The Forcing Hull Number of a Graph

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Dedicated to Frank Harary
on the Occasion of His 78th Birthday
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ABSTRACT

For two vertices u and v of a connected graph G, the set $H(u,v)$ consists of all those vertices lying on a $u-v$ geodesic in G. Given a set S of vertices of G, the union of all sets $H(u,v)$ for $u,v \in S$ is denoted by $H(S)$. A convex set S satisfies $H(S) = S$. The convex hull $[S]$ is the smallest convex set containing S. The hull number $h(G)$ is the minimum cardinality among the subsets S of $V(G)$ with $[S] = V(G)$. A set S is a geodetic set if $H(S) = V(G)$; while S is a hull set if $[S] = V(G)$. The minimum cardinality of a geodetic set of G is the geodetic number $g(G)$. A subset T of a minimum hull set S is called a forcing subset for S if S is the unique minimum hull set containing T. The forcing hull number $f(S,h)$ of S is the minimum cardinality among the forcing subsets of S, and the forcing hull number $f(G,h)$ of G is the minimum forcing hull number among all minimum hull sets of G. The forcing geodetic number $f(S,g)$ of a minimum geodetic set S in G and the forcing geodetic number $f(G,g)$ of G are defined in a similar fashion. The forcing hull numbers of several classes of graphs are determined. It is shown that for integers a,b with $0 \le a \le b$, there exists a connected graph G such that $f(G,h) = a$ and $h(G) = b$. We investigate the realizability of integers $a,b \ge 0$ that are the forcing hull and forcing geodetic numbers, respectively, of some graph.

1 Introduction

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is called a *$u-v$ geodesic*. The set $H(u,v)$ consists of of all vertices lying on some $u-v$ geodesic of G, while for $S \subseteq V(G)$,

$$H(S) = \bigcup_{u,v \in S} H(u,v)$$

In [1, 2] a set S of vertices of a connected graph G is defined to be a *geodetic set* if $H(S) = V(G)$. A geodetic set of minimum cardinality is a *minimum geodetic set*, and this cardinality is the *geodetic number* $g(G)$. In [4] a subset T of a minimum geodetic set S is called a *forcing subset* of S if S is the unique minimum geodetic set containing T. The *forcing geodetic number* $f(S,g)$ of S is the minimum cardinality of a forcing subset for S, while the *forcing geodetic number* $f(G,g)$ of G is the smallest forcing number among all minimum geodetic sets of G. For example, consider the graph G of Figure 1. It is easy to verify that $g(G) = 3$. There are four minimum geodetic sets in G, namely $S_1 = \{u,x,z\}$, $S_2 = \{v,y,w\}$, $S_3 = \{x,y,w\}$, and $S_4 = \{v,y,z\}$. Since S_1 is the only minimum geodetic set containing u, it follows that $f(S_1,g) = 1$. No other vertex of G belongs to only one minimum geodetic set, so $f(S_i,g) \geq 2$ for $i = 2,3,4$. Therefore, $f(G,g) = 1$.

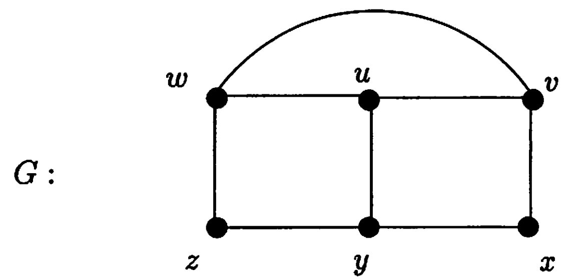

Figure 1: A graph G with forcing geodetic number 1

A set S of vertices in a connected graph G is *convex* if $H(S) = S$. The *convex hull* $[S]$ is the smallest convex set containing S. The convex hull $[S]$ of S can also be found from the sequence $\{H^k(S)\}, k \geq 0$, where $H^0(S) = S, H^1(S) = H(S)$, and $H^k(S) = H(H^{k-1}(S))$ for $k \geq 2$. From some term on, this sequence is constant. The convex hull $[S]$ is the set $H^p(S)$, where p is the smallest number such that $H^p(S) = H^{p+1}(S)$. A set S of vertices of G is defined in [3] as a *hull set* of G if $[S] = V(G)$, and a hull set of minimum cardinality is a *minimum hull set of* G. The cardinality of a minimum hull set in G is the *hull number* $h(G)$. Clearly, $2 \leq h(G) \leq n$ for every connected graph G of order $n \geq 2$. As an illustration of these concepts, consider the graph G of Figure 2. Let $S_1 = \{u,z\}$. Since $[S_1] = S_1$, which is a proper subset of $V(G)$, it follows that S_1 is not a hull set of G. On the other hand, let $S_2 = \{x,y\}$. Since $H(S_2) = \{x,y,u,v,w\}$ and $H(H(S_2)) = V(G)$, we have $[S_2] = V(G)$ and so $h(G) = 2$.

In this paper, we introduce forcing concepts for hull sets. A *forcing subset* of a minimum hull set S of G is a subset T of S with the property that S is the unique minimum hull set containing T. The *forcing hull number* $f(S,h)$ of S is the minimum cardinality of a forcing subset for S, while the *forcing hull number* $f(G,h)$ of G is the smallest forcing number

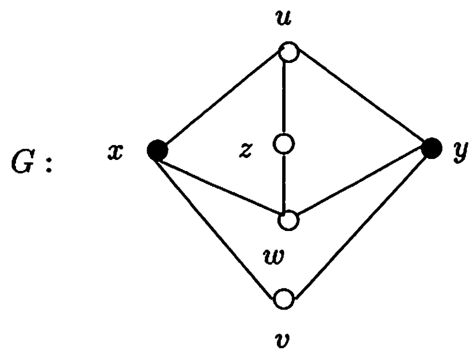

Figure 2: A graph G with minimum hull set $\{x, y\}$

among all minimum hull sets of G. Hence if G is a graph with $f(G, h) = a$ and $h(G) = b$, then $0 \leq a \leq b$ and there exists a minimum hull set S of cardinality b containing a forcing subset T of cardinality a. For example, in the graph $G = K_{2,3}$ of Figure 3, the sets $S_1 = \{x, y\}$ and $S_2 = \{u, v\}$ are minimum hull set of G. The remaining minimum hull sets are similar to S_2. Since S_1 is the unique minimum hull set containing x, it follows that $f(S_1, h) = 1$. On the other hand, S_2 is not the unique graph containing any of its proper subsets, $f(S_2, h) = 2$. Therefore, $f(G, h) = 1$.

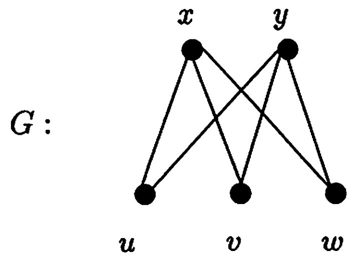

Figure 3: A graph G with forcing hull number 1

It is immediate that $f(G, h) = 0$ if and only if G has a unique minimum hull set. If G has no unique minimum hull set but contains a vertex belonging to only one minimum hull set, then $f(G, h) = 1$. We summarize these observations below.

Lemma 1.1 *For a graph G, the forcing hull number $f(G, h) = 0$ if and only if G has a unique minimum hull set. Moreover, $f(G, h) = 1$ if and* *only if G does not have a unique minimum hull set but some vertex of G belongs to exactly one minimum hull set.*

Figure 4 shows a graph G with hull number 2 and unique minimum hull set $\{t, z\}$. Thus $f(G,h) = 0$. The minimum geodetic sets for G are $\{t, w_1, x_2\}$, $\{t, w_2, y_2\}$, $\{t, w_3, z\}$, $\{t, w_4, y_1\}$, and $\{t, w_5, x_1\}$. Therefore, $g(G) = 3$ and $f(G,g) = 1$.

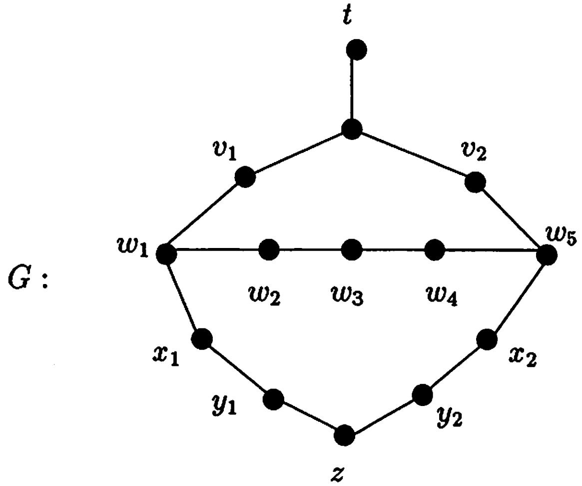

Figure 4: A graph G with $f(G,h) = 0$ and $f(G,g) = 1$

The following result is a direct consequence of Lemma 1.1.

Corollary 1.2 *For a graph G, the forcing hull number* $f(G,h) \geq 2$ *if and only if every vertex of each minimum hull set belongs to at least two minimum hull sets.*

2 Graphs With Prescribed Hull and Forcing Hull Numbers

We have already noted that if G is a graph with $f(G,h) = a$ and $h(G) = b$, then $0 \leq a \leq b$. We show, in fact, that every pair a, b of integers with $0 \leq a \leq b$ is realizable as the forcing hull and hull numbers, respectively, of some graph. In order to verify this fact, we first determine the forcing hull numbers of some well known graphs. The following two observations appeared in [3].

Theorem A If v is a vertex of a graph G such that $\langle N(v) \rangle$ is complete, then v belongs to every hull set of G.

An immediate consequence of Theorem A is that $f(K_n, h) = 0$ for all $n \geq 1$.

Corollary B Each end-vertex of a graph G belongs to every hull set of G.

Since the set of all end-vertices of a tree T is the unique minimum hull set of T (see [3]), it follows that the forcing hull number $f(T, h) = 0$ for all trees T.

Now we determine the forcing hull numbers of cycles.

Theorem 2.1 *The forcing hull number of C_n, $n \geq 3$, is*

$$f(C_n, h) = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

Proof. If n is even, then $h(C_n) = 2$ and every pair of antipodal vertices of C_n is a minimum hull set of C_n. Consequently, C_n does not have a unique minimum hull set but every vertex of C_n has a unique vertex antipodal to it, so $f(C_n, h) = 1$. If n is odd, then $h(C_n) = 3$. Again C_n has more than one minimum hull set. Moreover, every vertex of C_n belongs to at least two distinct minimum hull sets. By Corollary 1.2, $f(C_n, h) \geq 2$. On the other hand, for every pair u, v of adjacent vertices in C_n, there is a unique vertex w in C_n with $d(u, w) = d(v, w)$. Then $S = \{u, v, w\}$ is the unique minimum geodetic set containing $\{u, v\}$, implying that $f(S, h) = 2$. Therefore, $f(C_n, h) = 2$. ■

Next we study the forcing numbers of complete bipartite graphs. It was shown in [3] that if r and s are integers with $1 \leq r \leq s$, then

$$h(K_{r,s}) = \begin{cases} s & \text{if } r = 1 \\ 2 & \text{otherwise} \end{cases} \quad (1)$$

Theorem 2.2 *Let $K_{r,s}$ be a nontrivial complete bipartite graph with $r \leq s$. Then*

$$f(K_{r,s}, h) = \begin{cases} 0 & r = 1 \\ 1 & r = 2 \\ 2 & \text{otherwise} \end{cases}$$

Proof. Let $V_1 = \{u_1, u_2, \cdots, u_r\}$ and $V_2 = \{v_1, v_2, \cdots, v_s\}$ be the partite sets of $K_{r,s}$. If $r = 1$, then $K_{r,s}$ is a tree and so its forcing hull number is 0. If $r = 2$, then $K_{r,s}$ has more than one minimum hull set. So the forcing hull number is at least 1 by Lemma 1.1. Since V_1 is the unique minimum hull set containing u_1, it follows that the forcing hull number is 1. We now assume that $r \geq 3$. Then every minimum hull set S of $K_{r,s}$ is of the form $S = \{x, y\}$, where either $x, y \in V_1$, or $x, y \in V_2$. Since S is not the unique minimum hull set containing any of its proper subsets, $f(S, h) = 2$. Therefore, $f(K_{r,s}, h) = 2$. ■

We now present the aforementioned realization result.

Theorem 2.3 *Every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$ can be realized as the forcing hull number and hull number, respectively, of some graph.*

Proof. We consider two cases.

Case 1. $a = b$. Then $a \geq 2$. For each integer a, we construct a connected graph G_a with the desired property. If $a = 2$, then let $G_2 = K_{3,3}$. By (1) and Theorem 2.2, $h(G_2) = f(G_2, h) = 2$. For $a = 3$, let F_1, F_2, and F_3 be three copies of $K_{3,3}$, where F_i has partite sets U_i and V_i with $U_i = \{u_{i1}, u_{i2}, u_{i3}\}$ and $V_i = \{v_{i1}, v_{i2}, v_{i3}\}$, $1 \leq i \leq 3$. Then G_3 is obtained from F_1, F_2, and F_3 by adding the two edges $u_{13}u_{21}$ and $v_{23}v_{31}$. The graph G_3 is shown in Figure 5.

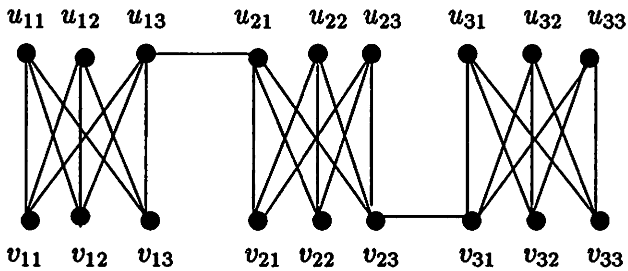

Figure 5: The graph G_3 with $h(G_3) = f(G_3, h) = 3$

We first show that $h(G_3) = 3$. Since $\{u_{11}, u_{22}, v_{33}\}$ is a hull set of G_3, it follows that $h(G_3) \leq 3$. Assume, to the contrary, that $h(G_3) = 2$ and let $W = \{x, y\}$ be a hull set of G_3. Then $V(F_i) \cap W = \emptyset$ for some i $(1 \leq i \leq 3)$. This implies that $V(F_i) \not\subseteq [W]$, contradicting the fact that W is a hull set. Therefore, $h(G_3) = 3$.

Next we show that $f(G_3, h) = 3$. Observe that if S is a minimum hull set of G_3, then, necessarily, $S = \{x, y, z\}$ for some vertices x, y and z, where $x \in \{u_{11}, u_{12}\}$, $y \in \{u_{22}, u_{23}, v_{21}, v_{22}\}$ and $z \in \{v_{32}, v_{33}\}$. Then S is not the unique minimum hull set containing any of its proper subsets. Therefore, $f(G_3, h) = 3$.

For $a \geq 4$, the structure of the graph G_3 described above can be extended to construct a graph G_a. In particular, let $F_1, F_2, \cdots, F_a$ be a copies of $K_{3,3}$. For each i, $1 \leq i \leq a$, let $U_i = \{u_{i1}, u_{i2}, u_{i3}\}$ and $V_i = \{v_{i1}, v_{i2}, v_{i3}\}$

be the partite sets of F_i. The graph G_a is obtained from $F_1, F_2, \cdots, F_a$ by adding $a-1$ new edges $u_{i3}u_{i+1,1}$ (i is odd) and $v_{i3}u_{i+1,1}$ (i is even) between F_i and F_{i+1}, where $1 \le i \le a-1$. A similar argument shows that $h(G_a) = f(G_a, h) = a$.

Case 2. $a < b$. There are two subcases.

Subcase 2.1. $a = 0$ or $a = 1$ For $a = 0$ and $b \ge 2$, $G = K_b$ has the desired property. For $a = 1$ and $b = 2$, let $G = C_{2n+1}$, where $n \ge 2$. By Theorem 2.1, $f(C_{2n+1}, h) = 1$ and $h(C_{2n+1}) = 2$ as desired. So we assume that $a = 1$ and $b \ge 3$. Consider the graph G of Figure 6, and let $U = \{u_1, u_2, \cdots, u_{b-2}\}$.

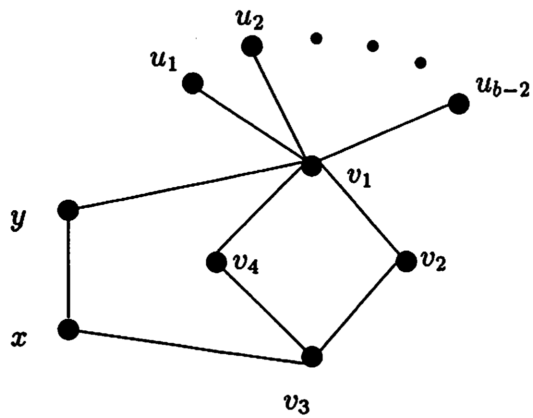

Figure 6: The graph G with $f(G, h) = 1$ and $h(G) = b$

We first show that $h(G) = b$. Since $S = U \cup \{v_3, x\}$ is a hull set of G, it follows that $h(G) \le b$. On the other hand, by Corollary B, every minimum hull set of G contains every vertex of U. Moreover, it is routine to verify that for each $z \notin U$, the set $U \cup \{z\}$ is not a hull set of G, implying that $h(G) \ge b$. Therefore, $h(G) = b$.

Next we show that $f(G, h) = 1$. Since S and $S' = U \cup \{v_3, y\}$ are two distinct minimum hull sets of G, it follows that $f(G, g) \ge 1$ by Lemma 1.1. Moreover, S' is the unique minimum hull set containing $\{y\}$. This implies that $f(S, h) = 1$. Therefore, $f(G, h) = 1$.

Subcase 2.2. $a \ge 2$. There are two subcases here.

Subcase 2.2.1. $b = a + 1$. We consider the graph G of Figure 7.

We first show that $h(G) = a + 1$. Since $S = \{u_2, u_3, \cdots, u_{a+1}, v_1\}$ is a hull set of G, it follows that $h(G) \le a+1$. Next we show that $h(G) \ge a+1$. For every i with $1 \le i \le a+1$, each of u_i and v_i lies only on geodesics with initial or terminal vertex u_i or v_i. This implies that if W is a minimum hull

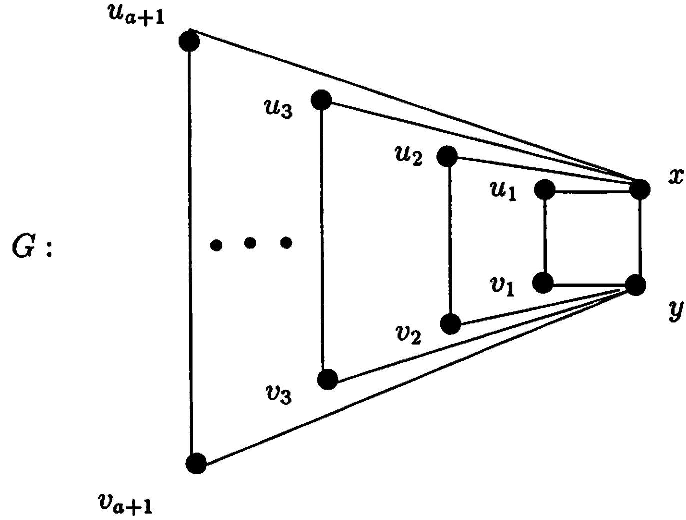

Figure 7: The graph G with $h(G) = f(G,h) + 1$

set of G, then W contains at least one vertex from each set $\{u_i, v_i\}$, where $1 \le i \le a+1$, implying that $h(G) \ge a+1$. Therefore, $h(G) = a+1$.

Next we show that $f(G,h) = a$. Observe that every minimum hull set W of G is a subset of $V(G) - \{x,y\}$ with $|W| = a+1$ such that

$$W \cap \{u_1, u_2, \cdots, u_{a+1}\} \ne \emptyset \quad \text{and} \quad W \cap \{v_1, v_2, \cdots, v_{a+1}\} \ne \emptyset$$

This implies that if W is a minimum hull set of G, then W is not the unique minimum hull set containing any of its subsets W' with $|W'| < a$; that is, $f(W,h) \ge a$ for every minimum hull set W of G. Therefore, $f(G,h) \ge a$. One the other hand, $W = \{u_2, u_3, \cdots, u_{a+1}, v_1\}$ is the unique minimum hull set containing $\{u_2, u_3, \cdots, u_{a+1}\}$, implying that $f(W,h) = a$. Therefore, $f(G,h) = a$.

Subcase 2.2.2. $b \ge a+2$. Consider the graph G of Figure 8.

We first show that $h(G) = b$. Since

$$X = \{u_2, u_3, \cdots, u_{a+1}, v_1, w_1, w_2, \cdots, w_{b-a-1}\}$$

is a hull set, $g(G) \le b$. Next we show that $h(G) \ge b$. By Corollary B, the end-vertices $w_1, w_2, \cdots, w_{b-a-1}$ belong to every hull set of G. Moreover, for every i with $1 \le i \le a+1$, each of u_i and v_i lies only on geodesics with initial or terminal vertex u_i or v_i. This implies that every minimum hull set of G must contain at least one vertex from each set $\{u_i, v_i\}$, where $1 \le i \le a+1$. Therefore, $h(G) \ge (b-a-1) + (a+1) = b$.

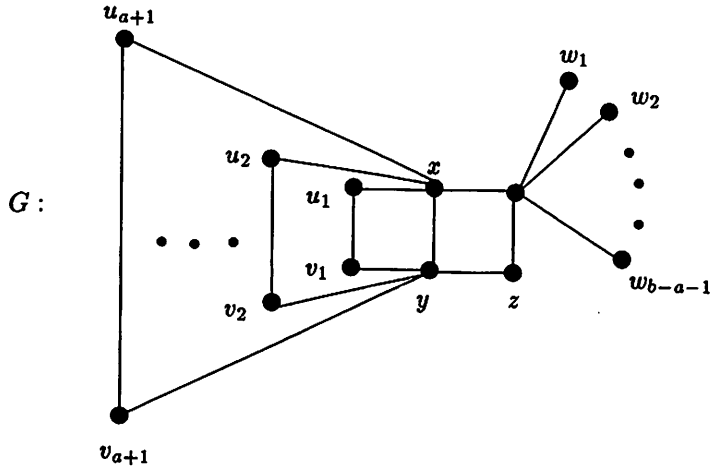

Figure 8: The graph G with $g(G) = b$ and $f(G, h) = a$

Next we show that $f(G, h) = a$. Every minimum hull set S of G has the form

$$S = U \cup V \cup W$$

where $U \subseteq \{u_1, u_2, \cdots, u_{a+1}\}$, $V \subseteq \{v_1, v_2, \cdots, v_{a+1}\}$ and $W = \{w_1, w_2, \cdots, w_{b-a-1}\}$ with $U \neq \emptyset$ and $V \neq \emptyset$. This implies that if S is a minimum hull set of G and $S' \subseteq S$ with $|S'| < a$, then S is not the unique minimum hull set containing S'. Therefore, $f(G, h) \geq a$. One the other hand, the set X defined above is the unique minimum hull set containing $\{u_2, u_3, \cdots, u_{a+1}\}$, implying that $f(X, h) = a$. Therefore, $f(G, h) = a$. ∎

3 Graphs With Prescribed Forcing Hull Number and Forcing Geodetic Number

In this section, we investigate the problem of determining which integers $a, b \geq 0$ are the forcing hull and forcing geodetic numbers, respectively, of some graph. In order to do this, we first present several results on the forcing geodetic numbers of some well known graphs (see [4]).

Theorem C For a nontrivial tree T, the forcing geodetic number $f(T, g)$ is 0.

Theorem D The forcing geodetic number of C_n, $n \geq 3$, is

$$f(C_n, g) = \begin{cases} 1 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

Theorem E Let $K_{r,s}$ be a nontrivial complete bipartite graph with $r \leq s$. Then

$$f(K_{r,s},g) = \begin{cases} 0 & r=1 \quad \text{or} \quad r=2,3 \text{ and } r<s \\ 1 & r=2,3 \quad \text{and} \quad r=s \\ 3 & r=4 \\ 4 & r \geq 5 \end{cases}$$

We now show that each nonnegative integer a is both the forcing geodetic number and forcing hull number of some graph.

Theorem 3.1 *For every integer $a \geq 0$, there exists a connected graph G such that $f(G,h) = a = f(G,g)$.*

Proof. For $a = 0$, a tree T has the desired property. For $a = 1$, an even cycle C_n has $f(C_n,h) = f(C_n,g) = 1$ by Theorems 2.1 and D. So we assume that $a \geq 2$. It was shown in [4] that the graph G of Figure 7 has forcing geodetic number a. Therefore, $f(G,h) = f(G,g) = a$ by Theorem 2.3. ■

Next we show that for many pairs a, b of integers with $0 \leq a < b$, there exist graphs G with $f(G,h) = a$ and $f(G,g) = b$.

Theorem 3.2 *For every integer $k \geq 2$, there exists a connected graph G such that*

$$f(G,h) = k \quad \textit{and} \quad f(G,g) = 2k+2$$

Proof. For each integer $k \geq 2$, we construct a connected graph G_k with the desired property. For $k = 2$, let F_1 and F_2 be two copies of $K_{5,5}$ with $V(F_1) = X \cup Y$ and $V(F_2) = U \cup V$, where $X = \{x_1, x_2, \cdots, x_5\}$, $Y = \{y_1, y_2, \cdots, y_5\}$, $U = \{u_1, u_2, \cdots, u_5\}$, and $V = \{v_1, v_2, \cdots, v_5\}$. Then the graph G_2 is formed from F_1 and F_2 by adding an edge x_5u_1 between F_1 and F_2. The graph G_2 is shown in Figure 9. It was shown in [4] that $f(G_2,g) = 6$.

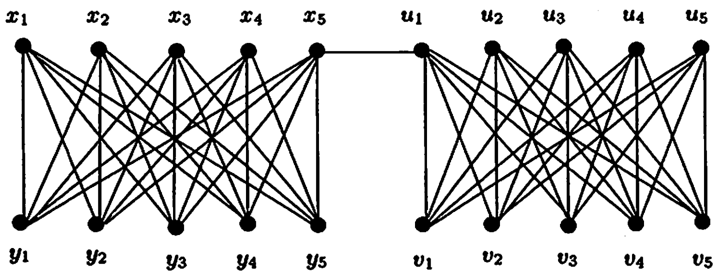

Figure 9: The graph G_2

We now show that $f(G_2,h)=2$. Observe that $h(G_2)=2$ as $\{x_1,u_5\}$ is a hull set. Moreover, every minimum hull set of G_2 is of the form $S=\{x,u\}$ where $x\in\{x_1,x_2,x_3,x_4\}$ and $y\in\{u_2,u_3,u_4,u_5\}$. This implies that S is not the unique minimum geodetic set containing any of its proper subsets. Therefore, $f(G_2,h)=2$.

The structure of G_2 can be extended to construct a graph G_k for all $k\geq 3$. Let $F_1,F_2,\cdots,F_k$ be k copies of $K_{5,5}$. For each i with $1\leq i\leq k$, assume that the partite sets of F_i are $U_i=\{u_{i1},u_{i2},\cdots,u_{i5}\}$ and $V_i=\{v_{i1},v_{i2},\cdots,v_{i5}\}$. Then the graph G_k is formed from $F_1,F_2,\cdots,F_k$ by adding $k-1$ new edges $u_{i5}u_{(i+1)1}$ between F_i and F_{i+1}, where $1\leq i\leq k-1$. It was shown in [4] that $f(G_k,g)=2k+2$. An argument similar to that for G_2 shows that $f(G_k,h)=k$ for all $k\geq 3$. ∎

Theorem 3.3 *For every integer $k\geq 2$, there exists a connected graph G such that*

$$f(G,h)=k \quad \text{and} \quad f(G,g)=3k$$

Proof. For each integer $k\geq 2$, let $H_1,H_2,\cdots,H_k$ be k copies of $K_{5,5}$. For each i with $1\leq i\leq k$, assume that the partite sets of F_i are $U_i=\{u_{i1},u_{i2},\cdots,u_{i5}\}$ and $V_i=\{v_{i1},v_{i2},\cdots,v_{i5}\}$. Then the graph G_k is formed from $F_1,F_2,\cdots,F_k$ by adding a new vertex v and k new edges vu_{i1} where $1\leq i\leq k$. The graph G_3 is shown in Figure 10. It can be verified that $f(G_k,g)=3k$ using a similar argument introduced in [4].

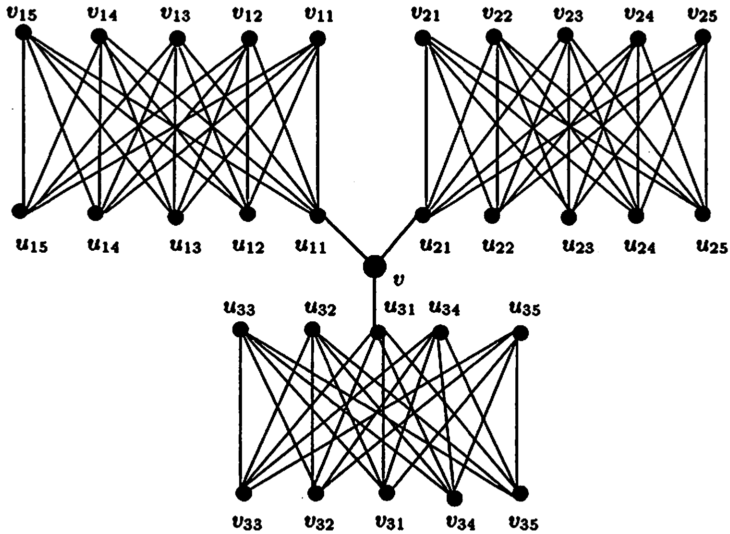

Figure 10: The graph G_3

We now show that $f(G_k, h) = k$. If $k = 2$, then $f(G_2, h) = 2$ by Theorem 3.2. For $k = 3$, we first show that $h(G_3) = 3$. Since $S = \{u_{12}, u_{22}, u_{32}\}$ is a hull set, $h(G_3) \leq 3$. Suppose, to the contrary, that $h(G_2) = 2$ and let $W = \{x, y\}$ be a hull set of G. Then $W \cap V(F_i) = \emptyset$ for some i $(1 \leq i \leq 3)$. This implies that $V(F_i) \not\subseteq [W]$, which is a contradiction. Therefore, $h(G_3) = 3$. We now show that $f(G_3, h) = 3$. Observe that every minimum hull set of G_3 is of the form $S = \{x, y, z\}$ where $x \in \{u_{12}, u_{13}, u_{14}, u_{15}\}$, $y \in \{u_{22}, u_{23}, u_{24}, u_{15}\}$, and $z \in \{u_{32}, u_{33}, u_{34}, u_{35}\}$. This implies that S is not the unique minimum geodetic set containing any of its proper subsets. Therefore, $f(G_3, h) = 3$. An argument similar to that for G_3 shows that $h(G_k) = f(G_k, h) = k$ for all $k \geq 4$. ■

We have seen, by Theorems 3.2 and 3.3, that for many integers a and b with $2 \leq a < b$, there exists a graph G with $f(G, h) = a$ and $f(G, g) = b$. Also, by Theorems 2.2 and E, $f(K_{4,4}, h) = 2$ and $f(K_{4,4}, g) = 3$; while $f(K_{5,5}, h) = 2$ and $f(K_{5,5}, g) = 4$. Hence for all integers a and b with $2 \leq a < b$, where $2|b$ or $3|b$, there exists a graph G with $f(G, h) = a$ and $f(G, g) = b$. However, numerous gaps exist. For example, we do not know if there is a graph G with $f(G, h) = 2$ and $f(G, g) = 5$. On the other hand, we have no general results for $a = 0$ or $a = 1$. The graph G of Figure 4 has $f(G, h) = 0$ and $f(G, g) = 1$. However, we are left with an interesting conjecture.

Conjecture For every nonnegative integer b, there exists a graph G having a unique minimum hull set and forcing geodetic number b.

The situation is even more complex however. Obviously, if G is a connected graph, then $h(G) \leq g(G)$. From our discussion, one might easily infer that $f(G, h) \leq f(G, g)$ for every nontrivial connected graph G, but this is not the case, as the following result shows.

Theorem 3.4 *For every pair a, b of integers with $a < b$ and $a \in \{0, 1, 3, 5, \cdots\}$, there exists a connected graph G such that $f(G, g) = a$ and $f(G, h) = b$.*

Proof. We consider two cases.

Case 1. $a = 0$. For $b = 1$, let $G = K_{2,3}$. Then, by Theorems E and 2.2, $f(G, g) = 0$ and $f(G, h) = 1$. For $b = 2$, let $G = K_{3,4}$. Similarly, $f(G, g) = 0$ and $f(G, h) = 2$. Therefore, we assume that $b \geq 3$. Suppose that F_1 and F_b are two copies of $K_{3,3}$ and $F_2, F_3, \cdots, F_{b-1}$ are $b - 2$ copies of $K_{3,4}$. Assume that U_i and V_i are the partite sets of F_i, where $U_i = \{u_{i1}, u_{i2}, u_{i3}\}$ $(1 \leq i \leq b)$, $V_i = \{v_{i1}, v_{i2}, v_{i3}\}$ $(i = 1, b)$, and $V_i = \{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}$ $(2 \leq i \leq b - 1)$. Then the graph G_b is obtained from $F_1, F_2, \cdots, F_b$ by adding $b - 1$ edges $u_{i3}u_{i+1,1}$ (i odd) and $v_{i4}u_{i+1,1}$ (i even) between F_i and F_{i+1}, where $1 \leq i \leq b - 1$. The graph G_b is shown in Figure 11. An argument similar to that of Theorem 2.3 will show that $f(G_b, h) = b$. Moreover, it is routine to verify that G_b has a unique minimum geodetic set. Therefore, $f(G_b, g) = 0$.

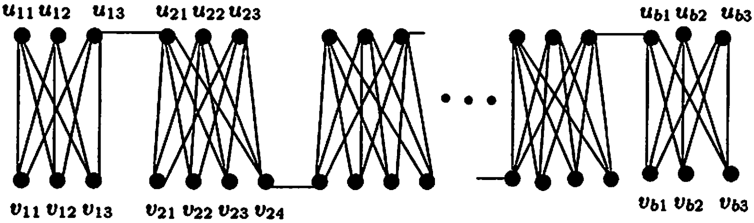

Figure 11: The graph G_b with $f(G_b, h) = b$ and $f(G_b, g) = 0$

Case 2. a is odd. Let $a = 2t - 1$, where $t \geq 1$ is an integer. We now consider the graph G of Figure 12.

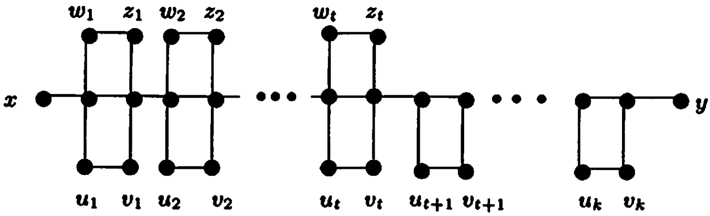

Figure 12: The graphs G with $f(G,h) = k + t$ and $f(G, g) = 2t - 1$

We first show that $f(G, g) = 2t - 1$. Observe that G has two minimum geodetic sets, that is,

$$
\begin{aligned}
S &= \{x, y, u_1, z_1, u_2, z_2, \cdots, u_t, z_t, u_{t+1}, v_{t+1}, \cdots, u_k, v_k\} \\
S' &= \{x, y, v_1, w_1, v_2, w_2, \cdots, v_t, w_t, u_{t+1}, v_{t+1}, \cdots, u_k, v_k\}
\end{aligned}
$$

Hence $g(G) = 2t + 2$. Since S is not the unique minimum geodetic set containing any of its proper subsets S' with $|S'| < 2t-1$, but S is the unique minimum geodetic set containing $\{z_1, u_2, z_2, \cdots, u_t, z_t\}$, it follows that $f(S, g) = 2t - 1$. Similarly, $f(S', g) = 2t - 1$. Therefore, $f(G, g) = 2t - 1$.

Next we show that $f(G, h) = k + t$. Observe that

$$W = \{x, y, u_1, z_1, u_2, z_2, \cdots, u_t, z_t, u_{t+1}, u_{t+2}, \cdots, u_k\}$$

is a minimum hull set of G and other minimum hull sets of G are similar to W. So $h(G) = k + t + 2$. Moreover, W is not the unique minimum geodetic set containing any of its proper subsets W' with $|W'| < k + t$ and W is the unique minimum hull set containing { u_1, z_1, u_2, z_2, $\cdots$, u_t, z_t, u_{t+1}, u_{t+2}, $\cdots$, u_k }; so $f(W,h) = k+t$. Therefore, $f(G,h) = k+t$. Leting $k = b - t = b - \frac{a+1}{2}$, we have $f(G,h) = b$. ■

Based on this information, we close with an even more intriguing conjecture.

Conjecture For every pair a, b of nonnegative integers, there exists a connected graph G with $f(G,g) = a$ and $f(G,h) = b$.

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