STS-Graphical Invariant for Perfect Codes

I.J. Dejter*
Department of Mathematics and Computer Science
University of Puerto Rico, Río Piedras, PR 00931-3355
email: dejterij@coqui.net

ABSTRACT. Let C be a perfect 1-error-correcting code of length 15. We show that a quotient $H(C)$ of the minimum distance graph of C constitutes an invariant for C more sensible than those studied up to the present, namely the kernel dimension and the rank. As a by-product, we get a nonlinear Vasil'ev C all of whose associated Steiner triple systems are linear. Finally, the determination of $H(C)$ for known families of C's is presented.

1 Introduction

Given $1 < n \in \mathcal{Z}$, let $I_n = \{1, \ldots, n\}$. The n-cube Q_n is the graph with vertex set $V = \Pi_{i=1}^n F_i$ and edge set $\cup_{i=1}^n f^i$, where $F_i = \mathcal{Z}_2 = \{0, 1\}$ and $f^i = \{(v, w) \in V \times V : (v - w)_j = 1 \text{ if and only if } j = i, (j \in I_n)\}$, for $i \in I_n$. The edges of f^i are said to have (or lie along) *direction* $i \in I_n$. A perfect (1-error-correcting) code C_r of length $n = 2^r - 1$, where $0 < r \in \mathcal{Z}$, can be seen as an isolated vertex-set of Q_n such that each vertex of $Q_n \setminus C_r$ is neighbor of exactly one vertex of C_r. This way, C_r has 2^{n-r} vertices and minimal distance 3. For every r as above there is at least a C_r, the linear one, see [7, 11, 14]. This C_r is unique for each $r < 4$. The situation changes for $r = 4$: there are many nonlinear C_4's, see [18, 12, 13, 4, 15, 17]. Historically, two invariants were used in trying to classify the C_r's: the dimension of the intersection of all maximal linear subcodes contained in C_r and the smallest linear subspace containing C_r, denoted respectively as the kernel dimension and the rank of C_4, [13].

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A quotient $H(C_4)$ of the minimal-distance graph depending on the associated Steiner triple systems on the 15 coordinates of Q_{15}, (or STS(15)'s, [11, 14, 15]), can be defined as an invariant with higher sensibility than kernel dimension and rank combined. On the other hand, $H(C_4)$ allows to see that there exists a nonlinear Vasil'ev C_4 all whose associated STS(15)'s are linear. The determination of $H(C_4)$ for known families of such C_4's is presented, in particular Vasil'ev ([6, 18]), Phelps-LeVan ([13, 8]), Rifà ([15]), and Phelps-Solov'eva ([12, 17]) codes, as obtained in programming work of [1, 10].

Let $M(C_r)$ be the minimal distance graph with vertex set C_r and any two vertices adjacent if their distance is 3, i.e. they are 3 coordinates apart. This $M(C_r)$ can be considered as an edge-labeled graph by assigning to each edge the triple of coordinates separating its endvertices. The edge-labels at any vertex v of $M(C_4)$ form an STS(15) denoted $S[v]$. Let $\mathcal{S}$ be the ordered set formed by the isomorphism classes of STS(15)'s, according to their listing in [8, 9, 19], so we denote each $S \in \mathcal{S}$ by its order number $\eta(S) \in I_{|\mathcal{S}|} = I_{80}$. If $S = S[v]$ for some $v \in C_4$, we abuse notation by taking S to denote also $\{w \in C_4; S = S[w]\}$. Let $\mathcal{S}_0$ be the resulting partition of C_4 into such classes S. For each $v \in C_4$ and $S \in \mathcal{S}_0$, let $\rho_0(v, S)$ be the cardinality of the set of neighbors u of v such that $S[u] = S$. Two vertices $v, w \in C_4$ are $\chi^{\mathcal{S}_0}$-equivalent if: (1) $S[v] = S[w]$; (2) $\rho_0(v, S) = \rho_0(w, S)$, for every $S \in \mathcal{S}_0$. In this case, we write $\rho_0(S[v], S) = \rho_0(v, S)$. If $v \in T \in \mathcal{S}_0$, let T_0^v be the $|\mathcal{S}_0|$-tuple formed by the numbers $\rho_0(v, S)$, where S varies in $\mathcal{S}_0$. If T_0^v does not depend on the selection of $v \in T$, then we write $T_0 = T_0^v$. Notice that the sum of the numbers in each T_0^v is the regular degree 35 of $M(C_4)$. We define $H(C_4)$ inductively. For $i \geq 1$, the i-th inductive step starts by checking if the $\chi^{\mathcal{S}_{i-1}}$-equivalence classes yield a (unique) well definition of each $|\mathcal{S}_{i-1}|$-tuple T_{i-1} of numbers $\rho_{i-1}(T, S)$ with $T \in \mathcal{S}_{i-1}$ fixed and S varying in $\mathcal{S}_{i-1}$, i.e. if the $|\mathcal{S}_{i-1}|$-tuple T_{i-1}^v associated to every vertex $v \in T$ does not depend on the selection of v, in which case a quotient graph $H^{\mathcal{S}_{i-1}}(C_4)$ of $M(C_4)$ is well-defined. Otherwise, (if at least one such tuple T_{i-1} is not unique, i.e. if two different vertices u, v of T have different $|\mathcal{S}_{i-1}|$-tuples T_{i-1}^u and T_{i-1}^v): (a) let $\mathcal{S}_i$ be the refinement of $\mathcal{S}_{i-1}$ into maximal classes of vertices $v \in C_4$ with T_{i-1}^v constant; (b) let $\rho_i(v, S)$ be defined similarly to ρ_0 with $\mathcal{S}_0$ replaced by $\mathcal{S}_i$; (c) apply the $(i+1)$-th inductive step. If $H(C_4)$ is determined totally in the i-th inductive step, we say that C_4 has *inductive invariant* $\iota(C_4) = i$.. The codes C_4 treated in the sections below have $\iota(C_4) \in \{1, 2\}$. If $\iota(C_4)$=1 then the vertex set of $H(C_4)$ is $\mathcal{S}_0$. If $\iota(C_4) > 1$ then a refinement of $\mathcal{S}_0$ yields the vertex set of $H(C_4)$.

Theorem 1. *$H(C_4)$ is a well-defined graph, for any C_4.*

Proof: Because of the finiteness of C_4, for some $i \geq 1$ we have that

$H^{S_{i-1}}(C_4)$ yields the desired $H(C_4)$.

With the notation of the proof above, we take $\chi = \chi^{S_j}$.

Proposition 2. *If $\chi(v), \chi(w)$ are adjacent in $H(C_4)$, so v, w are adjacent in $M(C_4)$, let $(e, v), (e, w)$ be the flags, (or incidence edge-vertex pairs), associated to the resulting edge e. Then the number $\chi(e, v, w)$, $[\chi(e, w, v)]$, of edges of $M(C_4)$ projecting onto $H(C_4)$ and joining v, $[w]$, to representatives of $\chi(w)$, $[\chi(v)]$, is constant.*

Proof: Notice that $\chi(e, v, w) = \rho_{i-1}(S[v], S[w])$ and $\chi(e, w, v) = \rho_{i-1}(S[w], S[v])$. Then the statement arises from the well definition of $H(C_4)$.

Now $\chi(e, v, w), \chi(e, w, v)$ are taken as labels for the flags associated to an edge e as above. In addition, each vertex v of $H(C_4)$ is labeled by means of: **(1)** the number $n(v) = \eta(S)$ of its associated $S \in \mathcal{S}$ and **(2)** the cardinality $c(v)$ of the χ-equivalence class it represents. With these edge and vertex labels, we say that $H(C_4)$ is (χ, n, c)-labeled.

Corollary 3. *$H(C_4)$ is well-defined as a (χ, n, c)-labeled graph, for any C_4.*

In determining $n(v) = \eta(S)$, the concept of fragment, or Pasch configuration, defined in [5, 8], was useful for the work of [1, 10]. A fragment $F = \{x_1x_2, x_3x_4, x_5x_6\}$, constituted by 3 ordered pairs x_1x_2, x_3x_4, x_5x_6 in I_{15}, is the subsystem of S formed by the 4 triples $\{x_1x_3x_5, x_1x_4x_6, x_2x_3x_6, x_2x_4x_5\}$, (we avoid parenthesis, keys and commas when possible). The order in which x_1x_2, x_3x_4, x_5x_6 appear is irrelevant: they yield the same F. By permuting the order of any two of x_1x_2, x_3x_4, x_5x_6, F stays unaltered. [5, 8] associate to any $S \in \mathcal{S}$ the expression formed by the following two objects, (concatenated): **(1)** the cardinality $\gamma(S)$ of the family of fragments of S; **(2)** a 15-tuple $\Gamma(S)$ whose terms are numbers $\gamma_i(S)$, $(i \in I_{15})$, *given in non-increasing order*, where $\gamma_i(S)$ is the cardinality of the subfamily of those fragments of S containing $i \in I_{15}$. This expression determines S uniquely in $\mathcal{S}$, see [8]. We consider a C_4 as a list of 2048 15-tuples on the alphabet $\{0, 1\}$. Each such a tuple or codeword v of C_4, determines the system $\mathcal{S}[v]$ represented by $\gamma(\mathcal{S}[v])\Gamma(\mathcal{S})[v])$. For example, the linear code of length 15 has each one of its vertices determining the $S \in \mathcal{S}$ produced by the projective geometry of dimension 3 over GF_2, to which we assign the invariant $\gamma(S)(\Gamma(S)) = 105(42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42, 42)$.

2 Hergert's Classification of Vasil'ev Codes

Let $C = C_r$. Let $g : C \to GF_2$ be any correspondence with $g(0) = 0$. Let $\pi : GF_2^n \to GF_2$ be the parity function: $\pi(v) =$ weight of v modulo 2. A Vasil'ev code V is given by

$$V = \{(u|u+v|\pi(u)+g(v)); u \in GF_2^n, v \in C\}. \qquad (1)$$

Then V is a C_{r+1}. Vasil'ev codes $V = V_4$ of length 15 were classified by F. Hergert, ([6]), into 19 equivalence classes encoded by means of respective binary 11-tuples whose supports H_i, $(i \in I_{19})$, in the parameter set $\{b_1 \dots b_{11}\}$, namely:

$H_1 = \emptyset$, $H_2 = b_8$, $H_3 = b_{11}$, $H_4 = b_8b_9$, $H_5 = b_6b_9$,
$H_6 = b_3b_{10}$, $H_7 = b_9b_{10}b_{11}$, $H_8 = b_3b_8b_{10}$, $H_9 = b_3b_7b_9b_{10}b_{11}$, $H_{10} = b_2$,
$H_{11} = b_2b_{11}$, $H_{12} = b_2b_3b_{10}$, $H_{13} = b_2b_3b_9b_{10}b_{11}$, $H_{14} = b_2b_3b_{10}b_{11}$, $H_{15} = b_1$,
$H_{16} = b_1b_{11}$, $H_{17} = b_1b_8$, $H_{18} = b_1b_6b_9$, $H_{19} = b_1b_8b_9$,

undergo the following changes of coordinates:

$$\begin{aligned}
&b_1 = w_1(w_2w_3w_4 + w_3w_4 + w_2w_4 + w_2w_3 + w_2 + w_3 + w_4),\\
&b_2 = w_2w_3w_4 + w_3w_4 + w_2w_4 \;\; b_3 = w_1w_2w_4, \;\; b_4 = w_1w_2w_4,\\
&b_5 = w_1w_2w_3, \;\; b_6 = w_3w_4, \;\; b_7 = w_2w_4, \;\; b_8 = w_2w_3,\\
&b_9 = w_1w_2, \;\; b_{10} = w_1w_3, \;\; b_{11} = w_1w_4;\\
&w_1 = v_1 + v_2 + v_3, \;\; w_2 = v_2, \;\; w_3 = v_3, \;\; w_4 = v_4
\end{aligned}$$

and $v_i = x_i + z_i$ for $i = 1,2,3,4$. Then a code representative of one of the 19 equivalence classes in question has the form:

$$V = \{(X, f_1(X), f_2(X), f_3(X), f_4(X)); x, z \in GF_2^4, y \in GF_2^3\},$$

where $X = (x,y,z) \in GF_2^{11}$, with $x = (x_1,x_2,x_3,x_4)$, $y = (y_1,y_2,y_3)$, $z = (z_1,z_2,z_3,z_4)$,

$$\begin{aligned}
f_1(x,y,z) &= x_2 + x_3 + x_4 + y_1 + z_2 + z_3 + z_4,\\
f_2(x,y,z) &= x_1 + x_3 + x_4 + y_2 + z_1 + z_3 + z_4,\\
f_3(x,y,z) &= x_1 + x_2 + x_4 + y_3 + z_1 + z_2 + z_4,\\
f_4(x,y,z) &= x_1 + x_2 + x_3 + x_4 + y_1 + y_2 + y_3 + \overline{g}(x,z)
\end{aligned}$$

and the extension $\overline{g}(x,z)$ of g in (1) is one of the 19 functions produced by the changes of coordinates. The $f_i(x,y,z)$'s are the scalar products of the rows of the Jacobian of V with (x,y,z). For the linear code, produced by H_1, this Jacobian 4×11-matrix followed to its right by the identity 4×4 matrix can be taken as the parity check matrix.

The resulting 19 Vasil'ev-Hergert codes V_i produced from their respective H_i, where $i \in I_{19}$, have their kernel dimensions $k(V_i)$, ranks $r(V_i)$ and graphs $H(V_i)$ as indicated in Figure 1, where vertices v are represented as circles containing $n(v)$ and $c(v)$ on top and bottom respectively, and where edges having two differing flag labels show indications of both; if they coincide, they are indicated only once, as is the case for loops.

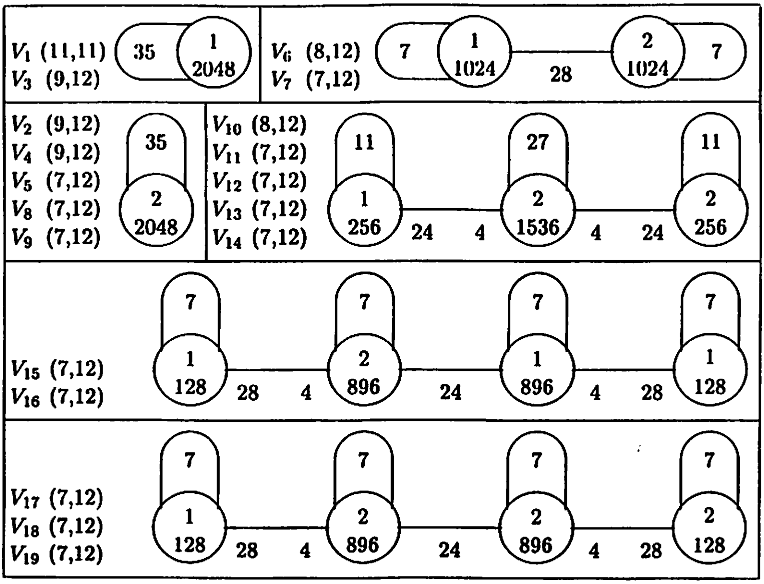

Figure 1. $(k(V_i), r(V_i))$'s and $H(V_i)$'s, for $i \in I_{19}$

In particular:

1. $H(V_1) = H(V_3)$, $[H(V_2) = H(V_4) = H(V_5) = H(V_8) = H(V_9)]$, has a unique vertex v and a unique edge: a loop ℓ with $\chi(\ell, v, v) = 35$. Moreover, $n(v) = 1$, $[n(v) = 2]$, and $c(v) = 2^{11}$, but $(k(V_1), r(V_1)) = (11, 11)$, for V_1 is the linear C_4, while $r(V_i) = 12$ for $i \in I_{19} \backslash \{1\}$, and $k(V_i) = 9$, for $i = 2, 3, 4$. In particular, the STS(15) associated to each vertex of V_3 is the linear one, but V_3 is not the linear C_4. On the other hand, $k(V_i) = 7$, for $i \in I_{19} \backslash \{1, 2, 3, 4, 6, 10\}$. Here, $\iota(V_i) = 1$, for $i = 1, 2, 3, 4, 5, 8, 9$.

2. $H(V_6) = H(V_7)$ has exactly two vertices u, v, loops ℓ_u, ℓ_v with $\chi(\ell_u, u, u) = \chi(\ell_v, v, v) = 7$, and an edge e with $\chi(e, u, v) = \chi(e, v, u) = 28$. Moreover, $n(u) = 1$, $n(v) = 2$, $c(u) = c(v) = 2^{10}$ and $k(V_6) = 8$. Here, $\iota(V_i) = 1$, for $i = 6, 7$.

3. $H(V_{10}) = H(V_{11}) = H(V_{12}) = H(V_{13}) = H(V_{14})$ has exactly 3 vertices u, v, w, loops ℓ_u, ℓ_v, ℓ_w with $\chi(\ell_u, u, u) = \chi(\ell_w, w, w) = 11$, $\chi(\ell_v, v, v) = 27$, and two edges e, f with $\chi(e, u, v) = \chi(f, w, v) = 24$, $\chi(e, v, u) = \chi(f, v, w) = 4$. Moreover, $n(u) = 1$, $n(v) = v(w) = 2$, $c(u) = c(w) = 2^8$, $c(v) = 2^{11} - 2^9$ and $k(V_{10}) = 8$. Here, $\iota(V_i) = 2$.

4. $H(V_{15}) = H(V_{16})$, $[H(V_{17} = H(V_{18}) = H(V_{19})]$, has exactly 4 vertices u, v, w, x, corresponding loops with edge labels 7, and edges

e, f, g with $\chi(e,u,v) = \chi(g,x,w) = 28$, $\chi(e,v,u) = \chi(g,w,x) = 4$, $\chi(f,v,w) = \chi(f,w,v) = 24$. Moreover, $n(u) = 1$, $n(v) = 2$, $n(w) = n(x) = 1$, $c(u) = c(x) = 2^7$, $c(v) = c(w) = 2^{10} - 2^7$. Again, $\iota(V_i) = 2$.

We see that several $k(V_i)$'s equal 7 and 8 and that the corresponding $H(V_i)$'s make further distinctions among some of them.

Corollary 4. *$H(C_4)$ has higher sensibility as an invariant than $(k(C_4), r(C_4))$.*

However, based on item 1., (codes V_1 and V_3), we have the following.

Corollary 5. *There exists a nonlinear C_4 with linear local STS(15)-behavior.*

3 Phelps-LeVan Codes

Phelps-LeVan codes ([8, 13]) use in their construction the following "switching" technique. Consider a $C = C_r$. Let T_i be the subspace generated by $\{v \in C; \text{weight}(v) = 3; v_i = 1\}$, for $i \in I_n$. Let $T_i + x \in C$, for some $x \in C$. A *switch* is the process of replacing $T_i + x$ by $T_i + x + e^i$, where $i \in I_n$ and $e^i_j = 1$ if and only if $i = j$, for $j \in I_n$. In [8, 13] it is proved that: (a) $C' = (C \backslash (T_i + x)) \cup (T_i + x + e^i)$ is a C_r; (b) if $i_1, \ldots, i_s$ are s independent elements of the projective space associated with the words of weight 3 in the linear $C_r = C$, then $T_{i_1} \cap \ldots \cap T_{i_r}$ has dimension 2^{r-s} and there exist $x^{i_1}, \ldots, x^{i_s}$ such that $T_{i_j} + x^{i_j}$ is disjoint from $T_{i_k} + x^{i_k}$, for $j, k \in I_s$ with $j \neq k$. This entails the result of [8, 13] that there exist nonlinear C_r's with kernels of all possible dimensions j, that is $j \in \{1, \ldots, 2^r - r - 3\}$. We restrict to present $H(L_j)$, where L_j is the C_4 obtained from the linear code V_1 with j independent switches, for $j \in I_4$. For example:

$$L_4 = [C \backslash \{(T_i + x) \cup (T_j + y) \cup (T_k + z) \cup (T_\ell + w)\}] \cup$$
$$[(T_i + x + e^i) \cup (T_j + y + e^j) \cup (T_k + z + e^k) \cup (T_\ell + w + e^\ell)],$$

where i, j, k, ℓ are independent and $x, y, z, w \in C$ are selected adequately.

The following results hold, ([1]), where $\iota(L_i) = 2$, for $i = 1, 2, 3, 4$:

1. $(H, k, r)(L_1) = (H, k, r)(V_{15}) = (H, k, r)(V_{16})$;
2. $(k(L_2), r(L_2)) = (4, 13)$; $H(L_2)$, depicted in Figure 2, has vertices 1_1, 1_2, 1_3, 2_1, 2_2, 2_3, 4_1, 4_2, denoted by their respective $n(v)$'s and distinctive subindices, with: (a) $c(1_1) = c(1_2) = c(2_1) = c(2_2) = 2^7$ and $c(1_3) = c(2_3) = c(4_1) = c(4_2) = 3 \times 2^7$; (b) loops at $1_1, 1_3, 2_2, 4_1$ $(1_2, 2_1)$, $[2_3, 4_2]$, with label 1, (5), [13]; (c) edges $1_12_1, 4_14_2, 1_32_3, 1_22_2$, $(1_12_2, 4_12_3, 4_21_3, 4_22_3)$, $[4_11_3]$, with unique label 6, (4), [16]; (d) edges $1_14_2, 2_14_1, 1_21_3, 2_22_3, 2_14_2, 1_22_3$, $(1_14_1, 2_21_3)$, $[2_12_3, 1_24_2]$, with corresponding flag label pair (6,2), (18,6), [(12,4)].

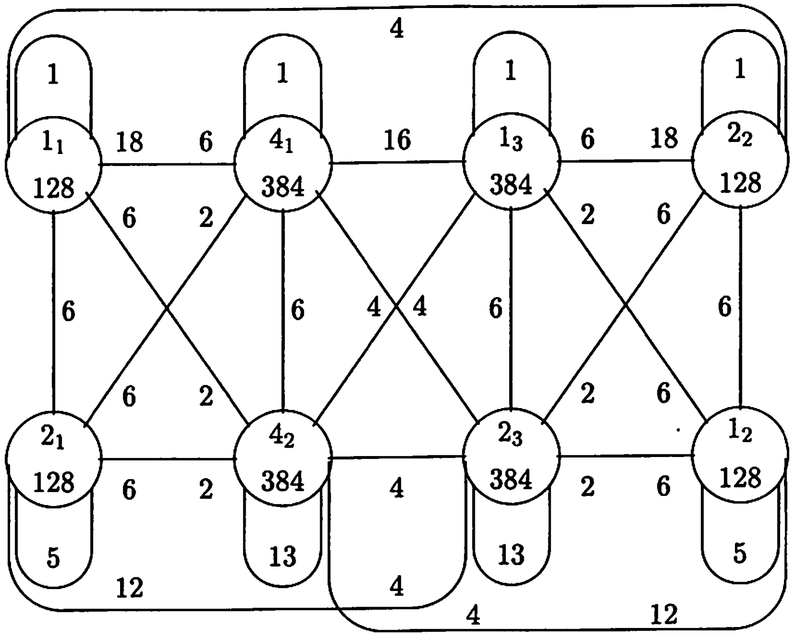

Figure 2. $H(L_2)$

3. $(k(L_3), r(L_3)) = (2, 14)$; $H(L_3)$, with 20 vertices, is presented by the following adjacency list whose vertex adjacency sublists are of the form

$$n(v)^{c(v)}_{j(v)} \text{ list}(n(w)_{\text{list}(j(w)^{\chi(v,w)})}; w \text{ adjacent to } v), \qquad (2)$$

where v is a vertex of $H(L_3)$; $\chi(v, w) = 1$ is not written; $n(v)$ ranges over $\{a = 1, b = 2, c = 4, d = 12, e = 9\}$, (with each STS(15)-type number, namely 1,2,4,12,9, replaced by a lowercase letter, namely a, b, c, d, e, respectively); $j(v)$ is a distinctive subindex and $c(v)$ ranges over $\{x = 64, y = 96, z = 192\}$:

$a_1^y b_{1^2236} c_{12^43^55^36^6} d_{1^4} e_{1^7}$
$a_2^x b_{5^37^6} c_{1^62^36^6} d_{1^5} e_{1^6}$
$a_3^y a_4 b_{1^3478} c_{2^44^55^26^7} d_{1^4} e_{1^6}$
$a_4^y a_{34^2} b_{1^545^46^27^38} c_{1^22^445^46^5}$
$b_1^y a_{1^23^34^5} b_{1^43^24^35^46^478} c_{2^446}$
$b_2^y a_1 b_{2^435^467^2} c_{1^32^43^54^45^16^2} e_{1^3}$
$b_3^y a_1 b_{1^223^64^267^2} c_{1^32^43^34^25^3} d_{1^4} e_1$
$b_4^y a_{34} b_{1^33^24^67^38} c_{1^22^43^24^35^26} d_{1^4}$
$b_5^z a_{24^2} b_{1^22^25^26^27^28^2} c_{1^22^73^24^25^26^2} d_1 e_{1^2}$
$b_6^y a_{14^2} b_{1^4235^46^27^2} c_{1^32^435^5} e_{1^5}$
$b_7^y a_{2^434^3} b_{12^23^24^35^46^27^68^3} c_{1^246}$
$b_8^y a_{34} b_{145^47^38^4} c_{1^22^43^44^56^3} e_{1^2}$
$c_1^y a_{12^44^2} b_{2^33^34^25^46^37^28^2} c_{1^635} e_1$
$c_2^z a_{1^223^24^2} b_{1^22^23^24^25^76^28^2} c_{2^23^24^25^2} d_1$
$c_3^y a_{1^5} b_{2^53^34^25^468^4} c_{12^43^24^25} e_1$
$c_4^y a_{3^54} b_{12^43^24^35^478^5} c_{2^43^24^26}$
$c_5^y a_{1^33^24^4} b_{23^34^25^46^5} c_{12^435^4} e_1$
$c_6^y a_{1^62^43^74^5} b_{12^245^478^3} c_4$
$d_1^x a_{1^62^53^6} b_{3^64^65^3} c_{2^3}$
$e_1^y a_{1^72^43^6} b_{2^335^46^58^2} c_{135}$

4. $(k(L_4), r(L_4)) = (1, 15)$; $H(L_2)$, with 48 vertices, is presented by an adjacency list as in 3. above, where $n(v)$ ranges over $\{a = 1, b = 2, c = 4, d = 12, e = 9, f = 24, g = 29\}$; j is a distinctive hexadecimal subindex and $c(v)$ ranges over $\{x = 16, y = 48\}$:

$a_1^x c_{5^38^3a^3d^3} d_{23^35^3} e_{2^33^35^3} f_1 g_{1^6}$
$a_2^x b_{d^3} c_{1^3b^3d^3} d_{13^35^4} e_{1^34^35^3} f_{1^3} g_{1^3}$
$a_3^y b_a c_{4679acd^2e} d_{124^25} e_{12^23^34^25^6} f_{1^2} g_{1^4}$
$a_4^y a_4 b_{78a^2bd} c_{1^23689^2a^2bcd^4fg^2} d_{1345} e_{2^245^3}$
$a_5^y b_{18} c_{1^3239^3a^2fg} d_{3^24^25} e_{1^2245^5} f_1 g_{1^6}$
$a_6^x b_{1^38^3c^3} c_{2^39^3a^3c^3d^3} d_{14^35} e_{1^35^3}$
$b_1^y a_{56} b_{1^32^23^248abcd} c_{1^323456789^3aefg^2} e_{15}$
$b_2^y b_{1^228^2cde} c_{123^246789^3ad} d_{123^24^2} e_{12^245} f_1 g$
$b_3^y b_{1^237bd^2} c_{1^32^23569adef^2} d_{23^24^25} e_{1234^25} g_{1^2}$
$b_4^x b_{1^3b^3} c_{1^33^3a^3b^3f^3} d_{23^3} e_{1^33^35^3} f_1$
$b_5^y b_{5^26ad} c_{1234^358^29ad^3e^2} d_{13^24^2} e_{123^24} g_{1^4}$
$b_6^y b_{5789a} c_{13^24^25^2679a^2eg} d_{1345} e_{1^22^23^34^25} g_1$
$b_7^y a_4 b_{36ad^2} c_{125^268^29a^3bd^2fg} d_{134} e_{1^2234^25} f_1 g_{1^2}$
$b_8^y a_{456} b_{12^268^29a^2bcd} c_{1^22346^28^29a^2cdef^2} d_4 e$
$b_9^y b_{689c} c_{1^22567^289bdefg^2} d_{23^2} e_{1^223^44^25^3} g_1$
$b_a^y a_{34^2} b_{15678^2a^4bc^2d} c_{2^23468^29acd^2g} e_{12^245}$
$b_b^y a_4 b_{1348a} c_{167^289a^2b^2cdfg} d_{3^245} e_{123^24^45^2} g_1$
$b_c^y a_6 b_{1289a^2} c_{1679abc^2d^2efg} d_{34^2} e_{1^22^4345^2} f$
$b_d^y a_{24} b_{123^257^28ad^5e} c_{134589a^2b^3c^2d} d_{3^245}$
$b_e^y b_{2de^2} c_{1^224^2579^3abc^3e^2} d_{2^234^35} e_{1^223} f_{1^2} g_1$
$c_1^y a_{24^25^3} b_{1^323^345678^29^2bcde^2} c_{1^22357abf}$
$c_2^y a_{56} b_{123^25789a^2e} c_{12^23^2456789abefg} d_{13^24} e$
$c_3^y a_{45} b_{12^23456^28ad} c_{12^23^25^26789cdefg} d_{34^2} e_{13} g_1$
$c_4^y a_3 b_{125^36^28ade^2} c_{245^36^28ce^2fg} d_{123^24^2} e_{12}$
$c_5^y a_1 b_{1356^27^29de} c_{123^24^35^378b^2def} d_{3^24} e_{34^2} g_1$
$c_6^y a_{34} b_{123678^29abc} c_{234^267abcd^2f^2g^3} e_{123^24} g$
$c_7^y a_3 b_{1269^2b^2ce} c_{123567^4abce^3f^2g} d_2 e_{1^22345}$
$c_8^y a_{14} b_{125^27^28^29a^2bd} c_{23458^2cdefg^2} d_{14^2} e_{12}$
$c_9^y a_{345^36} b_{1^32^3356789abcde^3} c_{2^239^2cef^2} e_1$
$c_a^y a_{134^25^26} b_{123456^27^38^2ab^2cd^2e} c_{1267bdeg} d_3$
$c_b^y a_{24} b_{479b^2cd^3e} c_{125^267ab^5cdefg} d_{25} e_{234^2}$
$c_c^y a_{346} b_{8abc^2d^2e^3} c_{346789bc^5deg} d_{12} e_{12^24} f_1$
$c_d^y a_{123^24^46} b_{235^37^289a^2bc^2d} c_{356^28abcdg} e_4$
$c_e^y a_3 b_{135^2689ce^2} c_{234^257^389abcf^2g^2} d_{34} e_{23^24}$
$c_f^y a_{45} b_{13^2478^29bc} c_{123456^27^289^2be^2f^2} d_3 e_{1^2235}$
$c_g^y a_{4^25} b_{1^2679^2abc} c_{2346^378^2abcde^2g} d_{34} e_{1234}$
$d_1^x a_{23^34^36} b_{2^35^36^37^3} c_{2^34^38^3c^3} e_{2^3}$
$d_2^x a_{13^3} b_{2^33^349^3e^6} c_{4^37^3b^3c^3} e_{1^3}$
$d_3^y a_{1245^2} b_{2^23^245^2679^2b^2cd^2e} c_{2^234^25^2aefg} d_3 e_4$
$d_4^y a_{3^245^26} b_{2^23^25^2678bc^2de^3} c_{23^24^258^2eg} d_4 e$
$d_5^x a_{1^32^43^34^35^36} b_{3^36^3b^3d^3e^3} c_{b^3}$
$e_1^y a_{235^26} b_{123456^27^29^2abc^2e^2} c_{3467^289cf^2g} d_2$
$e_2^y a_{13^24^25} b_{2^2356^279a^2bc^4e} c_{4678bc^2efg} d_{14} e_3$
$e_3^y a_{13^3} b_{345^26^379^4b^2ce} c_{2356^278be^2fg} e_{234}$
$e_4^y a_{23^245} b_{23^256^27^289^2ab^4c} c_{5^2678b^2cdeg} d_3 e_3$
$e_5^y a_{123^64^35^56} b_{12346789^3ab^2c^2} c_{7fg}$
$f_1^x a_{12^33^65^3} b_{2^347^3c^3e^6} c_{c^3e^3}$
$g_1^y a_{1^223^45^6} b_{2^23^25^46^27^29bce} c_{234568}$

4 Rifà Codes via Well-Ordered STS's

The following concepts from [15] lead to C_4's via well-ordered STS's, that we refer to as Rifà codes. Let $A = \{e^0, e^1, \dots, e^n\} \subset F = GF_2^n$, where e^0 is the zero vector and e^i, $(i \in I_n)$, has a 1 in the ith position and a 0 in the others. A *distance-compatible action* $\circ$ of A on GF_2^n is a map $\circ : A \times F \to F$ sending (e^i, v) onto $e^i \circ v$ such that there is an n-permutation π_v satisfying

$e^i \circ v = v + e^{\pi_v(i)}$, for every $v \in F$ and such that the induced map $v \to e^i \circ v$ is one-to-one. This is the case of the translations $(e^i, v) \to e^i + v$. Assuming there are actions $\circ, *$ of A on F, G respectively, an A-homomorphism $h : F \to G$ is a map such that $f \circ a = h(f) * a$, for every $a \in A, f \in F$.

The $S \in \mathcal{S}$ associated with $v \in C_4$ determines an operation $* : A \times A \to A$ such that both $x * y = x * z$ and $y * x = z * x$ only if $y = z$, for every $x \in A$. This makes $(A, *)$ into a *quasigroup*. Moreover, $0 * a = a * 0 = a$ and the operation is *totally symmetric*, for $a * b = c$ is preserved under permutations of $\{a, b, c\}$. This makes the quasigroup $(A, *)$ into a *Steiner loop*, or *Sloop*. [15] proves that an $S \in \mathcal{S}$ yields canonically a Sloop and viceversa. This is used to generate some C_4's not containing the linear $S \in \mathcal{S}$ as an STS(15).

Assume that F is partitioned into classes D which are translated C_4's. Let the class containing e^0 be denoted by C. Every $D \neq C$ is represented by the $e^i \in D$ at distance 1 from e^0, which allows to identify the partition with A. [15] proves that given such a partition A of F, there exists a distance-preserving action $*$ of A on F making A into a Sloop $\phi(F)$, where ϕ is an A-homomorphism. Moreover, $e^i * e^j = e^k$, where e^k represents the class containing $e^i + e^j$. Given $c \in C$, define $e^i \circ c$ as the only element of class e^i at distance 1 from c. Then any $v \in F$ can be written uniquely as $v = e^i \circ c$, for some $c \in C$, and there is a distance-compatible action $\circ : A \times F \to F$ such that $e^j \circ v = w$ is the vector of class e^k at distance 1 from v, where $e^i * e^j = e^k$. The map $\phi : F \to A$ is given by $\phi(v) = e^i$ if and only if $v \in$ class e^i. In fact, if $\phi(e^j \circ v) = e^k$, then $e^i \circ e^j = e^k$, where $\phi(v) = e^i$, so $\phi(e^j) * \phi(v) = e^k$ and $\phi(e^j \circ v) = \phi(e^j) * \phi(v)$.

The converse of this is, ([15]): if there exists an A-homomorphism $\phi : F \to A$ extending the identity map of A, then $\phi^{-1}(e^i) \subset F$ is a C_4, for each $e^i \in A$. To apply this, an order on $A \backslash e^0$ is fixed, say $e^1 < \ldots < e^n$. If $x = x_1 \ldots x_n \in F$, then the *ordered support* of x is $s^x = \{e^{a_1} < \ldots < e^{a_r}\}$, where $e^{a_i} \in s^x$ if and only if $x_{a_i} = 1$. Given the Sloop in A associated to an $S \in \mathcal{S}$ and $e^i, e^j \in A \backslash \{0\}$: (a) if $e^i \neq e^j$ then $e^i * e^j = e^k$, where $e^i, e^j, e^k \in S$; (b) if $e^i = e^j$ then $e^i * e^j = 0$, $e^i * e^0 = e^i$, $e^0 * e^j = e^j$. Now $\phi : F \to A$ is given by $\phi(x) = ((\ldots((e^{a_1} * e^{a^2}) * e^{a_3}) * \ldots) * e^{a_r})$.

Given $c^1, \ldots, c^r$, let $[c^1 \ldots c^r]$ stand for $((\ldots((c^1 * c^2) * \ldots) * c^r)$. Given $a, x, y \in A$, the equation $(a * x) * y = (a * \overline{y}) * x$ has a unique solution: $\overline{y} = [axyxa]$. An $S \in \mathcal{S}$ is a *well-ordered* STS(15) if there is an order on A such that $x < y$ if and only if $x < \overline{y}$, where $a, x, y \in A$ and $\overline{y} = [axyxa]$. [15] proves that if S is a well-ordered STS(15) and if A is the Sloop associated to S, then there is a distance-compatible action of A on F such that the corresponding ϕ is an A-homomorphism. Not only this yields a partition of F into C_4's but the action of A on F extends to an action of F on F. If $(a*x)*y \neq (a*y)*x$, then there is a unique $\overline{y}$ such that $(a*x)*y \neq (a*\overline{y})*x$ and if S is well-ordered, then $\overline{y}$ has the same order relation both with x and y. Let $q_{xy} \doteq \{\overline{y} = [axyxa] \in A; a \in A\}$, for $x, y \in F$. The components of q_{xy} are $> x$, $[< x]$, if $y > x$, $[y < x]$. Let $q_i = |\{(x,y); |q_{xy} = i\}|$, for $i \in I_n$. Then $(q_1, \dots, q_n)$ is a complete invariant for S. Let

$$q = |\{(a^0, a^1, a^2) \in (A\backslash\{0\})^3; a^{i+1}, a^{i+2} \in q_{a^i y^i}, y^i \in A\backslash\{0\}, i = 0,1,2\}|,$$

where additions are mod 3. [15] shows that if S is well-ordered then $q = 0$. A list of 18 well-ordered STS(15)'s and corresponding well-orders $\mathcal{O}_m$, ($m \in I_{18}$), is contained in [15], with the following properties: $\mathcal{O}_m = (1, \dots, 15)$ is the natural order, for $m = 1,2,3,5,7,16$; otherwise, $\mathcal{O}_m$ arises from $\mathcal{O}_1$ by resetting after $8 \in \mathcal{O}_m$:

$$\begin{aligned} &(11,9,10,12,15,13,14), && \text{for } m = 4; \\ &(11,9,10,12,13,14,15), && \text{for } m = 6; \\ &(11,13,14,9,10,12,15), && \text{for } m = 8,13,14,17; \\ &(14,9,15,10,12,11,13), && \text{for } m = 9; \\ &(14,10,12,9,15,11,13), && \text{for } m = 10; \\ &(11,12,15,9,10,13,14), && \text{for } m = 15; \\ &(12,9,13,10,14,11,15), && \text{for } m = 18. \end{aligned}$$

These orders yield corresponding Rifà codes R_m, for $m \in I_{18}$. Borges [2] showed that R_m is *homogeneous*, (i.e. $H(R_m)$ has a unique vertex v, with $n(v) = m$), for $m = 1, 2, 3, 5, 7, 13, 14, 16, 17$. In view of $H(R_9) = H(R_{10})$ and $H(R_{15}) = H(R_{18})$, we just want to represent R_m, for $m = 4,6,8,9,15$. The flag labels of R_m are represented by means of a square matrix table T_m, where: (a) there is a row and a column, for each vertex v of $H(R_m)$, whose intersection entry, on the main diagonal, is $\chi(\ell, v, v)$ if v is incident to the loop ℓ, and 0 otherwise; (b) the entry at the row headed by a vertex u and the column headed by a vertex $v \neq u$ is $\chi(e,u,v)$, if there exists an edge e between u and v, and 0 otherwise.

Each vertex v is denoted by $n(v)^{\log_2 c(v)}$ if $c(v)$ is a power of 2, or by $n(v)^{\log_2 c_1(v) \log_2 c_2(v)}$ if $c(v)$ is a difference $c_1(v) - c_2(v)$ of powers of 2, where logarithms are expressed in hexadecimal notation. The T_m's obtained are, (where $\iota(R_i)$ is 2 for $i = 6$ and is 1 for the other i's):

R_4	4^a	2^8	3^8	5^9
4^a	19	4	4	8
2^8	16	11	0	8
3^8	16	0	11	8
5^9	16	4	4	11

R_6	6^a	5^9	5^8	7^8
6^a	19	8	4	4
5^9	16	11	4	4
5^8	16	8	11	0
7^8	16	8	0	11

R_8	8^{b9}	16^8	3^8
8^{b9}	27	4	4
16^8	24	11	0
3^8	24	0	11

R_9	9^9	13^9	14^8	10^8	8^8	4^8
9^9	11	8	4	4	4	4
13^9	8	11	4	4	4	4
14^8	8	8	11	4	0	4
10^8	8	8	4	11	4	0
8^8	8	8	0	4	11	4
4^8	8	8	4	0	4	11

R_{15}	15^9	8^9	18^9	13^8	17^8
15^9	11	8	8	4	4
8^9	8	11	8	4	4
18^9	8	8	11	4	4
13^8	8	8	8	11	0
17^8	8	8	8	0	11

In [15] it is shown that $(k(R_m), r(R_m))$ is (11,11), for $m = 1$; (9,12), for $m = 2$; (8,13), for $m = 3, 5, 7$; (7,13), for $m = 4$; (6,13), for $m = 6$; (7,14), for $m = 8$; (8,14), for $m = 16$; (6,14), for $m = 9, 10, 13, 14, 15, 17, 18$.

5 Phelps-Solov'eva Codes

The Phelps-Solov'eva codes of length 15 can be presented as follows. Every C_r yields an extended perfect code D_r of length $n + 1 = 2^r$ by adjoining to each one of its codewords w a parity $(n + 1)$-th coordinate entry taking w into an even $(n + 1)$-word $\overline{w}$. Let A, B be two C_3's. Let $\{A = A_0, A_1, \dots, A_7\}$ and $\{B = B_0, B_1, \dots, B_7\}$ be any two partitions of Q_7 into translated C_3's. Let A_i^*, B_i^* denote the extended codes and let π be a permutation of $\{0\} \cup I_7$. Define D^* as follows:

$$(a|b) \in D^* \Leftrightarrow a \in A_i^*, b \in B_j^*, \pi(i) = j.$$

Then Theorem 2.1 of [12] implies that D^* is a D_4. If we assume that $\pi(0) = 0$, then the 0-vector is in D^* and both A^*, B^* are subcodes of D^*. Puncturing D^*, (deleting say the last coordinate), yields a C_4. The coordinates of extended codewords are presented in the order 1,2,3,4,5,6,7,0, so that puncturing means deletion of coordinate 0.

We apply $H(C_4)$ to the following insubstancial variation of this C_4. Let E be the set of even-weight vectors of Q_8. Let $\mathcal{A} = \{A_0, A_1, \dots, A_7\}$, $[\mathcal{B} = \{B_0, B_1, \dots, B_7\}]$, be a partition of E, $[Q_8 \setminus E]$, into translated extended perfect codes. Then there is an extended perfect code $\{(b_0|a_1); b_0 \in B_i, a_1 \in A_j, \pi(i) = j\}$. By eliminating coordinate 0, a C_4 is obtained that we refer to as a Phelps-Solov'eva code, or PS-code, [12, 17].

[12] shows that there are at least 6 nonisomorphic partitions of Q_7, denoted with Roman numerals I to VI. (Phelps found recently that there are exactly 11 such partitions). The following π's will be considered, one for each cycle structure type:

$\pi_1 = (12)$	$\pi_2 = (12)(34)$	$\pi_3 = (12)(34)(56)$
$\pi_4 = (12)(34)(56)(70)$	$\pi_5 = (123)$	$\pi_6 = (123)(45)$
$\pi_7 = (123)(45)(67)$	$\pi_8 = (123)(456)$	$\pi_9 = (123)(456)(70)$
$\pi_{10} = (1234)$	$\pi_{11} = (1234)(56)$	$\pi_{12} = (1234)(56)(70)$
$\pi_{13} = (1234)(567)$	$\pi_{14} = (1234)(5670)$	$\pi_{15} = (12345)$
$\pi_{16} = (12345)(67)$	$\pi_{17} = (12345)(670)$	$\pi_{18} = (123456)$
$\pi_{19} = (123456)(70)$	$\pi_{20} = (1234567)$	$\pi_{21} = (12345670)$

Notation for the C_4's obtained from the partitions and permutations above: By using letters $x, y \in \{a, b, c, d, e, f\}$, where a, b, c, d, e, f stand respectively for the nonisomorphic partitions I,II,III,IV,V,VI of [12], assume that a PS-code C_4 is obtained by means of the partitions $\mathcal{B}, \mathcal{A}$ of $E, Q_8 \backslash E \subset Q_8$, respectively, and the permutation π_z , $(z \in I_{21})$. This C_4 is denoted $\mathcal{B}\mathcal{A}_z$.

The table below completes the classification of the $H(C_4)$'s for these PS-codes. Newly appearing $H(C_4)$'s, i.e. not present in previous sections, are denoted as follows. If a code is homogeneous, then we refer to it as h_i, where $i = \eta(S)$, for fixed $S \in \mathcal{S}$. Four of the new $H(C_4)$'s are triangles, each one of which can be presented as a list

$$(F_1(f_{11}, f_{12}, f_{13}), F_2(f_{21}, f_{22}, f_{23}), F_3(f_{31}, f_{32}, f_{33}))\chi_1, \chi_2, \chi_3,$$

where F_1, F_2, F_3 are the participating $S \in \mathcal{S}$, the $f_{i,j}$ are the entries of the adjacency matrix of $H(C_4)$ and the χ_i's are the corresponding cardinalities. Thus, the four mentioned triangles can be denoted by:

$$\begin{aligned}
T &= (12(11,12,12), 9(8,19,8), 4(8,8,19))512, 768, 768,\\
T' &= (20(11,12,12), 18(8,19,8), 10(8,8,19))512, 768, 768,\\
T'' &= (22(5,28,2), 22(14,7,14), 21(2,28,5))512, 1024, 512,\\
T''' &= (6(19,12,4), 5(16,15,4), 7(16,12,7))1024, 768, 256.
\end{aligned}$$

Following this notation style, we further denote:

$S^{2,1} = (2(31,4), 1(28,7))1792, 256$, and likewise $S^{21,61}, S^{21,62}$;
$S_{3,5} = (3(7,28), 5(28,7))1024, 1024$, and likewise $S_{5,7}, S_{13,17}, S_{13,14}, S_{22,61}$;
$S'_{21,22} = (21(19,16), 22(16,19))1024, 1024$.

Some Rifà codes, (Section 4 above), may be redenoted by grouping the $S \in \mathcal{S}$ according to their behaviors: $R_{15} = (15, 8, 18; 13, 17), R_9 = (9, 13; 14, 10, 8, 4)$. Likewise, we denote

$$R' = (11, 10, 9; 6, 15), \quad R'' = (12, 9, 4; 9, 4),$$

$$R''' = (20, 18, 10; 18, 10)), \quad R'''' = (9, 10; 4, 8, 14, 13).$$

The graph on 5 vertices given as

$$(4(19,4,4,4,4),4(16,11,4,0,4),4(16,4,3,4,8),$$

$$2(16,0,4,11,4),3(16,4,8,4,3))2^{10}2^{8}2^{8}2^{8}2^{8},$$

will be denoted $P=(4;4,4,2,3)$. We get, likewise, a graph $P'=(4;5,5,2,3)$.

These $H(C_4)$'s are assigned in the table below to expressions of the form $\mathcal{B}\mathcal{A}^{k,r}_{z_1,\dots,z_s}$ that stand each for a $\mathcal{B}\mathcal{A}_{z_i}$, with additional dots after a number $t\in I_{21}$ in the subindex part $z_1,\dots,z_s$ representing subsequent numbers to t in I_{21}, (e.g. $ad^{6,14}_{1\dots,11,13.}=ad^{6,14}_{1,2,3,4,11,13,14}$), and where k,r stand for common kernel dimension and rank. The remaining needed $H(C_4)$'s are presented below and in **http://home.coqui.net/dejter/gihidden.zip** by means of adjacency lists using (2) of Section 3 with: **(a)** $n(v)$ in place of $\jmath(v)$; **(b)** the symbols used for the now tacit $\jmath(v)$, (and so for the $\jmath(w)$'s), run from a through subsequent letters if necessary up to z; **(c)** vertex adjacency sublists having a common STS(15)-type in one of these adjacency lists are alphabetically ordered from left to right and then from top to bottom; **(d)** the logarithmic upper-index notation for $c(v)$ by the end of Section 4 is used.

Of the remaining adjacency tables of Phelps-Solov'eva codes, we present below the simplest four. The remaining ones may be found, as mentioned, in an extension of this section located in the URL address cited above. Again, the ι's of the PS codes are in $\{1,2\}$.

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$aa^{9,12}_{1,3,5,7,11,16,20}$	h_2	$aa^{11,11}_{2,4,10,12,14}$	h_1	$aa^{8,13}_{6,8,15,19,21}$	h_3
$aa^{8,14}_{9,13,17.}$	h_{16}	$ab^{7,14}_{1.}$	V_{10}	$ab^{7,14}_{3,10,12.,21}$	h_2
$ab^{7,13}_{4,14}$	h_1	$ab^{7,14}_{5,7.,11,15,19..}$	h_3	$ab^{7,14}_{6,9,18}$	h_{16}
$ab^{7,14}_{16.}$	R_8	$ac^{6,13}_{1,3,10,12.}$	h_5	$ac^{6,13}_{2,4,11,14}$	h_3
$ac^{6,13}_{5}$	h_7	$ac^{6,14}_{6,9,16}$	h_{17}	$ac^{6,14}_{7.,15,17,19.}$	h_{13}
$ac^{6,14}_{18,21}$	h_{14}	$ad^{6,14}_{1...,11,13.}$	R_4	$ad^{6,14}_{6.,9,15.,19}$	R_{15}
$ad^{6,14}_{8,17.,20.}$	R_9	$ad^{6,14}_{5,10,12}$	R_6	$ae^{5,14}_{1....,13.,20}$	T
$ae^{5,14}_{7,9,15,17..}$	T'	$ae^{5,14}_{6,8,10..,16,21}$	R'	$af^{5,14}_{1,3,6,9.,12.,17,21}$	$S^{21,61}$
$af^{5,14}_{2,4.,7.,11,14..,18..}$	$S'_{21,22}$	$bb^{8,12}_{1...,6,8.,15,18.}$	$S^{2,1}$	$bb^{7,13}_{5,10..}$	V_{10}
$bb^{6,14}_{7,13.,16.,20.}$	bb_7	$bc^{6,13}_{1...,10..,14}$	$S_{3,5}$	$bc^{6,13}_{5,13}$	$S_{5,7}$
$bc^{5,14}_{6...,16,20}$	$S_{13,17}$	$bc^{5,14}_{7,15,17..,21}$	$S_{13,14}$	$bd^{5,13}_{1...}$	P
$bd^{5,13}_{5,10..}$	bd_5	$bd^{5,14}_{6...,15.,18..}$	bd_6	$bd^{6,13}_{13.}$	R_4
$bd^{5,14}_{17,21}$	R''''	$be^{4,14}_{1....,13.,20}$	R''	$be^{4,14}_{6,8.}$	be_6
$be^{4,14}_{7,15,17..}$	R'''	$be^{4,14}_{10..,16,21}$	R'	$bf^{4,14}_{1...,6.......,16...,21}$	bf_1
$bf^{4,14}_{5,14.,20}$	T''	$cc^{8,13}_{1,3,10,12.}$	h_5	$cc^{8,13}_{2,4,11,14}$	h_3
$cc^{8,13}_{5}$	h_7	$cc^{4,14}_{6,9,16}$	h_{17}	$cc^{4,14}_{7.,15,17,19.}$	h_{13}
$cc^{4,14}_{18,21}$	h_{14}	$cd^{6,13}_{1...,11,13}$	P'	$cd^{6,13}_{5,10,12}$	R_6
$cd^{4,14}_{6.,9,15.,19}$	R_{15}	$cd^{4,14}_{8,14,17,18,20.}$	R''''	$ce^{3,14}_{1....,13.,20}$	T
$ce^{3,14}_{6,8,10..,16,21}$	R'	$ce^{3,14}_{7,9,15,17..}$	T'	$cf^{3,14}_{1,3,6,9.,12.,16,18,21}$	$S^{21,61}$
$cf^{3,14}_{2,4.,7.,11,14.,17,19.}$	$S'_{21,22}$	$dd^{7,13}_{1.,4}$	bd_{13}	$dd^{5,13}_{3}$	dd_3
$dd^{6,13}_{5,10,12}$	T'''	$dd^{4,14}_{6.,9,15.,19}$	R_{15}	$dd^{4,14}_{8,17.,20.}$	dd_8
$dd^{4,13}_{11,13.}$	cd_1	$de^{3,14}_{1,5...,10..,15.,18}$	de_1	$de^{3,14}_{2..,9,13.,17,19..}$	R''
$df^{3,14}_{1,6.,9,11,15.,19,21}$	$S'_{21,22}$	$df^{3,14}_{2...,8,10,12..,18,20}$	df_2	$df^{3,14}_{17}$	$S_{22,61}$
$ee^{4,14}_{1,4.,7}$	de_1	$ee^{3,14}_{2.,10..,14,16,19}$	ee_2	$ee^{2,14}_{6,8.,13,15,17.,20.}$	ee_6
$ef^{2,14}_{1,4,8,10.,15,19..}$	ef_1	$ef^{2,14}_{2.,5..,12.,14,16.}$	ef_2	$ef^{2,14}_{9,18}$	ef_9
$ff^{2,14}_{1,6,16,21}$	ff_1	$ff^{2,14}_{2,4.,10.,13,15,17}$	ff_2	$ff^{2,14}_{3,7,9,12,18}$	ff_3
$ff^{2,14}_{8,14,19.}$	ff_8				

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