Two New 1-rotational (36,9,8) and (40,10,9) RBIBDs

Luis B. Morales
IIMAS, Universidad Nacional Autónoma de México
Apdo. Postal 70-221, México, DF, 04510, México
lbm@servidor.unam.mx

Abstract

We formulate the construction of 1-rotational difference families as a combinatorial optimization problem. A tabu search algorithm is used to find an optimal solution to the optimization problem for various 1-rotational difference family parameters. In particular, we construct two new 1-rotational difference families which lead to an equal number of new 1-rotational RBIBDs with parameters: $(36, 9, 8)$ and $(40, 10, 9)$. Our algorithm also was able to construct six nonisomorphic $(36, 9, 8)$ and three $(40, 10, 9)$ RBIBDs.

1 Introduction

In a previous paper [5] the construction of difference families was formulated as an optimization problem. The author of [5] constructed many difference families using a tabu search (TS) algorithm. In particular, he constructed six new difference families which lead to an equal number of new balanced incomplete block designs (BIBDs). In this paper, we also formulate the construction of resolvable 1-rotational difference families as a discrete optimization problem. We present an algorithm based on tabu search [3] to tackle the problem. Our new adaptation of TS is similar to the algorithm developed in the paper [5]; however, the feasible solutions, neighborhood function and tabu list concepts defined here are different from those used in [5].

Our TS algorithm was able to construct many resolvable 1-rotational difference families. In particular, we constructed two new resolvable 1-rotational difference families (see [1]) which lead to an equal number of new resolvable balanced incomplete block designs (RBIBDs) with parameters:

(36, 9, 8) and (40, 10, 9) (see [4]). We also consider the construction of nonisomorphic designs for these parameters.

To conclude this introduction, we give three types of design definitions.

A BIBD is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of b k-subsets (*blocks*) of V, $k < v$, such that each element of V is contained in exactly r blocks, and each pair of elements of V occur together in exactly λ blocks. The numbers v, b, r, k and λ are *parameters* of the BIBD. It is easy to see that these numbers satisfy

$$vr = bk, \quad r(k-1) = \lambda(v-1). \tag{1.1}$$

So, the three parameters v, k and λ determine the remaining two as $r = \lambda(v-1)/(k-1)$ and $b = vr/k$. Hence, a BIBD with parameters v, b, r, k and λ is sometimes called a (v, k, λ)-BIBD.

A (v, k, λ)-BIBD $(V, \mathcal{B})$ is *resolvable* if the collection $\mathcal{B}$ of blocks can be divided into r subsets called *parallel classes* of size $q = b/r$ such that every element of V appears in one block from each class.

A (v, k, λ)-BIBD $(V, \mathcal{B})$ is 1-rotational if $V = \{\infty\} \cup Z_{v-1}$, and the mapping ϕ from i to $i+1 \pmod v$, fixing ∞, is an automorphism of the design.

2 1-Rotational Difference Families

Definition 2.1. *Let $S_1 \ldots, S_s, T_1, \ldots, T_t$ be k-subsets of $V = \{\infty\} \cup Z_m$ such that*

(1) *Each $S_1, \ldots, S_s$ contains only elements from Z_m, and each $T_1, \ldots, T_t$ contains ∞ with elements from Z_m.*

(2) *Let $S_i = \{s_{i1}, \ldots, s_{ik}\}$ $(1 \le i \le s)$ and $T_j = \{t_{j1}, \ldots, t_{jk-1}, \infty\}$ $(1 \le j \le t)$. Then, every nonzero element of Z_m occurs λ times among the differences $s_{ix} - s_{iy}$ $(i = 1, \ldots, s;\ x, y = 1, \ldots, k)$ and $t_{jz} - t_{jw}$ $(j = 1, \ldots, t;\ z, w = 1, \ldots, k-1)$.*

Then, we say that $S_1, \ldots, S_s, T_1, \ldots, T_t$ form a 1-rotational difference family with parameters m, s, t, k and λ. The sets $S_1, \ldots, S_s, T_1, \ldots, T_t$ are called base blocks.

If the sets $S_1 \ldots, S_s, T_1, \ldots, T_t$ satisfy only condition (1) of Definition 2.1, we say that these sets form a *1-rotational family*.

Let us now define the concept of resolvability for difference families. A 1-rotational difference family with parameters m, s, t, k and λ is called *resolvable* if the set of the (s+t) base blocks can be divided into t subsets $\{S_{11}, \ldots, S_{1q}, T_1\}, \ldots, \{S_{t1}, \ldots, S_{tq}, T_t\}$ called *parallel classes* of size $q+1 =$

$(s+t)/t$ such that every element of $V = \{\infty\} \cup Z_m$ appears in one block from each class. Afterwards, $q = s/t$.

If $D = \{d_1, \ldots, d_k\}$ is a k-subset of Z_m then the set $D + g = \{d_1 + g, \ldots, d_k + g\}$ $(g \in Z_m)$ (mod m) is called a *translate* of D.

The following theorem is due to Bose [2].

Theorem 2.1. *Let $S_1 \ldots, S_s, T_1, \ldots, T_t$ be a 1-rotational difference family with parameters m, s, t, k and λ. Then the sets $S_i + g$ $(i = 1 \ldots, s;$ $g = 0, \ldots, m-1)$ and $T_j + g$ $(j = 1 \ldots, t;$ $g = 0, \ldots, m-1)$, with the convention that $\infty + g = \infty$, form a 1-rotational design with parameters $v = m+1$, $b = m(s+t)$, $r = mt$, k and λ.*

It follows from (1.1) and Theorem 2.1 that necessary conditions for the existence of a 1-rotational difference family with parameters (m, s, t, k, λ) are

$$(m+1)t = (s+t)k, \quad \lambda = t(k-1). \tag{2.1}$$

Moreover, if the difference family is resolvable then t divides s. This and the first equation of (2.1) show that

$$m = kq + k - 1. \tag{2.2}$$

If the 1-rotational difference family with parameters m, s, t, k and λ in Theorem 2.1 is resolvable, then it is not hard to prove that it leads to a resolvable 1-rotational $(m+1, k, t(k-1))$ design. Further, the sets of blocks

$$\{S_{11}, \ldots, S_{1q}, T_1\}, \ldots, \{S_{t1}, \ldots, S_{tq}, T_t\}$$

form parallel classes of the design, and the translates give $r(= mt)$ parallel classes in all.

The data structure we will use to store a resolvable 1-rotational family with parameters m, s, t, k and λ is a $t \times (q+1) \times m$ array $A = (a_{h\ell g})$, where

$$a_{h\ell g} = \begin{cases} 1, & \text{if } g \in S_{h\ell}, \text{ or } g \in T_h \text{ and } \ell = q+1, \\ 0, & \text{otherwise.} \end{cases}$$

We call A the *incidence* array of the 1-rotational family. Clearly, the array A satisfies the following relations:

$$\sum_{g=0}^{m-1} a_{h\ell g} = k \ (\ell = 1, \ldots, q), \quad \sum_{g=0}^{m-1} a_{h\ell g} = k - 1 \ (\ell = q+1), \tag{2.3}$$

for $h = 1 \ldots, t$. Moreover, if the 1-rotational family is resolvable we have

$$\sum_{\ell=1}^{q+1} a_{h\ell g} = 1 \ (g = 0, \ldots, m-1; h = 1, \ldots, t). \tag{2.4}$$

For a 1-rotational family $S_1, \dots, S_s, T_1, \dots, T_t$, we define an m-vector $B = (b_g)$ associated to it, where b_g is the number of times that $g \in Z_m - \{0\}$ is a difference arising from the $s + t$ sets, and $b_0 = \lambda$. Note that a 1-rotational family is a 1-rotational difference family if and only if each entry of the vector B is λ.

Let us now formulate the problem of construction of resolvable 1-rotational difference families as an optimization problem. Fix the integers m, s, t, k and λ satisfying (2.1). A feasible solution is a $t \times (q+1) \times m$ array A of zeros and ones that satisfies (2.3) and (2.4). Define the objective function as

$$f(A) = \sum_{g=0}^{m-1} (b_g - \lambda)^2, \tag{2.5}$$

where the vector (b_g) is associated with the sets $S_1, \dots, S_s, T_1, \dots, T_t$ defined by the array A. A 1-rotational difference family can be constructed if and only if the value of $f(A)$ is 0 in a global minimum.

3 Tabu Search

The tabu search method is a stochastic procedure for solving combinatorial optimization problems of a general type:

$$\text{minimize } f(x) \quad \text{subject to } x \in X,$$

where f is an objetive function and X is a set of feasible solutions. The procedure has been applied to a wide variety of problems.

TS is an iterative process. It starts from an initial feasible solution and tries to reach a global minimum by moving from one solution to another, subject to define the *neighborhood* function $N : X \to 2^X$. For each feasible solution x, the neighborhood $N(x)$ is the set of all feasible solutions directly reachable from x by a simple move. During each iteration, we move from the actual solution x to the best solution x^* in $N(x)$, whether or not $f(x^*) < f(x)$. If there are multiple minimal solutions, the tie is broken randomly.

To prevent cycling, a queue called the *tabu list* T of *length* $|T|$ is provided. Its aim is to forbid moves between solutions that reinstate certain attributes of past solutions. After $|T|$ iterations, such moves are removed from the list, and are free to be reinstated. However, it may happen that an interesting move (such as a move that improves the best solution found so far) In order to cancel the tabu status of such moves, an *aspiration criterion* is introduced.

Now a stopping rule should also be defined; in many cases (including our optimization problem) a lower bound of the objective function is known in advance. As soon as we have reached this bound, we may interrupt the procedure. In general, however, this bound is not available with sufficient accuracy, so a fixed maximum number of iterations is given.

4 Adaptation of Tabu Search

In the TS adaptation for constructing a resolvable 1-rotational difference family with parameters m, s, t, k and λ, we consider to be a feasible solution any $t \times (q+1) \times m$ array A of zeros and ones that satisfy (2.3) and (2.4). The objective function is the function (2.5).

Two resolvable 1-rotational families

$$S_{11}, \dots, S_{1q}, T_1, \dots, S_{t1}, \dots, S_{tq}, T_t$$

and

$$S'_{11}, \dots, S'_{1q}, T'_1, \dots, S'_{t1}, \dots, S'_{tq}, T'_t$$

with parameters m, s, t, k and λ are said to be *neighbors* if they are identical for every parallel class but one, and in the exceptional parallel class h, exactly two elements u and w of Z_m switch blocks. A move is entirely defined by the vector (h, u, w). In terms of the incidence array, A' is a neighboring solution of A if, for some h, i, j, u, w,

$$a_{hiu} = 1,\ a_{hiw} = 0,\ a_{hju} = 0,\ a_{hjw} = 1,$$

$$a'_{hiu} = 1,\ a'_{hiw} = 0,\ a'_{hju} = 0,\ a'_{hjw} = 1$$

and

$$a'_{dpg} = a_{dpg} \quad \text{for all} \quad (d,p,g) \neq (h,i,u),(h,i,w),(h,j,u),(h,j,w).$$

Let us now calculate the size of the neighborhood. In each parallel class h, there are two forms to choose the two blocks involved in the switch: S_{hi} and S_{hj}, or S_{hi} and T_h. For the first, there are $\binom{q}{2}$ ways to choose the two sets, and k ways to choose an element from each of the two sets. For the second form, there are q ways to choose the blocks S_{hi} and T_h, and k and $k-1$ ways to choose a finite element from the sets S_{hi} and T_h, respectively. Since there are t parallel classes, the size of the neighborhood is $tk\left[\binom{q}{2}k + q(k-1)\right]$.

An initial solution is generated as follows: Since $kq + k - 1 = m$ (see (2.2)), then in each parallel class h, we can randomly split the m elements of Z_m into the sets $S_{h1}, \dots, S_{hq}, T_h$, with k elements of the sets S_{hi} and the set T_h with $k-1$ elements and ∞. Note that every element of $V = \{\infty\} \cup Z_m$ appears in one set of the parallel class $\{S_{h1}, \dots, S_{hq}, T_h\}$. Hence, the t parallel classes $\{S_{11}, \dots, S_{1q}, T_1\}, \dots, \{S_{t1}, \dots, S_{tq}, T_t\}$ form a resolvable 1-rotational family with parameters m, s, t, k and $\lambda = t(k-1)$.

Whenever the elements u and w switched blocks at a parallel class, the tabu list forbids, during the preceding $|T|$ iterations, any exchange of the elements u or w. Formally, the tabu list consists of vectors (h, u, w), where the elements u or w could not be changed by any element in the parallel class h. The length of the tabu lists was adjusted experimentally depending on the difference family parameters. For the tabu list, the best tabu lengths seem to be integers somewhere between 3 and 7.

A very simple aspiration criterion was used. This criterion allows the tabu status of a move from A to A' to be overridden if the value $f(A')$ is strictly better than the best value obtained so far. This means that the tabu status of a move from A to A' may be dropped if $f(A') < f(A^o)$, where A^o is the best solution found so far.

The process stops if the objective function has reached the global minimum 0. However, since TS is a heuristic technique, it does not always guarantee reaching an optimal solution, and the search process will be stopped if the number of iterations used without improving the best solution is greater than a $nimax = 800$ limit.

5 Results

The TS algorithm described above was implemented in C language. It is available for free downloading from the website

`http://www.mcc.unam.mx/~lbm`

All the runs of the tabu search procedure were carried out with a random initial solution.

Although TS was also tested on many 1-rotational difference family parameters, we give here only the results on seven RBIBDs whose existence remained unknown so far, and the best value found by our algorithm was less or equal to 4. Fortunately, our technique was able to construct two of them. Now, we give the parameters of the two 1-rotational resolvable designs constructed by means of the tabu search method.

For constructing $(36, 9, 8)$-RBIBDs, we used the resolvable 1-rotational difference family with parameters $m = 35$, $s = 3$, $t = 1$, $k = 9$, $\lambda = 8$. For these difference family parameters, six optimal solutions were found after 25000 runs. Moreover, all $(36, 9, 8)$-RBIBDs constructed are nonisomorphic. The base blocks of the six difference families generated by TS are presented in Table 1. The base blocks of each row form a parallel class.

For constructing $(40, 10, 9)$-RBIBDs, we used the resolvable 1-rotational difference family with parameters $m = 39$, $s = 3$, $t = 1$, $k = 10$, $\lambda = 9$. For these difference family parameters, three optimal solutions were found after 60000 runs. Moreover, all (40, 10, 9)-RBIBDs constructed are nonisomorphic. The base blocks of the three difference families generated by TS are presented in Table 2. The base blocks of each row form a parallel class.

Table 1. The base blocks of six nonisomorphic (36, 9, 8)-RBIBDs

1	$\{0,9,16,17,18,19,22,24,30\}$, $\{3,4,6,12,21,23,27,31,34\}$,	$\{2,5,7,8,14,20,28,32,33\}$, $\{1,10,11,13,15,25,26,29,\infty\}$
2	$\{7,11,12,15,17,20,22,28,29\}$, $\{0,1,2,5,9,16,21,24,34\}$,	$\{3,4,6,13,23,26,27,30,32\}$, $\{8,10,14,18,19,25,31,33,\infty\}$
3	$\{2,16,18,19,23,25,29,32,33\}$, $\{1,3,4,7,10,12,26,27,28\}$,	$\{6,9,11,17,21,22,24,30,34\}$, $\{0,5,8,13,14,15,20,31,\infty\}$
4	$\{0,5,6,11,15,24,26,33,34\}$, $\{9,12,19,22,23,25,27,30,31\}$,	$\{1,3,8,13,17,18,20,21,32\}$ $\{2,4,7,10,14,16,28,29,\infty\}$
5	$\{0,7,14,16,24,25,29,30,32\}$, $\{1,6,9,10,15,18,21,22,34\}$,	$\{2,4,8,11,12,13,26,31,33\}$, $\{3,5,17,19,20,23,27,28,\infty\}$
6	$\{2,4,6,12,18,27,28,30,31\}$, $\{5,10,17,21,22,23,25,26,34\}$,	$\{1,3,8,9,16,19,24,29,33\}$, $\{0,7,11,13,14,15,20,32,\infty\}$

Table 2. The base blocks of three nonisomorphic (40, 10, 9)-RBIBDs

1	$\{6,10,11,14,16,18,19,20,32,38\}$, $\{0,1,3,5,15,21,24,30,31,37\}$,	$\{2,4,8,13,17,25,27,28,35,36\}$ $\{7,9,12,22,23,26,29,33,34,\infty\}$
2	$\{1,5,7,8,12,17,19,21,27,29\}$, $\{2,13,16,20,22,23,28,35,36,38\}$,	$\{0,9,11,14,15,18,25,26,30,31\}$ $\{3,4,6,10,24,32,33,34,37,\infty\}$
3	$\{0,4,6,8,12,13,15,24,25,35\}$, $\{1,3,14,18,19,21,28,32,37,38\}$,	$\{9,10,11,16,17,22,23,26,31,34\}$ $\{2,5,7,20.27,29.30,33,36,\infty\}$

Even starting from many (around 50000) random initial solutions, modifying the values of tabu length and the maximum number of iterations, TS did not produce the theoretical optimal solution for the resolvable 1-rotational difference family parameters: (14, 2, 1, 5, 4), (23, 2, 1, 8, 7), (29, 2, 1, 10, 9), (33, 2, 1, 11, 10), and (38, 2, 1, 13, 12), and their existence remains unknown. However, for these parameters the best value found was 4. This value is very close to the global minimum zero. Lastly, we believe that these problems pose a formidable challenge for further investigations in combinatorial global techniques.

6 Conclusions

We have formulated the problem for constructing resolvable 1-rotational difference families as a discrete optimization problem, where 0 is the global minimum. We have shown that it is possible to find optimal solutions to this problem using a procedure based on TS for many instances. In particular, it produced some new resolvable 1-rotational difference families which lead to six nonisomorphic $(36, 9, 8)$ and three $(40, 10, 9)$ RBIBDs whose existence was still undecided. Unfortunately, our procedure did not produce optimal solution for many of the tested problem instances. However, for the resolvable 1-rotational difference families with parameters $(14, 2, 1, 5, 4)$, $(23, 2, 1, 8, 7)$, $(29, 2, 1, 10, 9)$, $(33, 2, 1, 11, 10)$, and $(38, 2, 1, 13, 12)$ the local minimum cost of 4 was found. This value is very close to the global minimum zero.

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