Non-Existence of a Nested BIB Design NB(10, 15, 2, 3)

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ABSTRACT. A computer search shows that there does not exist a nested BIB design NB(10, 15, 2, 3).

1 Introduction

A balanced incomplete block (BIB) design $B(k, \lambda; v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a set of v treatments, $\mathcal{B}$ is a collection of k-subsets, called blocks, of $\mathcal{V}$, such that every pair of distinct treatments of $\mathcal{V}$ occurs in exactly λ blocks of $\mathcal{B}$.

A triple $(\mathcal{V}, \mathcal{B}_1, \mathcal{B}_2)$ is called a nested BIB design, denoted by $NB(v, b, s, k)$, if the triple has the following properties:

(i) $(\mathcal{V}, \mathcal{B}_1)$ is a $B(sk, \lambda_1; v)$, where $\lambda_1 = bsk(sk-1)/[v(v-1)]$,

(ii) $(\mathcal{V}, \mathcal{B}_2)$ is a $B(k, \lambda_2; v)$, where $\lambda_2 = bsk(k-1)/[v(v-1)]$,

(iii) Each block (superblock) of $\mathcal{B}_1$ can be partitioned into s subblocks with k treatments such that the resulting collection of subblocks coincides with the collection $\mathcal{B}_2$.

There are other definitions of nested BIB designs (see C. J. Colbourn and M. J. Colbourn [1]). Recently, Kageyama and Miao [2, 3] unified these nested designs and introduced a general definition. According to their notation, the present $NB(v, b, s, k)$ is denoted by $NB(sk, \lambda_1; v)$ of form $(k^s, \lambda_2)^1$. Though their notation can represent a general type of nested designs, we adopt the notation $NB(v, b, s, k)$ for simplicity.

Nested BIB designs were introduced in the statistical literature by Preece [7], who gave examples of their prior use in agricultural experiments, fully outlined the analysis, and provided a table of smaller designs. Since then, various constructions of these designs have been studied by many people.

Now, it is known that there exists a nested BIB design for $v \leq 14$ and $r \leq 30$ except for $NB(10, 15, 2, 3)$ (see Morgan [5, p.955]), where $r = \lambda(v-1)/(k-1)$. Incidentally, the existence of $NB(v, b, s, k)$ of any form has been entirely investigated for $sk \leq 5$ (see Kageyama and Miao [2, 3]). In the present case $sk = 6$.

In this paper, we will show the non-existence of a nested BIB design $NB(10, 15, 2, 3)$, by computer search. Section 2 will discuss a special case, while Section 3 will treat a general case, based on the observations given in Section 2.

2 A systematic design with some structure

This section deals with a $NB(10, 15, 2, 3)$ with a cyclic or block structure. The non-existence will be shown for both of cyclic and block structures.

One way to represent a $B(k, \lambda; v)$ is to use an $v \times b$ incidence matrix $A = (a_{ij})$, where $b = \lambda v(v-1)/[k(k-1)]$ is the number of blocks. Here the columns represent blocks and the rows represent treatments in the design. The entry a_{ij} is 1 if the ith treatment is included in the jth block, otherwise, 0. The matrix A has a constant column sum k, and the inner product of every pair of distinct rows is λ.

Firstly, it is certain that there is no cyclic NB(10, 15, 2, 3). In fact, a cyclic NB(10, 15, 2, 3) must consist of the following two base blocks : $B_1 = \{(0, a_1, a_2), (a_3, a_4, a_5)\}$, $B_2 = \{(0, a_6, a_7), (5, a_6+5, a_7+5)\}$ (mod 10) for $a_i \in Z_{10}$, $i = 1, 2, \ldots, 7$. However, some tedious calculations reveal that there are no such a_i's which form a cyclic NB(10, 15, 2, 3).

Next, the other structure will be considered. The existence of B(6, 5; 10) is equivalent to that of the complementary design B(4, 2; 10). The existence of B(4, 2; 10) is shown by Nandi [6] who proved that there exist exactly three non-isomorphic B(4, 2; 10). The solutions are displayed in Table 2. Thus three non-isomorphic B(6, 5; 10) can be obtained by exchanging 0 and 1 in Table 2. Furthermore, by applying certain permutations to rows and columns of these incidence matrices, we can make incidence matrices as in Table 3. Such permutations are presented at the marginal of each matrix. The incidence matrices given in Table 3 can be partitioned into 3×3 submatrices except for the first rows. Here these incidence matrices coincide with each other except for the two submatrices of elements underlined.

For each of incidence matrices given in Table 2, one of the way to divide the superblock of size 6 into two subblocks of block size 3 may be shown as follows:

$$N = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
 & & & & & & * & * & * & * & * & * & * & * & * \\ \hline
 & 1 & 2 & & 1 & 2 & 1 & & & & * & * & & * & * \\
 & 1 & 2 & 2 & & 1 & & 1 & & * & & * & * & & * \\
 & 1 & 2 & 1 & 2 & & & & 1 & * & * & & * & * & \\ \hline
2 & & 1 & & 1 & 2 & & * & * & 1 & & & & * & * \\
2 & & 1 & 2 & & 1 & * & & * & & 1 & & * & & * \\
2 & & 1 & 1 & 2 & & * & * & & & & 1 & * & * & \\ \hline
1 & 2 & & & 1 & 2 & & & & & & & 1 & & \\
1 & 2 & & 2 & & 1 & & B & & & B & & & 1 & \\
1 & 2 & & 1 & 2 & & & & & & & & & & 1
\end{array}\right]. \qquad (1)$$

Here in N, the element 1 or 2 implies that the treatment is included in the first or the second subblocks, "$*$" stands for 1 or 2, and the blanks denote 0's. Furthermore, the submatrix B is given by

$$\begin{bmatrix} * & * & \\ * & & * \\ & * & * \end{bmatrix}, \quad \begin{bmatrix} * & * & \\ & * & * \\ * & & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & * & * \\ * & & * \\ * & * & \end{bmatrix}, \qquad (2)$$

according to the types of the parent BIB designs (i), (ii) and (iii).

In the rest of this section, the following theorem will be established.

Theorem 1. It is impossible to fill the $*$ of N in (1) by 1 or 2 so that the resulting design is a NB(10, 15, 2, 3).

Lemma 2. Let $n((i,j),t)$ be the $((i,j),t)$th element, which are not zero(blank) of N. Then

(i) $n((i,j),t) \neq n((i,j'),t)$ holds for $j \neq j'$ and $t = 7, 8, \ldots, 15$, that is, in every column of any 3×3 cell with asterisks, each of 1 and 2 occurs once.

(ii) $n((i,j),t) \neq n((i',j),t)$ holds for $i \neq i'$ and $t = 7, 8, \ldots, 15$.

(iii) For any (i,j) and (i',j') with $i \neq i'$ and $j \neq j'$, there are exactly 3 blocks such that neither $((i,j),t)$ nor $((i',j'),t)$ are zero for $7 \leq t \leq 15$. Furthermore, if $n((i,j),t) \neq n((i',j'),t)$ in one of such 3 blocks, then in the other two blocks $n((i,j),t') = n((i',j'),t')$ must hold.

Proof. By numbering the 2nd, 3rd, ... , 10th rows (treatments) of N as (0,0), (0,1), (0,2), (1,0), (1,1), ... , (2,2), the proof is completed, since $\lambda_2 = 2$.

Applying Lemma 2, Theorem 1 can be shown by some tedious but straight-forward method.

Proof of Theorem 1.

For simplicity, we use the numbering 2, 3, ... , 10 instead of (0,0), (0,1), ... , (2,2) for representing rows of N. Denote the (i,j)th element of N by $n(i,j)$.

Case 1 : For the incidence matrix N with the submatrix B corresponding to the BIB design (i) in Table 3, the non-existence is shown as follows. At first, it holds that $n(8,7) = n(10,9) = 2$. Thus we have

$$n(9,7) = n(9,9) = 1; \; n(6,7) = n(6,9) = 2; \; n(7,7) = n(5,9) = 1.$$

Hence by counting λ_2 as coincidence numbers among treatments 2, 4 and 6, we have $n(2,11) = n(4,11) = 1$, which is impossible, since treatments 2 and 4 occur three times in subblocks.

Case 2 : For the incidence matrix N with B corresponding to the BIB design (ii) in Table 2, the non-existence is shown as follows. At first, it holds that $n(8,7) = n(9,8) = n(10,9) = n(8,10) = n(9,11) = n(10,12) = 2$ and $n(8,8) = n(9,9) = n(10,7) = n(8,11) = n(9,12) = n(10,10) = 1$. Thus the following can be obtained in turn:

$$n(2,11) = n(4,10) = n(5,8) = n(7,7) = 2.$$

$$n(4.11) = n(3.10) = n(6.7) = n(7.8) = 1;$$

$$n(2.12) = n(5.9) = 1. n(3.12) = n(6.9) = 2;$$

$$n(2, 14) = 2.$$

By counting λ_2 as coincidence numbers among treatments 4, 5 and 9, we have $n(4, 14) = n(5.14) = 1$, which is impossible, since treatments 5 and 9 occur three times in subblocks.

Case 3 : For N with B corresponding to the BIB design (iii) in Table 2, two cases are considered, depending on the values of the $(2, 11)$th element.

(Case 3-1) Assume that the $(2, 11)$th element is 1. Then

$$n(4.11) = 2;\ n(6, 9) = 1,\ n(5, 9) = 2;$$

$$n(4, 10) = 1.\ n(3.10) = 2; \quad n(3, 12) = 1.\ n(2, 12) = 2;$$

$$n(5, 8) = 1,\ n(7, 8) = 2; \quad n(7, 7) = 1,\ n(6, 7) = 2;$$

$$n(8, 8) = 2, n(8, 9) = 1, n(10, 8) = 1, n(9, 9) = 2. n(9, 7) = 1, n(10, 7) = 2;$$

$$n(8, 11) = n(9, 12) = n(10, 10) = 2.\ n(8, 12) = n(9, 10) = n(10, 11) = 1;$$

$$n(7, 14) = 1, n(5, 14) = 2, n(5.15) = 1, n(6, 15) = 2, n(6, 13) = 1, n(7, 13) = 2.$$

By counting λ_2 as coincidence numbers among treatments 3, 6 and 10, we have $n(3, 13) = 2$ and $n(3, 15) = 1$, which is impossible, since treatments 3 and 10 occur three times in subblocks.

(Case 3-2) Assume that the $(2, 11)$th element is 2. Then $n(4, 11) = 1$;

$$n(6, 7) = 1.\ n(7, 7) = 2; \quad n(2, 12) = 1.\ n(3, 12) = 2;$$

$$n(7, 8) = 1.\ n(5.8) = 2; \quad n(3, 10) = 1.\ n(4.10) = 2;$$

$$n(5, 9) = 1.\ n(6.9) = 2; \quad n(8, 8) = 1.\ n(8, 9) = 2;$$

$$n(8.11) = 1.\ n(8.12) = 2: \quad n(10, 8) = 2.\ n(9, 9) = 1;$$

$$n(10, 11) = 2.\ n(9.12) = 1; \quad n(10, 7) = 1.\ n(10, 10) = 1;$$

$$n(9, 7) = 2.\ n(9.10) = 2:$$

$$n(7, 13) = n(5, 14) = n(6, 15) = 1:$$

$$n(6.13) = n(7.14) = n(5.15) = 2:$$

$$n(4.13) = 2. n(4.14) = 1, n(3.13) = 1.$$

By counting λ_2 as coincidence numbers among treatments 2, 5 and 10, we have $n(2.14) = 2$ and $n(3.15) = 1$. which is impossible, since treatments 2 and 10 occur three times in subblocks. Thus, Theorem 1 has been proved.

3 Non-existence of a design by computer search

The non-existence of a NB(10, 15, 2, 3) will be shown by using computer. Here, we explain briefly the algorithms of our two computer programs. The algorithms are simple. To construct NB(10, 15, 2, 3), without loss of generality, it can be assumed that the incidence matrix of NB(10, 15, 2, 3) is given by

$N =$

						1	1	1	1	1	1	1	1	1
	1	1		1	1	*				*	*		*	*
	*	*	1		*		*		*		*	*		*
	*	*	*	*				*	*	*		*	*	
1		*		*	*		*	*	*				*	*
*		*	*		*	*		*		*		*		*
*		*	*	*		*	*				*	*	*	
*	*			*	*							*		
*	*		*		*		B			B			*	
*	*		*	*										*

where $*$'s are 1 or 2 and B's are 3×3 matrices in (2). The patterned incidence matrix, baud on the argument made in Section 2, is helpful to simplify the search by the present procedures. Then to construct NB(10, 15, 2, 3), the values of $*$'s has to be determined.

There are $10 = \binom{5}{2}$ ways to divide a block of size 6 of B(6, 5; 10) into two subblocks of size 3. For all pattern search, in principle 10^{15} trials are necessary to be carried out, which takes extremely long time. Thus some efficient algorithm has to be devised.

We prepare two algorithms to make sure the result of the non-existence of NB(10, 15, 2, 3). In the first algorithm, a backtracking method is used. On the halfway, if it becomes clear that the incidence matrix can not satisfy the condition of $\lambda_2 = 2$, then we stop it and try to find another way of determining $*$ by backtracking. In the second algorithm, ***a multistage all pattern search method*** is adopted. We divide N into 3 submatrices N_1 of size 10 × 3, N_2 of size 10 × 3, N_3 of size 10 × 9 with N_1 having the first three columns of N, N_2 having the next three columns of N and then N_3 7th to 15th columns, that is, $N = (N_1 : N_2 : N_3)$. In the first stage, we begin by filling each $*$ of N_3 with 1 or 2 so that the collection of subblocks satisfies the condition $\lambda_2 \le 2$. We collect all candidates of such N_3 and save them in a file. In the second stage, for each of candidates obtained by the first stage, we add N_2 and try to fill the $*$ in N_2 so that the collection of subblocks of $(N_2 : N_3)$ satisfies the condition $\lambda_2 \le 2$. Again, we list all candidates of such $(N_2 : N_3)$ and save them into another file. In the last stage, we add N_1, and finally fill the $*$'s in N_1 so that it satisfies $\lambda_2 = 2$. In Table 3, the numbers of candidates of the first and second stages are listed for each incidence matrix of Table 2.

Table 1: The numbers of submatrices N_3 and $(N_2 : N_3)$

	N_3	$(N_2 : N_3)$
(i)	164	4130
(ii)	135	2848
(iii)	184	8912

The computer we used is Sun microsystems Ultra2 and Sunsoft Solaris 2.5.1. We developed both of search programs by C Language. The first algorithm took about 20 hours to check each matrix in Table 2, while the second algorithm took about 30 hours. As a result, these programs found that no NB(10, 15, 2, 3) exist. The reader can get these programes through the Internet from the following URL:

`http://www.jim.info.gifu-u.ac.jp/~hishida/rep/nest.tar.Z`

Because of the use of computer in process, this result may not be considered as a theoretical "proof". But the possibility of the existence of NB(10, 15, 2, 3) is extremely small unless there are unexpected hardware trouble or bugs in both of the programs.

References

[1] C. J. Colbourn and M. J. Colbourn. Nested triple systems, *Ars Combin.* **16** (1983) 27-34.

[2] S. Kageyama and Y. Miao, Existence of nested designs with block size five, *J. Statist. Plann. Inference* (1998), to appear.

[3] S. Kageyama and Y. Miao. Nested designs of superblock size four, *J. Statist. Plann. Inference* (1998). to appear.

[4] J. Q. Longyear. A survey of nested designs. *J. Statist. Plann. Inference* **5** (1981) 181-187.

[5] J. P. Morgan, Nested Designs. in: S. Ghosh and C. R. Rao, eds., *Handbook of Statistics 13* (North-Holland. 1996) 939-976.

[6] H. K. Nandi. Enumeration of non-isomorphic solutions of balanced incomplete block designs. *Sankhyā* **7** (1946) 305-312.

[7] D. A. Preece, Nested balanced incomplete block designs. *Biometrika* **43** (1967) 479-486.

Table 2: Incidence matrices of three non-isomorphic B(4, 2; 10)

(i)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	0	0	0	1	1	0	0	0	0	0	0
2	1	1	0	0	1	1	0	0	0	1	1	0	0	0	0
3	1	0	1	0	1	0	1	0	0	0	0	1	1	0	0
4	1	0	0	1	0	1	1	0	0	0	0	0	0	1	1
5	0	1	1	0	0	0	0	0	0	1	0	1	0	1	1
6	0	1	0	0	0	0	1	1	0	0	1	0	1	1	0
7	0	0	1	0	0	1	0	0	1	0	1	0	1	0	1
8	0	0	0	1	0	1	0	1	0	1	0	1	1	0	0
9	0	0	0	1	1	0	0	0	1	0	1	1	0	1	0
10	0	0	0	0	1	0	1	1	1	1	0	0	0	0	1

(ii)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	0	0	0	1	1	0	0	0	0	0	0
2	1	1	0	0	1	1	0	0	0	1	1	0	0	0	0
3	1	0	1	0	1	0	1	0	0	0	0	1	1	0	0
4	1	0	0	1	0	1	1	0	0	0	0	0	0	1	1
5	0	1	0	0	0	0	1	1	0	1	0	1	0	1	0
6	0	1	0	0	0	0	1	0	1	0	1	0	1	0	1
7	0	0	1	0	0	1	0	1	0	0	1	0	1	1	0
8	0	0	1	0	0	1	0	0	1	1	0	1	0	0	1
9	0	0	0	1	1	0	0	1	0	0	1	1	0	0	1
10	0	0	0	1	1	0	0	0	1	1	0	0	1	1	0

(iii)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	0	0	0	1	1	0	0	0	0	0	0
2	1	1	0	0	1	1	0	0	0	1	1	0	0	0	0
3	1	0	1	0	1	0	1	0	0	0	0	1	1	0	0
4	1	0	0	1	0	1	1	0	0	0	0	0	0	1	1
5	0	1	1	0	0	0	0	0	0	1	0	1	0	1	1
6	0	1	0	0	0	0	1	1	0	0	1	0	1	1	0
7	0	0	1	0	0	1	0	0	1	0	1	0	1	0	1
8	0	0	0	1	1	0	0	1	0	1	0	0	1	0	1
9	0	0	0	1	1	0	0	0	1	0	1	1	0	1	0
10	0	0	0	0	0	1	1	1	1	1	0	1	0	0	0

Table 3: Incidence matrices of B(6, 5; 10)

(i)

	1	11	10	2	5	6	7	4	3	9	13	14	8	12	15
2	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
1	0	1	1	0	1	1	1	0	0	0	1	1	0	1	1
3	0	1	1	1	0	1	0	1	0	1	0	1	1	0	1
4	0	1	1	1	1	0	0	0	1	1	1	0	1	1	0
6	1	0	1	0	1	1	0	1	1	1	0	0	0	1	1
9	1	0	1	1	0	1	1	0	1	0	1	0	1	0	1
7	1	0	1	1	1	0	1	1	0	0	0	1	1	1	0
5	1	1	0	0	1	1	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	1	0	0
10	1	1	0	1	0	1	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	0	1	0
8	1	1	0	1	1	0	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	0	0	1

(ii)

	1	11	10	2	5	6	7	4	3	8	13	15	9	12	14
2	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
1	0	1	1	0	1	1	1	0	0	0	1	1	0	1	1
3	0	1	1	1	0	1	0	1	0	1	0	1	1	0	1
4	0	1	1	1	1	0	0	0	1	1	1	0	1	1	0
6	1	0	1	0	1	1	0	1	1	1	0	0	0	1	1
9	1	0	1	1	0	1	1	0	1	0	1	0	1	0	1
7	1	0	1	1	1	0	1	1	0	0	0	1	1	1	0
5	1	1	0	0	1	1	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	1	0	0
10	1	1	0	1	0	1	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	0	1	0
8	1	1	0	1	1	0	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	0	0	1

(iii)

	1	11	10	2	5	6	7	4	3	9	13	14	8	12	15
2	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
1	0	1	1	0	1	1	1	0	0	0	1	1	0	1	1
3	0	1	1	1	0	1	0	1	0	1	0	1	1	0	1
4	0	1	1	1	1	0	0	0	1	1	1	0	1	1	0
6	1	0	1	0	1	1	0	1	1	1	0	0	0	1	1
9	1	0	1	1	0	1	1	0	1	0	1	0	1	0	1
7	1	0	1	1	1	0	1	1	0	0	0	1	1	1	0
5	1	1	0	0	1	1	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	1	0	0
8	1	1	0	1	0	1	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	0	1	0
10	1	1	0	1	1	0	$\underline{1}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$	0	0	1