approach yields 4-designs with $k = 6$: Define the blocks of $\mathcal{B}_3$ to be the sets of the shape

$$S \cup \{T_*(S)\} \cup \{Q\}, \tag{4}$$

where $Q \in T_0(S)$. Then $\mathcal{B}_3$ is a 4-design with $\lambda = 60$ (see Theorem 3). If blocks are defined to be the sets of the shape

$$S \cup \{Q, Q'\}, \text{ where } \{Q, Q'\} \subset T_0(S), \tag{5}$$

we get a 4-design $\mathcal{B}_4$ with $\lambda = 90$ (see Theorem 3).
These designs are blockwise disjoint. Both are disjoint from the design with $k = 6, \lambda = 10$ as constructed in [2]. As a Corollary we get designs with parameters $4-(2^f+1, 6, \lambda)$ for every $\lambda \in \{10, 60, 70, 90, 100, 150, 160\}$. It turns out that the necessary and sufficient condition for our construction to work is $gcd(f, 6) = 1$ in each case. The full automorphism group of our designs is $P\Gamma L_2(q)$.
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2 Constructions and proofs

We fix notation: $q = 2^f, gcd(f, 6) = 1, G = PGL_2(q)$. The fact that elements in $\mathbb{F}_q - \{0, 1\}$ cannot satisfy polynomial equations with coefficients 0 or 1 of degree smaller than 5 will be freely used in the sequel. We will make frequent use of the following property of the trace:

Lemma 2 *Let $q = 2^f, tr : \mathbb{F}_q \longrightarrow \mathbb{F}_2$ the* **trace**. *The quadratic equation $x^2 + x + a = 0$, where $a \in \mathbb{F}_q$, has no solution if $tr(a) = 1$, it has two solutions if $tr(a) = 0$.*

For every 4-subset S of the projective line $\mathcal{P}_1(q)$ consider $T_*(S)$ and $T_0(S)$ as defined in the Introduction. Because of the triple transitivity of G we can choose without restriction

$$S = S_a = \{\infty, 0, 1, a\}, \text{ where } a \in \mathbb{F}_q - \mathbb{F}_2.$$

Recall that $V = G(S) =< \sigma_1, \sigma_2 >$ is a four-group, where

$$\sigma_1 = (\tau \longrightarrow a/\tau), \sigma_2 = (\tau \longrightarrow (\tau + a)/(\tau + 1)).$$

Lemma 3 *If* $S_a = \{\infty, 0, 1, a\}$, *then*

$$T_*(S_a) = \sqrt{a}, T_0(S_a) = \{a + 1, a/(a + 1), 1/a, a^2\}.$$

Proof: Clearly $\sqrt{a}$ is the fixed-point of V. As the stabilizer of $\{\infty, 0, 1, a, a+1\}$ is nontrivial, this set is a block of Alltop's design $\mathcal{B}_0$. Thus $a + 1 \in T_0(S_a)$. The operation of V produces $T_0(S_a)$. ∎

We denote $T_0(S_a)$ by $T(a)$.

Lemma 4 *Let H be the stabilizer of $S \cup T_0(S)$, where S is any 4-subset of $\mathcal{P}_1(q)$. Then H is elementary abelian of order 8.*

Proof: Let $S = S_a$. Clearly $V \subseteq H$. Let $\sigma_3 = (\tau \longrightarrow \frac{(a+1)\tau+a}{\tau+a+1})$. Then $H \supseteq< \sigma_1, \sigma_2, \sigma_3 >$, and the latter group is elementary abelian of order 8. By Lemma 1 $|H|$ divides $8 \cdot 7 \cdot 6$. As f is odd, elements of order 3 are fixedpointfree on the projective line and cannot be present in H. As f is not a multiple of 3, there is no element of order 7 in G. Thus H is a 2-group. Because of the sharp triple transitivity of G, every involution of H is fixedpointfree on $S_a \cup T(a)$. Thus the order of H cannot exceed 8.∎

Lemma 5 $T_*(T_0(S)) = T_*(S), T_0(T_0(S)) = S$ *for every 4-subset* $S \subset \mathcal{P}_1(q)$.

Proof: The operation of σ_3 shows that $T(a) \cup \{\infty\}$ is a block of Alltop's design, i.e. $\infty \in T_0(T(a))$. The Lemma follows.∎

Lemma 6 *If $B = S \cup T_0(S) = R \cup T_0(R)$ for 4-subsets S and R of $\mathcal{P}_1(q)$, then either $R = S$ or $R = T_0(S)$.*

Proof: Let $S = S_a$. Assume first R intersects S_a or $T(a)$ in three points. We can assume $R = \{\infty, 0, 1, t\} = S_t$, where $t \in T(a)$.

- If $t = a + 1$, then $(a + 1)/a \in T(a + 1), \notin B$.

- If $t = a/(a+1)$, then $1/(a+1) \in T(t), \notin B$.
- If $t = 1/a$, then $(a+1)/a \in T(1/a), \notin B$.
- If $t = a^2$, then $a^2 + 1 \in T(t), \notin B$,

contradiction in each case. We can assume $| R \cap S_a |=| R \cap T(a) |= 2$, and $\infty \in R \cap S_a$.

case 1: $R \cap S_a = \{\infty, 0\}$. As we have σ_1 at our disposition, only four subcases arise:

1. $R = \{\infty, 0, a+1, a/(a+1)\}$. Then $1 \in T_0(R)$. The operation $\tau \mapsto \tau/(a+1)$ shows $1/(a+1) \in T(a/(a^2+1)) = \{(a^2+a+1)/(a^2+1), a/(a^2+a+1), (a^2+1)/a, a^2/(a^4+1)\}$, which is not the case.
2. $R = \{\infty, 0, a+1, 1/a\}$. This time we get $1/(a+1) \in T(a^{-1}(a+1)^{-1})$ via the same operation, contradiction.
3. $R = \{\infty, 0, a+1, a^2\}$. The same procedure yields $1/(a+1) \in T(a^2/(a+1))$, which is not true.
4. $R = \{\infty, 0, 1/a, a^2\}$. Then $1 \in T_0(R)$. The mapping $\tau \mapsto a\tau$ shows $a \in T(a^3) = \{a^3+1, a^3/(a^3+1), a^{-3}, a^6\}$. We get a contradiction again. Observe that $a^5 \neq 1$ as f is odd.

The remaining cases $R \cap S_a = \{\infty, 1\}$ and $R \cap S_a = \{\infty, a\}$ are handled in an analogous fashion.■

Definition 1 *Let $gcd(f, 6) = 1$. We define the blocks of designs $\mathcal{B}_1$ and $\mathcal{B}_2$ to be the sets of the form $S \cup T_0(S)$ and $S \cup T(S)$, respectively.*

It follows from Lemma 6 that $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same number $\binom{q+1}{2}/2$ of blocks. We will show that both are 4-designs. A counting argument shows that necessarily then $\lambda(\mathcal{B}_1) = 35, \lambda(\mathcal{B}_2) = 63$.

Theorem 1 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. Then $\mathcal{B}_1$ is a design $4-(q+1, 8, 35)$.*

Proof: We count the blocks $B = U \cup T_0(U)$ of $\mathcal{B}_1$ containing S_a. As U and $T_0(U)$ occur in symmetric position in B we can assume $|S_a \cap U| \geq 2$.

1. Let $U = S_a$. Then $B = S_a \cup T(a)$ is uniquely determined.
2. If $|U \cap S_a| = 3$, then without restriction $U \cap S_a = \{\infty, 0, 1\}$. Thus $U = S_x$ (where $x \neq a$) and $a \in T(x)$. For each $y \in T(x)$, the equation $a = y$ has a unique solution x. We count 16 blocks under case 2.

3. Let $|U \cap S_a| = 2$. Because of the symmetric position of U and $T_0(U)$, we can assume $\infty \in U$. We get three possibilities:

   - $S_a \cap U = \{\infty, 0\}, S_a \cap T_0(U) = \{1, a\}$. We have to find x, y such that $\{1, a\} \subset T_0(\{\infty, 0, x, y\})$. The operation $\tau \mapsto \tau/x$ shows that this is equivalent with $\{1/x, a/x\} \subset T(y/x) = \{(y+x)/x, y/(y+x), x/y, y^2/x^2\}$. Every ordered pair of elements in $T(y/x)$ yields a system of two equations for x, y : We have to consider these 12 equations in turn. Let us number them as $\alpha_i, i = 1, 2, \ldots, 12$.

$$\left.\begin{array}{rcl} 1/x & = & (y+x)/x \\ a/x & = & y/(y+x) \end{array}\right\} \quad (\alpha_1)$$

     This is equivalent with the quadratic equation $x^2 + x + a = 0$. It has two solutions if $tr(a) = 0$, no solution if $tr(a) = 1$ (see Lemma 2). The situation is paradigmatic for all our proofs. The solvability of most of our equations depends on a trace-condition.

$$\left.\begin{array}{rcl} 1/x & = & (y+x)/x \\ a/x & = & x/y \end{array}\right\} \quad (\alpha_2)$$

     This is equivalent with $(x/a)^2 + (x/a) + a^{-2} = 0$. The solvability is determined by the trace-condition $tr(a^{-2}) = 0$, or equivalently $tr(1/a) = 0$.
     The ninth case in lexicographical order is an example of a unique solution:

$$\left.\begin{array}{rcl} 1/x & = & x/y \\ a/x & = & y^2/x^2 \end{array}\right\} \quad (\alpha_9)$$

     This is equivalent with $x^3 = a$. As f is odd, 3 does not divide $q - 1$ and we have a unique solution.
     We summarize the results in the 12 cases at hand: two cases have unique solutions, trace-conditions $tr(a) = 0$ and $tr(1/a) = 0$ occur five times each.

   - Let $\{\infty, 1\} \subset U$. We give only the statistics in this case: Two of the 12 subcases have a unique solution, trace-conditions $tr(1/(a+1)) = 0$ and $tr(a) = 1$ occur five times each.

   - Let $\{\infty, a\} \subset U$. Two of the 12 subcases have a unique solution, the trace-conditions $tr(1/a) = 1$ and $tr(1/(a+1)) = 1$ occur five times each.

We conclude that, whatever the values of $tr(a), tr(1/a)$ and $tr(1/(a+1))$ may be, we do get a total of 36 solutions. As we have counted ordered pairs (x, y) where we should have counted unordered pairs, we count 18 blocks containing S_a in case 3. Thus
$\lambda(\mathcal{B}_1) = 1 + 16 + 18 = 35$.

■

Theorem 2 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. Then $\mathcal{B}_2$ is a design $4-(q+1, 9, 63)$.*

Proof: Because of Theorem 1 it suffices to show that each 4-set S is contained in exactly 28 blocks $B = U \cup T_0(U) \cup \{T_*(U)\}$ satisfying $T_*(U) \in S$. As the stabilizer of S is transitive on S, we can choose $S = S_{\sqrt{a}}$ and we have to show that $\{\infty, 0, 1\}$ is contained in exactly seven blocks B of $\mathcal{B}_1$ satisfying $T_*(U) = \sqrt{a}$, where $B = U \cup T_0(U)$. As U and $T_0(U)$ are in symmetric position in B, it suffices to consider the following two cases: $\{\infty, 0, 1\} \subset U$ and $|\{\infty, 0, 1\} \cap U| = 2$. In the former case put $\sigma = (\tau \mapsto a/\tau)$. Then σ stabilizes U. Thus $\sigma(1) = a \in U$ and U is uniquely determined.
Assume now $|\{\infty, 0, 1\}| = 2$. We have to consider the following three subcases:

1. $\{\infty, 0\} \subset U, 1 \in T_0(U)$. Put $\sigma = (\tau \mapsto a/\tau)$.
2. $\{\infty, 1\} \subset U, 0 \in T_0(U)$. Put $\sigma = (\tau \mapsto (\tau + a)/(\tau + 1))$.
3. $\{0, 1\} \subset U, \infty \in T_0(U)$. Put $\sigma = (\tau \mapsto a(\tau + 1)/(\tau + a))$.

In each of these cases we have $\sigma \in G$ stabilizing U. If $x \in U$ then also $\sigma(x) \in U$. We have to determine the values of x satisfying $1 \in T_0(U), 0 \in T_0(U)$ and $\infty \in T_0(U)$, respectively. In each of the three subcases two equations possess a unique solution x, the remaining two equations depend on trace-conditions. These trace-conditions are $tr(a) = 0$ and $tr(1/a) = 0$ in case 1. $tr(1/(a+1)) = 0$ and $tr(a) = 1$ under 2. and $tr(1/a) = 1, tr(1/(a+1)) = 1$ in case 3. We get a total of 12 solutions for x, independent of the values of the traces. In this way we have counted every block twice. It follows $\lambda(\mathcal{B}_2) = 35 + 4(1 + 6) = 63$.■

Corollary 1 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. Then $\mathcal{B}_2$ is disjoint from the block design with parameters $4-(q+1, 9, 84)$ as constructed in [3].*

Proof: The blocks of the design with $\lambda = 84$ are exactly those 9-subsets of the projective line having stabilizer S_3 in G. The blocks of $\mathcal{B}_2$ have an elementary abelian group of order 8 as stabilizer.■

Lemma 7 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. Then a 6-subset $M \subset PG(1, q)$ contains at most two blocks of Alltop's design $\mathcal{B}_0$.*

Proof: By Lemma 3 we can assume

$$M = \{\infty, 0, 1, a, a+1, x\}, \text{where } x \in T(a) - \{a+1\}.$$

We have to show that there is no block B of $\mathcal{B}_0$ containing $a+1$ and x, which is contained in M. Assume there is such a block. There are four cases:

1. $B = \{\infty, 0, 1, a+1, x\}, x \in T(a+1)$.
2. $B = \{\infty, 0, a, a+1, x\}$. Then $a^{-1}B \in \mathcal{B}_0$, hence $x \in aT((a+1)/a)$.
3. $B = \{\infty, 1, a, a+1, x\}$. Let $\sigma = (\tau \mapsto (\tau+1)/(a+1))$. Then $\sigma(B) = \{\infty, 0, 1, a/(a+1), (x+1)/(a+1)\}$, hence $x \in 1 + (a+1)T(a/(a+1))$.
4. $B = \{0, 1, a, a+1, x\}$. Put $\sigma = (\tau \mapsto a(\tau+1)(a+1)^{-1}\tau^{-1}$. Then $\sigma(B) = \{\infty, 0, 1, a^2/(a^2+1), a(x+1)(a+1)^{-1}x^{-1}\} \in \mathcal{B}_0$, hence $\frac{a(x+1)}{(a+1)x} \in T(a^2/(a^2+1))$.

We have to derive a contradiction in each case, for example $(T(a) - \{a+1\}) \cap T(a+1) = \emptyset$ in case 1. and $(T(a) - \{a+1\}) \cap aT((a+1)/a) = \emptyset$ in case 2. The verification is routine. Note that the assumption $gcd(f, 6) = 1$ has to be used all the time.■

Definition 2 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. We define block-designs $\mathcal{B}_3$ and $\mathcal{B}_4$. Blocks of $\mathcal{B}_3$ are the 6-subsets of the form $B = S \cup \{T_*(S)\} \cup \{Q\}$ of $\mathcal{P}_1(q)$, where $Q \in T_0(S)$ and $|S| = 4$. Blocks of $\mathcal{B}_4$ are the 6-subsets of the form $B = S \cup \{Q, Q'\}$, where $\{Q, Q'\} \subset T_0(S), |S| = 4$.*

It follows from Lemma 7 that the 4-set S in the description of the blocks of $\mathcal{B}_3$ and $\mathcal{B}_4$ is uniquely determined. The makes it easy to count the blocks. The proof of the following Lemma is by now trivial.

Lemma 8
- *The stabilizer of B in G is trivial if $B \in \mathcal{B}_3$, it has order 2 if $B \in \mathcal{B}_4$.*

- *The number of blocks of $\mathcal{B}_3$ and $\mathcal{B}_4$ are*

$$b(\mathcal{B}_3) = 4 \cdot \binom{q+1}{4},\ b(\mathcal{B}_4) = 6 \cdot \binom{q+1}{4}.$$

It follows that if $\mathcal{B}_3, \mathcal{B}_4$ should be 4-designs then necessarily $\lambda(\mathcal{B}_3) = 60, \lambda(\mathcal{B}_4) = 90$.

Theorem 3 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. Then $\mathcal{B}_3$ is a design $4 - (q+1, 6, 60)$.*

Proof: We count the blocks $B = S \cup \{T_*(S)\} \cup \{Q\}$ containing S_a. Four cases have to be considered.

1. $U = S_a$. There are four choices for B in this case.

2. $|U \cap S_a| = 3, T_*(U) \in S_a$. We can assume without restriction $a = T_*(U)$. Thus $U = \{\infty, 0, 1, x\} = S_x$. It follows $a = \sqrt{x}$, hence $x = a^2$. We count $4 \cdot 4 = 16$ blocks B in this case.

3. $|U \cap S_a| = 3, Q \in S_a$. Without restriction $Q = a \in T_0(U)$, where $U = S_x$, hence $a \in T(x)$. There are four solutions x. We count 16 blocks B again.

4. $|U \cap S_a| = 2$. Without restriction $a = T_*(U)$. We have to consider three subcases.

   - $U = \{\infty, 0, x, y\}, 1 \in T_0(U)$. It follows $1/x \in T(y/x)$ and $a/x = \sqrt{y/x}$. The second condition yields $xy = a^2$, the first condition reads then $y/a^2 \in T(y^2/a^2)$. In two cases there are unique solutions, the others depend on the conditions $tr(a) = 0$ and $tr(1/a) = 0$, respectively.
   - $U = \{\infty, 1, x, y\}, 0 \in T_0(U)$. We use $\sigma = (\tau \mapsto (\tau+1)/(x+1))$. Then $\sigma(U) = S_{(y+1)/(x+1)}, \sigma(a) = (a+1)/(x+1), \sigma(0) = 1/(x+1)$. The conditions are the following:
     (i) $(a+1)/(x+1) = \sqrt{(y+1)/(x+1)}$, or equivalently $(x+1)(y+1) = a^2+1$.
     (ii) $1/(x+1) = (y+1)/(a^2+1) \in T((y+1)/(x+1)) = T((y^2+1)/(a^2+1))$.
     Again two of the four equations have a unique solution y, the remaining two depend on the conditions $tr(1/(a+1)) = 0$ and $tr(a) = 1$, respectively.
   - $U = \{0, 1, x, y\}, \infty \in T_0(U)$. Using $\sigma = (\tau \mapsto x(\tau+1)(x+1)^{-1}\tau^{-1})$ we get $\sigma(U) = S_{x(y+1)(x+1)^{-1}y^{-1}}, \sigma(\infty) = x/(x+$

1), $\sigma(a) = x(a+1)(x+1)^{-1}a^{-1}$. The conditions are the following:
(i) $\sigma(a) = \sqrt{x(y+1)(x+1)^{-1}y^{-1}}$. Solving this equation yields $y = a^2(x+1)/(x+a^2)$.
(ii) $x/(x+1) \in T(x^2(a^2+1)a^{-2}(x^2+1)^{-1})$.
As before there are two cases of unique solution. Two trace conditions appear: $tr(1/a) = 1$ and $tr(1/(a+1)) = 1$.

The same reasoning as in the preceding Theorems shows that we are done.■

Theorem 4 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. Then B_4 is a design $4-(q+1, 6, 90)$.*

Proof: As before we count the blocks $B = U \cup \{Q, Q'\}$ containing S_a.

1. $S_a = U$. There are six choices for B.

2. $|U \cap S_a| = 3$. Without restriction $U \cap S_a = \{\infty, 0, 1\}, Q = a$. If $U = S_x$, then we must have $a \in T(x)$. As each of the resulting four equations has a unique solution x and as we can choose Q' from a set of three elements in this case, we get $4 \cdot 4 \cdot 3 = 48$ choices for B in this case.

3. $|U \cap S_a| = 2$. There are six subcases. The transitivity of the stabilizer of S_a on S_a shows that only three of these need to be considered:

   - $U = \{\infty, 0, x, y\}, \{1, a\} \subset T_0(U)$.
   - $U = \{\infty, 1, x, y\}, \{0, a\} \subset T_0(U)$.
   - $U = \{\infty, a, x, y\}, \{0, 1\} \subset T_0(U)$.

   We transform this into a standard situation with the help of an operation $\sigma \in G$, where $\sigma = (\tau \mapsto \tau/x)$ in the first case, $\sigma = (\tau \mapsto (\tau+1)/(x+1))$ in the second case and $\sigma = (\tau \mapsto (\tau+a)/(x+a)$ in the third case. The condition is then $\{\sigma(1), \sigma(a)\} \subset T(y/x)$ in the first case, analogously in the other cases.
   Consider the first case again. For every ordered pair (u, v) of elements of $T(y/x)$ one has to determine the solutions (x, y) of the system of equations $\sigma(1) = u, \sigma(a) = v$, likewise in the other cases. It turns out that in each of the three cases, two of the twelve equations possess a unique solution (x, y), whereas the solvability of the remaining 10 equations is determined by trace-conditions, each condition appearing exactly five times. The trace-conditions are the usual ones, $tr(a) = 0$ and $tr(1/a) = 0$ in the first case, $tr(a) = 1$ and $tr(1/(a+1)) = 0$ in

the second case, $tr(1/a) = 1$ and $tr(1/(a+1)) = 1$ in the third case. As before we see that there are exactly 36 solutions, independent of the actual values of the traces involved. Thus case 3. yields 36 choices of B.

In case 1. we have dropped a factor of 2 by considering only the cases satisfying $\infty \in U \cap S_a$. In case 2. we have considered ordered pairs (x, y), where unordered pairs should have been counted. We conclude $\lambda(\mathcal{B}_4) = 90$.■

Corollary 2 *Let $q = 2^f$, where $gcd(f, 6) = 1, f \geq 5$. The designs $\mathcal{B}_3$ and $\mathcal{B}_4$ are disjoint, and both are disjoint from the design with $k = 6, \lambda = 10$ as constructed in [2].*

Proof: Blocks in different designs have G-stabilizers of different orders (1,2 and 6, respectively).■

References

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