On Orbital Domination Numbers of Graphs

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Abstract

If the distance between two vertices u and v in a graph G is k, then u and v are said to k-step dominate each other. A set S of vertices of G is a k-step dominating set if every vertex of G is k-step dominated by some vertex of S. The minimum cardinality of a k-step dominating set is the k-step domination number $\rho_k(G)$ of G. A sequence $s: \ell_1, \ell_2, \ldots, \ell_k$ of positive integers is called an orbital dominating sequence for G if there exist distinct vertices $v_1, v_2, \ldots, v_k$ of G such that every vertex of G is ℓ_i-step dominated by v_i for some i $(1 \leq i \leq k)$. An orbital dominating sequence s is minimal if no proper subsequence of s is an orbital dominating sequence for G. The minimum length of a minimal orbital dominating sequence is the orbital domination number $\gamma_o(G)$, while the maximum length of such a sequence is the upper orbital domination number $\Gamma_o(G)$ of G.

It is shown that for every pair i, j of positive integers with $i < j$, there exist graphs G and H such that both $\rho_i(G) - \rho_j(G)$ and $\rho_j(H) - \rho_i(H)$ are arbitrarily large. Also, there exist graphs G of arbitrarily large radius such that $\gamma_o(G) < \rho_i(G)$ for every integer i $(1 \leq i \leq \operatorname{rad} G)$. All trees T with $\gamma_o(T) = 3$ are characterized, as are all minimum orbital sequences of length 3

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for graphs. All graphs G with $\Gamma_o(G) = 2$ are characterized, as are all trees T with $\Gamma_o(T) = 3$.

1 Introduction

One of the major areas of research in graph theory in recent years has been domination in graphs. Indeed, the book by Haynes, Hedetniemi, and Slater [3] is devoted entirely to this subject. A vertex is said to *dominate* its neighbors as well as itself. The *neighborhood* $N(v)$ of a vertex v in a graph G is the set of vertices adjacent to v; while the *closed neighborhood* $N[v]$ is defined by $N[v] = N(v) \cup v$. Thus, a vertex dominates each vertex in its closed neighborhood. A set S of vertices in a graph G is a *dominating set* if every vertex of G is dominated by at least one vertex of S. The minimum cardinality of a dominating set in G is called the *domination number* $\gamma(G)$ of G.

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the minimum length of a u–v path in G. For a nonnegative integer k, the *k-neighborhood* $N_k(v)$ of a vertex v is the set of all vertices at distance k from v; while the *closed k-neighborhood* $N_k[v]$ is defined as $N_k[v] = \{u \in V(G) \mid d(u,v) \leq k\}$. Thus, $N_0(v) = N_0[v] = \{v\}$, $N_1(v) = N(v)$, and $N_1[v] = N[v]$. The *eccentricity* $e(v)$ of a vertex v in G is the distance from v to a vertex furthest from v. Thus, for every vertex v in a connected graph G, it follows that $N_{e(v)}[v] = V(G)$. The minimum eccentricity among the vertices of G is called the *radius* $\operatorname{rad} G$ of G and the maximum eccentricity is its *diameter* $\operatorname{diam} G$.

For a positive integer k, a vertex v in a graph G is said to *k-dominate* a vertex u if $d(u,v) \leq k$. Therefore, v k-dominates all vertices in its closed k-neighborhood $N_k[v]$. A set S of vertices in G is a *k-dominating set* if every vertex of G is k-dominated by some vertex of S. The *k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a k-dominating set. A survey of distance domination in graphs has been written by Henning [4]. If $d(u,v) = k$, then u

and v are said to *k-step dominate* each other. A set S of vertices in G is a *k-step dominating set* for G if every vertex of G is k-step dominated by some vertex of S. The *k-step domination number* $\rho_k(G)$ is the minimum cardinality of a k-step dominating set for G. The parameter $\rho_1(G)$ is also referred to as the *open domination number* of G. Clearly, $\gamma(G) \leq \rho_1(G)$ for every graph G. It was shown by Hayes, Schultz, and Yates [2] that $\rho_k(G)$ is well-defined if and only if $\operatorname{rad} G \geq k$.

A sequence $s\colon \ell_1, \ell_2, \ldots, \ell_k$ of positive integers is called an *orbital sequence* for a graph G if G contains distinct vertices $v_1, v_2, \ldots, v_k$ such that $\cup_{i=1}^{k} N_{\ell_i}(v_i) = V(G)$. Equivalently, s is a orbital sequence for G if every vertex of G is ℓ_i-step dominated by v_i for some i $(1 \leq i \leq k)$. We refer to ℓ_i as the *step* of v_i and write step $v_i = \ell_i$.

An orbital sequence s for a graph G is *minimal* if no proper subsequence of s is an orbital sequence for G. A *minimum orbital sequence* for G is a (minimal) orbital sequence of minimum length. The length of a minimum orbital sequence for G is called the *orbital domination number*, or more simply the *orbital number* of G, and is denoted by $\gamma_o(G)$. These concepts were introduced by Hayes, Schultz, and Yates [2].

Obviously, $\gamma_o(G) \leq \rho_k(G)$ for every graph G and every positive integer $k \leq \operatorname{rad} G$. For the graph G of Figure 1a, $\gamma_o(G) = 3$ while $\rho_1(G) = \rho_2(G) = 4$. Figure 1b shows a minimum orbital sequence for G; while Figures 1c and 1d show a minimum 1-step dominating set and minimum 2-step dominating set, respectively, for G.

This terminology comes from selecting certain vertices of G, which we call the planets of G. Each planet has an associated radius and those vertices whose distance from a given planet is the radius of that planet, constitute the orbit of the planet. Our goal is to select appropriate planets with suitable radii so that every vertex of G lies on the orbit of some planet.

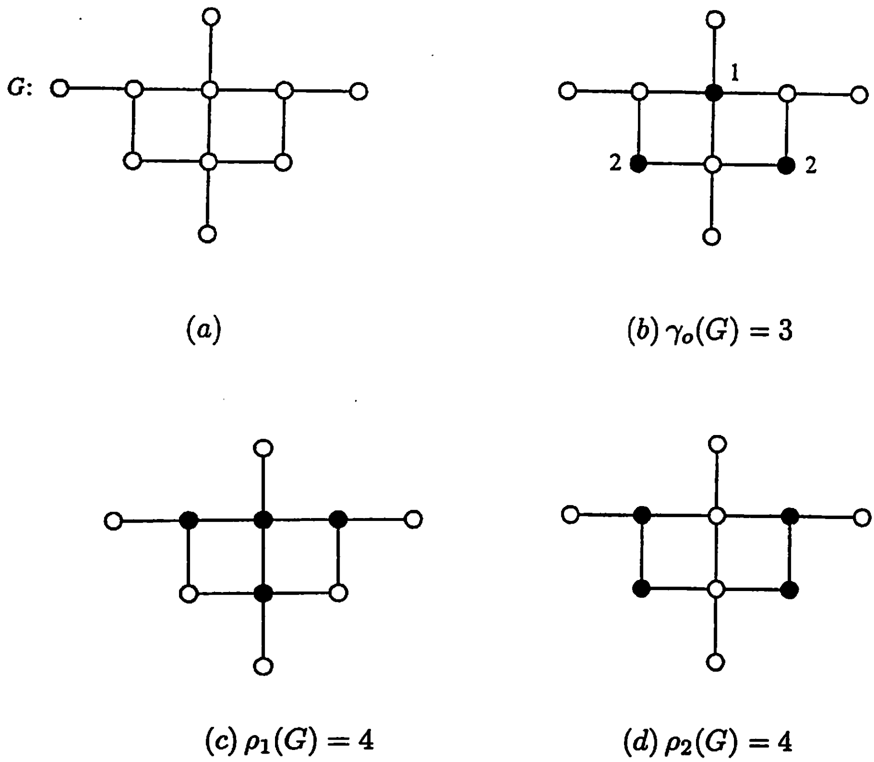

Figure 1.

2 Orbital and k-Step Domination Numbers

First, we show that there is no relationship among the numbers $\rho_i(G)$, $i = 1, 2, \ldots$, in general for an arbitrary graph G.

Theorem 1 *For positive integers i, j, and n with $i < j$, there exist graphs G and H such that $\rho_i(G) - \rho_j(G) \geq n$ and $\rho_j(H) - \rho_i(H) \geq n$.*

Proof. Let G be the tree obtained by subdividing each edge of the star $K_{1,n+2j}$ a total of $j-1$ times. Thus G is produced by identifying end-vertices of $n+2j$ paths of length j. Let v be the central vertex of G. If we assign step j to all vertices of one of these paths (including v) and assign step j to all vertices x on a second path for which

$d(v,x) < j$, then we have a j-step dominating set of cardinality $2j$; so $\rho_j(G) \leq 2j$. On the other hand, each end-vertex z of G can be i-step dominated only by the vertex y on that path for which $d(y,z) = i$. Consequently, $\rho_i(G) \geq n + 2j$, and so $\rho_i(G) - \rho_j(G) \geq n$.

Next let H be the tree obtained by subdividing each edge of the star $K_{1,2n+2}$ a total of $3j-1$ times. Let v be the central vertex of H, and let $Q_1, Q_2, \ldots, Q_{2n+2}$ be the $2n+2$ disjoint paths of length $3j-1$ that do not contain v. Then any j-step dominating set of H must contain at least $2j$ vertices from each path Q_ℓ, $\ell = 1, 2, \ldots, 2n+2$. (See Figure 2.) Thus, $\rho_j(H) \geq (4n+4)j$, or, equivalently, $\rho_j(H) \geq p - (2n+2)j - 1$ where $p = |V(H)|$.

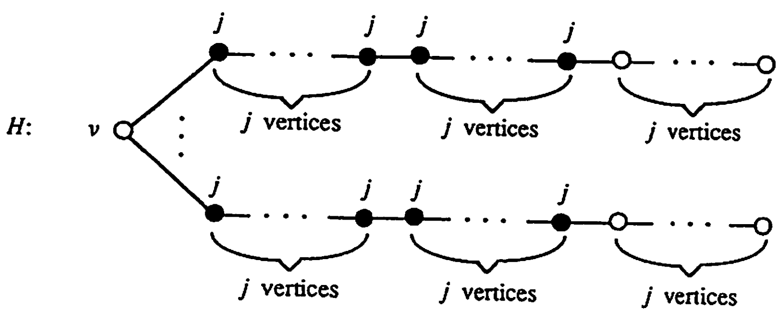

Figure 2.

If $4i \geq 3j$, then assign step i to all vertices x at distance at least i but at distance at most $3i$ ($< 3j$) from an end-vertex of H to i-step dominate all vertices of H except possibly v (if $4i = 3j$). (See Figure 3.) Thus, $\rho_i(H) \leq (4n+4)i + 1 \leq 4(n+1)(j-1) + 1 \leq \rho_j(H) + 1 - 4n - 4$, so $\rho_j(H) - \rho_i(H) \geq 4n + 3 > n$.

So we may assume that $4i < 3j$. Thus $3j = 4ki + r$ for some integers $k \geq 1$ and $0 \leq r < 4i$. We consider four possibilities.

Case 1. $3i \leq r < 4i$. Assign step i to all vertices x for which $r - 3i + 1 \leq d(x,v) \leq r - i$ or for which $d(x,v) = (4t+1)i + r + s$ where $0 \leq t \leq k-1$ and $1 \leq s \leq 2i$ to i-step dominate all vertices of H. (See Figure 4.) Thus, $\rho_i(H) \leq p - (2n+2)(r - 2i + 2ki) - 1$.

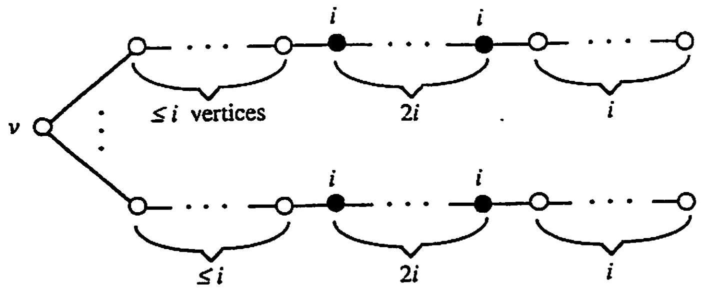

Figure 3.

Hence,

$$\begin{aligned}
\rho_j(H) - \rho_i(H) &\geq (p - 2(n+1)j - 1) - (p - 2(n+1)r + 4ni + 4i \\
&\quad -4(n+1)ki - 1) \\
&= -2nj - 2j + 2nr + 2r - 4ni - 4i + 4nki + 4ki \\
&= -2(n+1)j + 2(n+1)r - 4(n+1)i + \\
&\quad (n+1)(3j - r) \quad (\text{since } k = (3j-r)/4i) \\
&= (n+1)j + (n+1)r - 4(n+1)i \\
&= (n+1)(j + r - 4i) \geq (n+1)(i + 1 + 3i - 4i) \\
&= n + 1 > n.
\end{aligned}$$

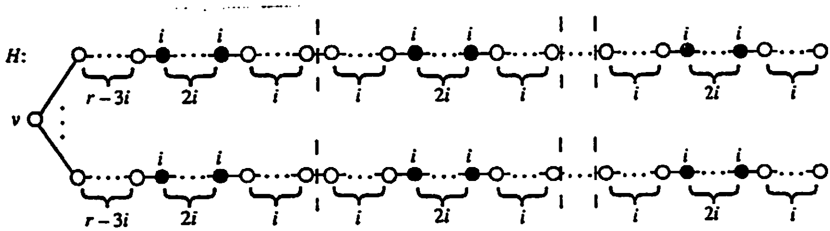

Figure 4.

Case 2. $2i \le r < 3i$. Assign step i to all vertices x for which $d(x,v) \le r-i$ or for which $d(x,v) = (4t+1)i+r+s$ where $0 \le t \le k-1$ and $1 \le s \le 2i$ to i-step dominate all vertices of H. (See Figure 5.) Thus, $\rho_i(H) \le p - 2(n+1)(i+2ki)$. Hence,

$$\begin{aligned}
\rho_j(H) - \rho_i(H) &\ge (p - 2nj - 2j - 1) - (p - 2ni - 2i - 4nki - 4ki) \\
&= -2nj - 2j - 1 + 2ni + 2i + 4nki + 4ki \\
&= -2(n+1)j - 1 + 2(n+1)i + (n+1)(3j - r) \\
&= (n+1)j + 2(n+1)i - (n+1)r - 1 \\
&= (n+1)(j + 2i - r) - 1 \\
&\ge (n+1)(3i + 1 - r) - 1 \\
&\ge 2n + 1 > n.
\end{aligned}$$

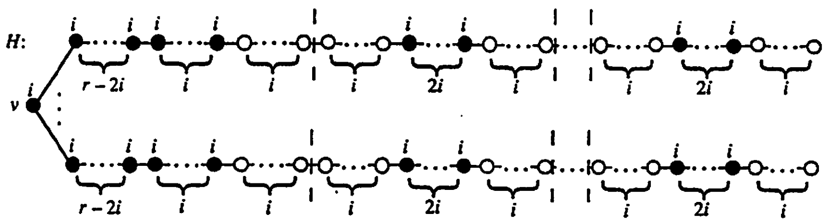

Figure 5.

Case 3. $i < r < 2i$. Assign step i to all vertices x on the path Q_1 for which $d(x,v) \le 3i - r - 1$, and step i to all vertices x not on Q_1 for which $d(x,v) \le r - i$. Furthermore, assign step i to all vertices x for which $d(x,v) = (4t+1)i + r + s$ where $0 \le t \le k-1$ and $1 \le s \le 2i$. (See Figure 6.) This produces an i-step dominating set of H. Thus, $\rho_i(H) \le p - (2n+1)(i+2ki) - [(k-1)2i + (2r - i + 1)] = p - (2n+1)(i+2ki) - (2ki + 2r - 3i + 1)$. Hence,

$$\begin{aligned}
\rho_j(H) - \rho_i(H) &\geq (p - 2nj - 2j - 1) - [p - (2n+1)(i + 2ki) - \\
&\quad (2ki + 2r - 3i + 1)] \\
&= -2nj - 2j + (2n+1)(i + 2ki) + (2ki + 2r - 3i) \\
&= -2(n+1)j + 2(n+1)i + 4(n+1)ki + 2r - 4i \\
&= -2(n+1)j + 2(n+1)i + (n+1)(3j - r) + 2r - 4i \\
&= (n+1)j + 2(n+1)i - (n+1)r + 2r - 4i \\
&\geq (n+1)(j + 2i - r) + 2 - 2i \quad (\text{since } r \geq i+1) \\
&\geq (n+1)(3i + 1 - r) + 2 - 2i \\
&\geq (n+1)(i+2) + 2 - 2i \\
&= 2n + 4 + i(n-1) \\
&\geq 2n + 4 > n.
\end{aligned}$$

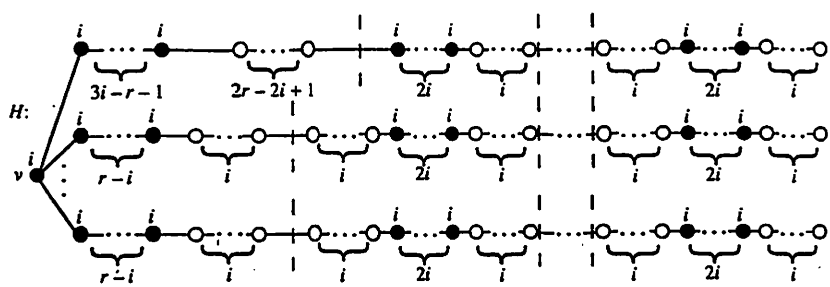

Figure 6.

Case 4. $0 \leq r \leq i$. Assign step i to exactly one vertex at distance i from v, and step i to all vertices x for which $i + 1 \leq d(x, v) \leq r + i$ or for which $d(x, v) = (4t + 1)i + r + s$ where $0 \leq t \leq k - 1$ and $1 \leq s \leq 2i$ to i-step dominate all vertices of H. (See Figure 7.) Thus, $\rho_i(H) \leq p - 4(n+1)ki$. Hence,

$$
\begin{aligned}
\rho_j(H) - \rho_i(H) &\geq [p - 2(n+1)j - 1] - [p - 4(n+1)ki] \\
&= 4(n+1)ki - 2(n+1)j - 1 \\
&= (n+1)(3j - r) - 2(n+1)j - 1 \\
&= (n+1)(j - r) - 1 \\
&\geq n \quad (\text{since } j \geq i+1 \geq r+1). \quad \square
\end{aligned}
$$

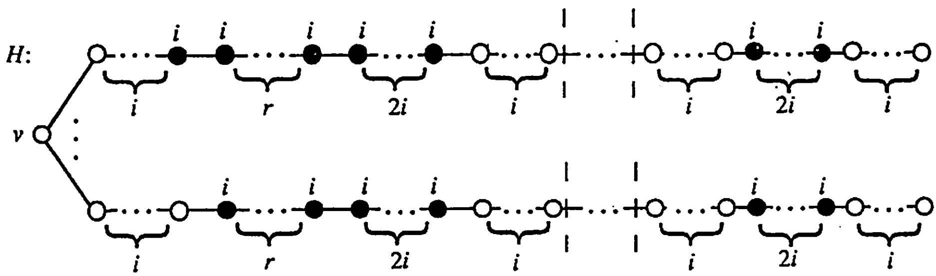

Figure 7.

We now have an immediate consequence of this result.

Corollary 2 *For every positive integer n, there exists a graph G such that*

$$\max\{\rho_i(G) \mid 1 \leq i \leq rad G\} - \gamma_o(G) \geq n.$$

Therefore, the orbital number of a graph G can be arbitrarily smaller than an i-step domination number of G. Next we show that the orbital number of a graph can be distinct from and consequently less than all i-step domination numbers of the graph.

Theorem 3 *There exist graphs G of arbitrarily large radius such that $\gamma_o(G) < \rho_i(G)$ for every integer i $(1 \leq i \leq rad G)$.*

Proof. Let $k \geq 2$ be an integer. For $j = 1, 2, \ldots, 2k-2$, let Q_j denote a path of length 1; while for $j = 2k-1, 2k, \ldots, 4k-3$, let Q_j denote a path of length $j - 2k + 3$. Select an end-vertex of each path Q_j $(1 \leq j \leq 4k-3)$, and let T be the rooted tree obtained by identifying these end-vertices resulting in a root v. Thus, T is a subdivision of the star $K_{1,4k-3}$ whose root v has degree $4k-3$. The rooted tree T has radius and height $2k$. The vertices at distance ℓ from v $(0 \leq \ell \leq 2k)$ are said to lie in level ℓ. The tree T is indicated in Figure 8.

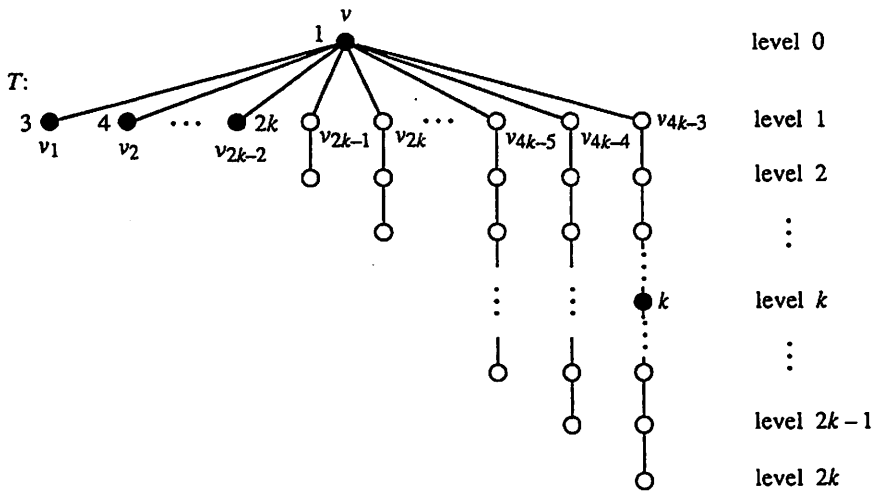

Figure 8.

We now assign step 1 to the root v of T and steps $3, 4, \ldots, 2k$ to the $2k-2$ end-vertices at level 1. In addition, step k is assigned to the vertex of Q_{4k-3} belonging to level k, which k-step dominates both v (the only vertex at level 0) and the only vertex at level $2k$. The end-vertex assigned step i $(3 \leq i \leq 2k)$ i-step dominates all vertices at level $i-1$; while v 1-step dominates all vertices at level 1. Hence, every vertex of T is i-step dominated by one of the $2k$ vertices assigned steps for some i $(1 \leq i \leq 2k)$. Therefore, $\gamma_o(T) \leq 2k$.

It remains to show that $\rho_i(T) \geq 2k+1$ for every integer i ($1 \leq i \leq 2k$). Let i ($1 \leq i \leq 2k$) be a fixed integer. Suppose that step i is assigned to a vertex x at level ℓ belonging to the path Q_j ($2k-1 \leq j \leq 4k-3$). If $i+\ell \leq j-2k+3$, then x i-step dominates a unique vertex at level $i+\ell$, namely, the vertex of Q_j at level $i+\ell$. Unless $i+\ell = 2k$, there are vertices of T at level $i+\ell$ that are not i-step dominated by x. If $\ell - i \geq 0$, then x i-step dominates a unique vertex at level $\ell - i$, namely, the unique vertex of Q_j at level $\ell - i$. Unless $\ell - i = 0$, there are vertices of T at level $\ell - i$ that are not i-step dominated by x. If $\ell - i < 0$, then x i-step dominates all vertices of T not on Q_j that are at level $i - \ell$. In any case, a vertex assigned step i cannot i-step dominate all vertices on more than one level unless it i-step dominates both v and the unique vertex of T at level $2k$. However, then, $i = k$; but, in this case, no single vertex of T can i-step dominate all vertices at level 1. Since T has $2k+1$ levels, $\rho_i(T) \geq 2k+1$. □

A set $S = \{v_1, v_2, \ldots, v_t\}$ of vertices in a graph G is a *step dominating set* for G if there exist *nonnegative* integers $k_1, k_2, \ldots, k_t$, where k_i is called the *step* of v_i ($1 \leq i \leq t$), so that each vertex in G is k_i-step dominated by v_i for exactly one i ($1 \leq i \leq t$). The *step domination number* $\gamma_s(G)$ of G is the minimum number of vertices in a step dominating set for G. We now show that the orbital domination number of a graph is bounded above by its step domination number.

Theorem 4 *If G is a connected graph, then $\gamma_o(G) \leq \gamma_s(G)$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_t\}$ be a step dominating set for G of minimum cardinality with corresponding step sequence $k_1, k_2, \ldots, k_t$. If this sequence has only nonzero terms, then $k_1, k_2, \ldots, k_t$ is a orbital sequence and $\gamma_o(G) \leq \gamma_s(G)$. Assume then that some $k_i = 0$. If $t < |V(G)|$, then let u be a vertex of G not in S and suppose that $d(u, v_i) = k_i'$. Then $k_1, k_2, \ldots, k_{i-1}, k_i', k_{i+1}, \ldots, k_t$ is a orbital sequence for G. Repeating this procedure produces a orbital sequence for G. If $t = |V(G)|$, then by definition, $\gamma_o(G) \leq t$. In either case, $\gamma_o(G) \leq \gamma_s(G)$. □

With the aid of Theorem 4, we can determine the orbital number of each cycle.

Theorem 5 *For an integer $n \geq 3$, $\gamma_o(C_n) = (n+2)/2$ if $n \equiv 2 \pmod 4$, and $\gamma_o(C_n) = \lceil n/2 \rceil$ if $n \not\equiv 2 \pmod 4$.*

Proof. Since $|N_i(v)| = 2$ for every integer i with $1 \leq i \leq \lfloor n/2 \rfloor$ and for each vertex $v \in V(C_n)$, it follows that $\gamma_o(C_n) \geq \lceil n/2 \rceil$. It was shown in [1] that if $n \not\equiv 2 \pmod 4$, then $\gamma_s(C_n) = \lceil n/2 \rceil$. Therefore, by Theorem 4, $\gamma_o(C_n) \leq \gamma_s(C_n) = \lceil n/2 \rceil$. This establishes the result if $n \not\equiv 2 \pmod 4$.

Now, assume that $n \equiv 2 \pmod 4$. Since $\gamma_1(C_n) \leq (n+2)/2$, it follows that $n/2 \leq \gamma_o(C_n) \leq (n+2)/2$. Assume that $\gamma_o(C_n) = n/2$. Let $\ell_1, \ell_2, \ldots, \ell_{n/2}$ be a orbital sequence for C_n and let $v_1, v_2, \ldots, v_{n/2}$ be the corresponding vertices. Thus, $|N_{\ell_i}(v_i)| = 2$ for all i $(1 \leq i \leq n/2)$ and $\{v_1, v_2, \ldots, v_{n/2}\}$ is a step dominating set for C_n. However, $\gamma_s(C_n) = (n+2)/2$, producing a contradiction. Therefore, $\gamma_o(C_n) = (n+2)/2$. □

3 The Orbital Numbers of Trees

The orbital sequence of any connected graph G can never consist of exactly one term, so $\gamma_o(G) \geq 2$. Hayes, Schultz, and Yates [2] showed that the only orbital sequence of length 2 is 1, 1. An immediate consequence now follows.

Theorem A (Hayes et al. [2]) *For a nontrivial tree T, $\gamma_o(T) = 2$ if and only if T is isomorphic to a rooted tree of height at most 2 and of diameter at most 3.*

The orbital number of paths was given in [2].

Theorem B (Hayes et al. [2]) *For any integer* $n \geq 3$, $\gamma_o(P_n) = (n+2)/2$ *if* $n \equiv 2 \,(mod\, 4)$, *and* $\gamma_o(P_n) = \lceil n/2 \rceil$ *if* $n \not\equiv 2 \,(mod\, 4)$.

For a subset S of vertices in a connected graph G, the *Steiner distance* $d(S)$ of S in G is the smallest number of edges in a connected subgraph of G that contains S. Such a subgraph is necessarily a tree, called a *Steiner tree* for S.

A tree is a *double star* if it has exactly two vertices that are not end-vertices.

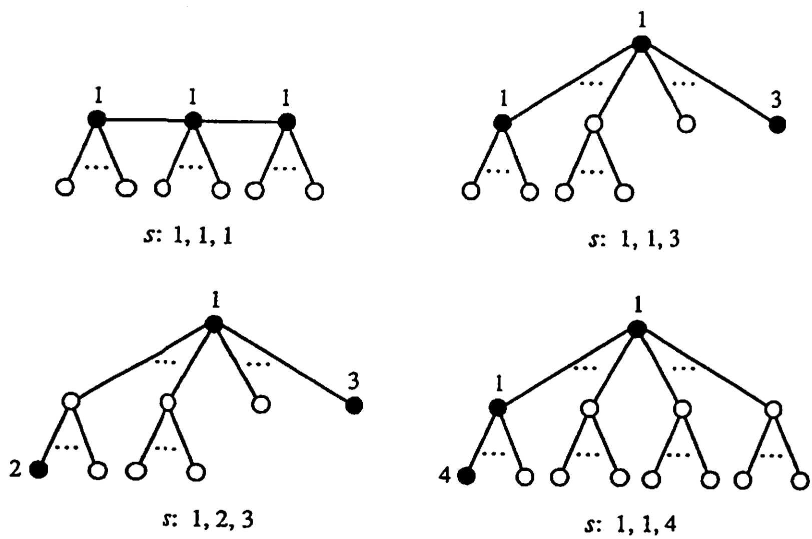

Figure 9.

Theorem 6 *Let* $s\colon \ell_1, \ell_2, \ell_3$ *be a minimum orbital sequence of a tree* T. *Then,*

(*a*) *s is* $1, 1, 1$ *and* T *is a caterpillar of diameter* 4, *or*

(*b*) *s is* $1, 1, 3$ *or* $1, 2, 3$ *and* T *is a rooted tree of height* 2 *and of diameter* 4 *with at least one leaf at height* 1, *or*

(*c*) *s is* $1, 1, 4$ *and* T *is a rooted tree of height 2 and of diameter 4.*

Proof. Let v_1, v_2, v_3 be vertices in T such that $\cup_{i=1}^{3} N_{\ell_i}(v_i) = V(T)$. Let $S = \{v_1, v_2, v_3\}$ and let T_S be the Steiner tree for S. We consider two cases.

Case 1. $d(S) = 2$. Without loss of generality, we may assume that T_S is the path v_1, v_2, v_3, and that v_2 is 1-step dominated by v_1. The vertex v_3 must then be 1-step dominated by v_2, so $\ell_1 = \ell_2 = 1$. If $\ell_3 \geq 5$, then a vertex at distance 4 from v_3 in T is not step dominated. Hence $\ell_3 \leq 4$. If $\ell_3 = 1$, then T must be a caterpillar of diameter 4 (possibly, $\deg v_2 = 2$). If $\ell_3 = 2$, then T must be a double star with v_1 and v_2 as central vertices. However, the sequence $s: 1, 1, 2$ is then not minimal since $1, 1$ is also a orbital sequence of T. Thus, $\ell_3 \neq 2$. If $\ell_3 = 3$, then T must be a rooted tree (with root v_2) of height 2 and of diameter 4 with at least one leaf (namely, v_3) at height 1. If $\ell_3 = 4$, then $\deg v_3 = 1$. Furthermore, the component of $T - v_1v_2$ containing v_1 must be a rooted tree of height 2 with root v_1. Hence T is a rooted tree (with root v_1) of height 2 and of diameter 4.

Case 2. $d(S) \geq 3$. First we show that T_S must be a path. If this is not the case, then T_S is obtained from a star $K_{1,3}$ by subdividing edges, if necessary. Let v be the vertex of degree 3 in T_S. If $d(S) = 3$, then $T_S \cong K_{1,3}$. Without loss of generality, we may assume that v_1 1-step dominates v, and that v_2 2-step dominates v_1; so $\ell_1 = 1$ and $\ell_2 = 2$. Thus, v_3 must 2-step dominate v_2; so $\ell_3 = 2$. It follows that T must be a double star with v_1 and v as central vertices. However, the sequence $s: 1, 2, 2$ is not a minimum orbital sequence for a double star, producing a contradiction. Hence $d(S) \geq 4$. Without loss of generality, we may assume that $d(v_1, v) \geq 2$ and that v_2 ℓ_2-step dominates v_1; so $\ell_2 \geq 3$. If v_1 (respectively, v_3) ℓ_1-step dominates v_2, then v_3 (respectively, v_1) must ℓ_3 -step dominate all vertices on the v–v_1 path different from v_1, which is impossible. We deduce, therefore, that T_S must be a path.

Without loss of generality, we may assume that T_S is a v_1–v_3 path and that $d(v_1, v_2) \geq 2$. If $d(v_2, v_3) \geq 2$, then, without loss of generality, we may assume that v_1 ℓ_1-step dominates v_2. If v_2

ℓ_2-step dominates v_1, then v_3 must ℓ_3-step dominate the vertices immediately following v_1 and v_2 on the v_1–v_3 path, which is impossible. Hence v_3 must ℓ_3-step dominate v_1. But then v_2 must ℓ_2-step dominate every vertex on the v_2–v_3 path different from v_2, which is impossible. Hence $d(v_2, v_3) = 1$.

Let v_1' be the vertex of T_S adjacent with v_1. If v_2 ℓ_2-step dominates v_1, then $\ell_2 \geq 2$ and v_1 must then ℓ_1-step dominate v_3. Thus, v_3 must ℓ_3-step dominate both v_1' and v_2, which is impossible. Hence, v_1 must be ℓ_3-step dominated by v_3. Thus, v_2 is ℓ_1-step dominated by v_1. It follows that v_2 must ℓ_2-step dominate v_3 and every internal vertex of the v_1–v_2 path. Consequently, T_S is the path v_1, v_1', v_2, v_3 and $\ell_1 = 2$, $\ell_2 = 1$, and $\ell_3 = 3$. Hence in T, $\deg v_1 = \deg v_3 = 1$ and $N_2(v_1') = N(v_2) - \{v_1'\}$. Since s is a minimum orbital sequence of T, it follows that T cannot be a double star; so the component of $T - \{v_1'v_2, v_2v_3\}$ containing v_2 must be a rooted tree with root v_2 of height 2. Consequently, T is a rooted tree with root v_2 of height 2 and of diameter 4 with at least one leaf (namely, v_3) adjacent to the root. This completes the proof of the theorem. □

As an immediate corollary of Theorem 6 we have the following results.

Corollary 7 *For a tree T, $\gamma_o(T) = 3$ if and only if T is isomorphic to a rooted tree of height 2 and of diameter 4.*

Proof. The necessity follows immediately from Theorem 6. For the sufficiency, let T be a rooted tree of height 2 and of diameter 4. By Theorem A, $\gamma_o(T) \geq 3$. However, $1, 1, 4$ is a orbital sequence of T as may be seen by assigning a step of 4 to any leaf at height 2, a step of 1 to its parent, and a step of 1 to the root. Hence $\gamma_o(T) \leq 3$. Thus, $\gamma_o(T) = 3$. □

Corollary 8 *The only minimum orbital sequences of length 3 in a tree are $(1,1,1)$, $(1,1,3)$, $(1,1,4)$, and $(1,2,3)$.*

Corollary 8 shows that if (ℓ_1, ℓ_2, ℓ_3) is a minimal orbital sequence with $\ell_1 \leq \ell_2 \leq \ell_3$, then $\ell_3 \leq 4$. This illustrates the following result.

Theorem 9 *If T is a tree with minimal orbital sequence $\ell_1, \ell_2, \ldots, \ell_r$ with $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_r$ $(r \geq 3)$, then*

$$\ell_r \leq \begin{cases} 2r-2 & \text{if } r \text{ is odd} \\ 2r-3 & \text{if } r \text{ is even} \end{cases}$$

Proof. Let v_i be assigned step ℓ_i $(1 \leq i \leq r)$ so that every vertex of T is ℓ_i-step dominated by v_i for some i $(1 \leq i \leq r)$. Let $P: v_r = u_0, u_1, \ldots, u_{\ell_r}$ be a path of length ℓ_r in T. Thus u_{ℓ_r} is step dominated by v_r, but no other vertex of P is step dominated by v_r. Hence each of the ℓ_r vertices $u_0, u_1, \ldots, u_{\ell_r - 1}$ is step dominated by some vertex v_i $(1 \leq i \leq r-1)$. However, each such vertex v_i can step dominate at most two vertices of P; so $\ell_r \leq 2r-2$.

Assume now that r is even and suppose, to the contrary, that $\ell_r = 2r-2$. Then Theorem B implies that for the path $P_{\ell_r}: u_0, u_1, \ldots, u_{\ell_r}$, we have $\gamma_o(P_{\ell_r}) = \gamma_o(P_{2r-2}) = r$, which in turn implies that $\gamma_o(T) \geq r+1$. This contradicts the fact that $\ell_1, \ell_2, \ldots, \ell_r$ is a minimal orbital sequence. Hence, when r is even, $\ell_r \leq 2r-3$. □

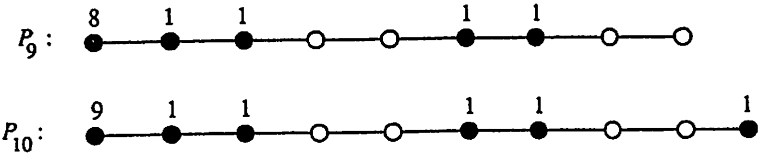

Figure 10.

To show that the bound presented in Theorem 9 is sharp, we note that when r is odd the sequence $\ell_1, \ell_2, \ldots, \ell_r$ $(r \geq 3)$ defined by $\ell_i = 1$ for $1 \leq i \leq r-1$ and $\ell_r = 2r-2$ is a minimal orbital sequence for the path P_{2r-1}. This is illustrated in Figure 10 for

$r = 5$. To see that the bound is sharp when r is even, we note that the sequence $\ell_1, \ell_2, \ldots, \ell_r$ ($r \geq 4$) defined by $\ell_i = 1$ for $1 \leq i \leq r-1$ and $\ell_r = 2r - 3$ is a minimal orbital sequence for P_{2r-2}. This is illustrated in Figure 10 for $r = 6$.

Theorem 10 *The only minimum orbital sequences of length three for graphs are* (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 2), (1, 2, 3), (2, 2, 2), (2, 2, 3), (2, 3, 3), *and* (3, 3, 3).

Proof. If s is one of the ten sequences in the statement of the theorem, then s is a minimum orbital sequence for some graph, as illustrated in Figure 11.

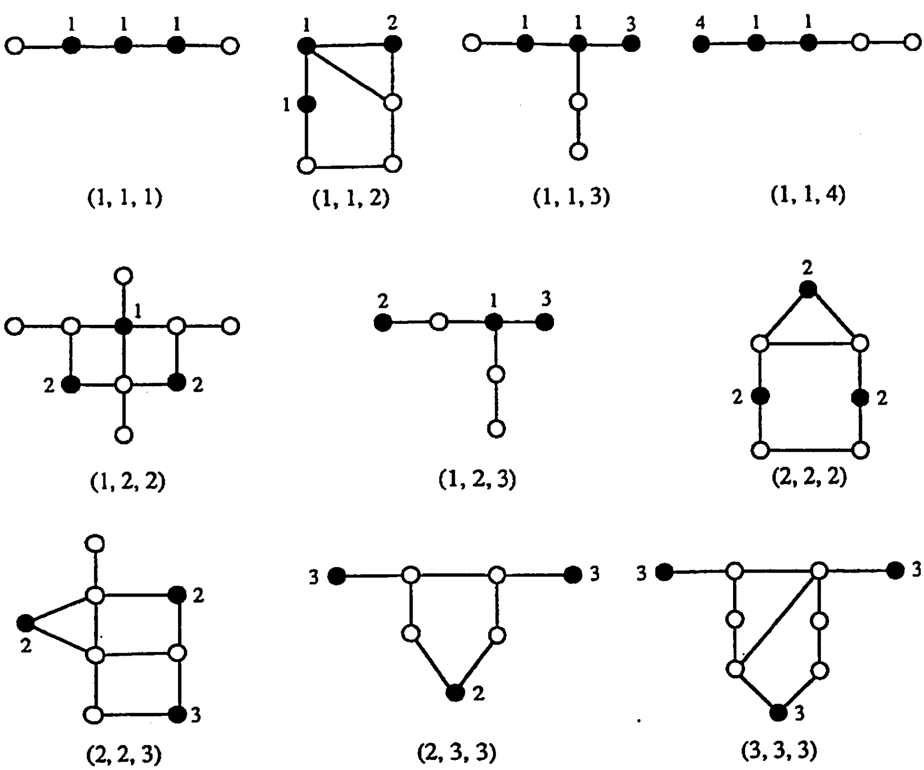

Figure 11.

Conversely, let $s: \ell_1, \ell_2, \ell_3$ be a minimum orbital sequence in a graph G with $\ell_1 \leq \ell_2 \leq \ell_3$. Let v_1, v_2, v_3 be vertices in G such that $\cup_{i=1}^{3} N_{\ell_i}(v_i) = V(G)$. So every vertex of G is ℓ_i-step dominated by v_i for some i $(1 \leq i \leq 3)$. The following lemma will prove to be useful.

Lemma 11 *If $\ell_1 \geq 2$ and $\ell_4 \geq 4$, then $d(v_1, v_2) = \ell_1$.*

Proof. Suppose that $d(v_1, v_2) \neq \ell_1$. Then, v_2 must be ℓ_3-step dominated by v_3. Let x be the vertex at distance 2 from v_2 on a shortest v_2–v_3 path (of length $\ell_3 \geq 4$). Then, $d(x, v_2) = 2 < \ell_2$ and $d(x, v_3) = \ell_3 - 2$. Thus, x must be ℓ_1-step dominated by v_1. Let y be the vertex adjacent to x on a shortest x–v_1 path (of length $\ell_1 \geq 2$). Then, $d(v_1, y) = \ell_1 - 1$, $d(v_2, y) \leq 3 < \ell_2$, and $d(v_3, y) \leq \ell_3 - 1$. Thus, y is not step dominated, which produces a contradiction. □

Before proceeding further, we prove four claims.

Claim 1 $\ell_1 \leq 3$.

Proof. Suppose that $\ell_1 \geq 4$. By Lemma 11, $d(v_1, v_2) = \ell_1$. Let x be the vertex at distance 2 from v_1 on a shortest v_1–v_2 path (of length ℓ_1). Then, $d(x, v_1) = 2$ and $d(x, v_2) = \ell_1 - 2 \leq \ell_2 - 2$. Thus, x must be ℓ_3-step dominated by v_3. Let y be the vertex adjacent to x on a shortest x–v_3 path (of length ℓ_3). Then, $d(y, v_1) \leq 3 < \ell_1$, $d(y, v_2) \leq \ell_2 - 1$, and $d(y, v_3) = \ell_3 - 1$. Thus, y is not step dominated, which is a contradiction. □

Claim 2 *If $\ell_1 = 1$, then $\ell_2 \leq 2$.*

Proof. Suppose that $\ell_2 \geq 3$. If $d(v_1, v_2) = 1$, then v_1 must be ℓ_3-step dominated by v_3; so $d(v_1, v_3) = \ell_3 \geq 3$. But then v_3 must be ℓ_2-step dominated by v_2. Let x be the vertex adjacent to v_3 on a shortest v_2–v_3 path (of length ℓ_2). Then $d(x, v_2) = \ell_2 - 1$ and $d(x, v_3) = 1$. Thus, x must be ℓ_1-step dominated by v_1, that is, $d(x, v_1) = 1$. But then,

$$d(v_1, v_3) \leq d(v_1, x) + d(x, v_3) = 2 < \ell_3,$$

producing a contradiction. Hence $d(v_1, v_2) > 1$. Now v_2 must be ℓ_3-step dominated by v_3. Let Q be a shortest v_2–v_3 path (of length ℓ_3), and let x be the vertex adjacent to v_2 on Q. Then x must be ℓ_1-step dominated by v_1, so $d(v_1, x) = 1$ and therefore $d(v_1, v_2) = 2$. Thus v_1 must be ℓ_3-step dominated by v_3, so $d(v_1, v_3) = \ell_3$. However, the vertex y at distance 2 from v_2 on Q must also be ℓ_1-step dominated by v_1; so $d(v_1, y) = 1$ and therefore $d(v_1, v_3) \leq \ell_3 - 1$, producing a contradiction. □

Claim 3 *If $\ell_1 = 2$, then $\ell_2 \leq 3$.*

Proof. Suppose that $\ell_2 \geq 4$. By Lemma 11, $d(v_1, v_2) = 2$. Thus v_1 must be ℓ_3-step dominated by v_3. But then the vertex adjacent to v_1 on a shortest v_1–v_3 path (of length ℓ_3) is at distance at most 3 ($< \ell_2$) from v_2 and is therefore not step dominated, a contradiction. □

Claim 4 *If $\ell_1 = 3$, then $\ell_2 = 3$.*

Proof. Suppose that $\ell_2 \geq 4$. Lemma 11, $d(v_1, v_2) = 3$. Thus v_1 must be ℓ_3-step dominated by v_3; so $d(v_1, v_3) = \ell_3 \geq 4$. Hence v_3 must be ℓ_2-step dominated by v_2. Let x be the vertex at distance 2 from v_2 on a shortest v_2–v_3 path (of length $\ell_2 \geq 4$). Then $d(x, v_2) = 2 < \ell_2$ and $d(x, v_3) = \ell_2 - 2 \leq \ell_3 - 2$. Thus, x must be ℓ_1-step dominated by v_1. But then the vertex adjacent to x on a shortest x–v_1 path (of length ℓ_1) is not step dominated, producing a contradiction. □

We now return to the proof of Theorem 10. By Claim 1, ℓ_1 is $1, 2$, or 3. We consider each case in turn.

Case 1. *Suppose that $\ell_1 = 1$.* Then, by Claim 2, $\ell_2 \leq 2$. Suppose, first, that $\ell_2 = 1$. Then, v_3 is 1-step dominated by v_1 or v_2, say v_1. If $d(v_1, v_2) > 1$, then v_1 is ℓ_3-step dominated by v_3; so $\ell_3 = d(v_1, v_3) = 1$. If $d(v_1, v_2) = 1$, then $d(v_2, v_3) \leq 2$. Now if $\ell_3 \geq 5$, then a vertex at distance 4 from v_3 is not step dominated. Hence if s is the sequence $1, 1, \ell_3$, then $\ell_3 \leq 4$.

Suppose, next, that $\ell_2 = 2$. If $d(v_1, v_2) > 1$, then v_2 must be ℓ_3-step dominated by v_3; so $d(v_2, v_3) = \ell_3$. Thus, v_3 is 1-step dominated by v_1. Now let x be the vertex adjacent to v_2 on a shortest v_2–v_3 path (of length ℓ_3). Then x must be 1-step dominated by v_1. Thus, $d(v_1, v_2) = 2$ and so $\ell_3 = d(v_2, v_3) \leq d(v_1, v_2) + d(v_1, v_3) = 3$. On the other hand, if $d(v_1, v_2) = 1$, then v_1 must be ℓ_3-step dominated by v_3; so $d(v_1, v_3) = \ell_3 \geq 2$. Thus, v_3 must be ℓ_2-step dominated by v_2, so $d(v_2, v_3) = 2$. Hence,

$$\ell_3 = d(v_1, v_3) \leq d(v_1, v_2) + d(v_2, v_3) = 3.$$

Thus, if s is the sequence $1, 2, \ell_3$, then $2 \leq \ell_3 \leq 3$.

Case 2. *Suppose that* $\ell_1 = 2$. Then, by Claim 3, $\ell_2 \leq 3$. Suppose, first, that $\ell_2 = 2$. Then v_3 is 2-step dominated by v_1 or v_2, say v_1. Thus, $d(v_1, v_3) = 2$. If $d(v_1, v_2) \neq 2$, then v_1 is ℓ_3-step dominated by v_3; so $\ell_3 = d(v_1, v_3) = 2$. If $d(v_1, v_2) = 2$, then v_3 must ℓ_3-step dominate every vertex x adjacent to both v_1 and v_2. Thus,

$$\ell_3 = d(x, v_3) \leq d(x, v_1) + d(v_1, v_3) = 3.$$

Hence if s is the sequence $2, 2, \ell_3$, then $2 \leq \ell_3 \leq 3$.

Suppose, next, that $\ell_2 = 3$. If $d(v_1, v_2) \neq 2$, then v_2 must be ℓ_3-step dominated by v_3, so $d(v_2, v_3) = \ell_3$. Now if v_3 is ℓ_2-step dominated by v_2, then $\ell_3 = \ell_2 = 3$. If v_3 is ℓ_1-step dominated by v_1, then $d(v_1, v_3) = \ell_1 = 2$. Thus, v_1 must be ℓ_2-step dominated by v_2; so $d(v_1, v_2) = \ell_2 = 3$. Let x be the vertex adjacent to v_1 on a shortest v_1–v_2 path (of length 3). Then x is ℓ_3-step dominated by v_3; so

$$\ell_3 = d(x, v_3) \leq d(x, v_1) + d(v_1, v_3) = 3.$$

Hence if $d(v_1, v_2) \neq 2$, then $\ell_3 = 3$. On the other hand, if $d(v_1, v_2) = 2$, then v_1 must be ℓ_3-step dominated by v_3, so $d(v_1, v_3) = \ell_3 \geq 3$. Thus, v_3 must be ℓ_2-step dominated by v_2, so $d(v_2, v_3) = 3$. Let x be the vertex adjacent to v_3 on a shortest v_2–v_3 path (of length 3). Then x must be 2-step dominated by v_1; so

$$\ell_3 = d(v_1, v_3) \leq d(v_1, x) + d(x, v_3) = 3.$$

Thus, if s is the sequence $2, 3, \ell_3$, then $\ell_3 = 3$.

Case 3. *Suppose that* $\ell_1 = 3$. Then, by Claim 4, $\ell_2 = 3$. Hence v_3 is 3-step dominated by v_1 or v_2, say v_1. Thus, $d(v_1, v_3) = 3$. If $d(v_1, v_2) \neq 3$, then v_1 is ℓ_3-step dominated by v_3; so $\ell_3 = d(v_1, v_3) = 3$. If $d(v_1, v_2) = 3$, then let x be the vertex adjacent to v_1 on a shortest v_1–v_3 path (of length 3). Then x must be 3-step dominated by v_2. Let y be the vertex adjacent to x on a shortest x–v_2 path (of length 3). Then $d(v_1, y) \leq 2$ and $d(v_2, y) = 2$. Thus y must be ℓ_3-step dominated by v_3; so

$$\ell_3 = d(v_3, y) \leq d(v_3, x) + d(x, y) = 3.$$

Hence if s is the sequence $3, 3, \ell_3$, then $\ell_3 = 3$. This completes the proof of Theorem 10. □

4 The Upper Orbital Domination Number of a Graph

The maximum length of a minimal orbital sequence for a graph G is called the *upper orbital domination number* or, more simply, the *upper orbital number* $\Gamma_o(G)$ of G. Consequently, for every integer i $(1 \leq i \leq \operatorname{rad} G)$,

$$\gamma_o(G) \leq \rho_i(G) \leq \Gamma_o(G).$$

The only minimal orbital sequence for K_n, $n \geq 2$, is $1, 1$; so $\gamma_o(K_n) = \Gamma_o(K_n) = 2$. For the path P_4, $\gamma_o(P_4) = 2$ and $\Gamma_o(P_4) = 4$. The minimal orbital sequences $1, 1$ and $2, 2, 2, 2$ are illustrated in Figure 12. Indeed, $\Gamma_o(T) = 4$ for every double star T.

Figure 12.

The difference between the upper orbital number and the orbital number of a graph can be arbitrarily large. We have seen that $\gamma_o(C4k) = 2k$. Since $\rho_k(C_{2k}) = 2k$, it follows that $\Gamma_o(C_{4k}) = 4k$ and so $\Gamma_o(C_{4k}) - \gamma_o(C_{4k}) = 2k$. The values of $\Gamma_o(C_{2k+1})$ are not known in general. The next result appears in [2].

Theorem C (Hayes et al. [2]) *For any integer $n \geq 2$,*

$$\Gamma_o(P_n) \leq \begin{cases} n & \text{for } n \text{ even} \\ n-1 & \text{for } n \text{ odd} \end{cases}$$

In the next result, we characterize those graphs having upper orbital number 2.

Theorem 12 *A connected graph G of order $n \geq 2$ has upper orbital number 2 if and only if $\text{rad}\, G = 1$ and $\delta(G) \geq n-2$.*

Proof. First, assume that G has radius 1 and minimum degree at least $n-2$. If $\delta(G) = n-1$, then $G = K_n$ and certainly the only minimal orbital sequence of G is $1, 1$; thus $\Gamma_o(G) = 2$. Hence we may assume that $\delta(G) = n-2$. Let u be a vertex of degree $n-2$ and let w be a vertex of eccentricity 1; so $\deg w = n-1$. If we assign step 1 to u and w, then every vertex of G is 1-step dominated. Consequently, $1, 1$ is a (minimal) orbital sequence of G.

Now let s be an arbitrary minimal orbital sequence of G. Since $\deg w = n-1$, it follows that w can only be step dominated by a vertex with step 1. Thus, without loss of generality, assign step 1 to a vertex x ($\neq w$). If $\deg x = n-1$, then x can only be step dominated by a vertex with step 1. Assigning step 1 to any vertex adjacent to x produces a (minimal) orbital sequence of G. Thus, $s\colon 1, 1$. Suppose, then, that $\deg x = n-2$ and s is not $1, 1$. So there is a unique vertex y ($\neq x$) that is not adjacent to x, and so $\deg y = n-2$. Hence if step 1 is assigned to x, then all vertices of G are step dominated except x and y. Since s is minimal, no other vertex of G can be assigned step 1. However, since x must be step dominated, y must be assigned step 2. However, then, y cannot be step dominated since x has already been assigned step 1 and no other vertex of G can be assigned step 1.

For the converse, assume that $\Gamma_o(G) = 2$. First, suppose, to the contrary, that $\operatorname{rad} G \geq 2$. Then $\rho_2(G)$ is defined. If some vertex v of G is assigned step 2, then no vertex assigned step 2 can simultaneously 2-step dominate v and all its neighbors. Thus $\rho_2(G) \geq 3$, which implies that $\Gamma_o(G) \geq 3$, producing a contradiction. Therefore, $\operatorname{rad} G = 1$.

Next, suppose that $\delta(G) < n - 2$. Let w be a vertex of eccentricity 1 and let x be a vertex with $\deg x < n - 2$. Hence there exist two vertices y and z distinct from x that are not adjacent to x. If we assign step 1 to y and step 2 to x and z, then every vertex of G is step dominated. Thus, $s\colon 1, 2, 2$ is a orbital sequence of G. Since $1, 1$ is the only orbital sequence of length 2, it follows that s must be minimal, completing the proof. □

As a consequence of Theorem 12, the only trees with upper orbital number 2 are K_2 and P_3. Next we characterize those trees with upper orbital number 3.

Theorem 13 *Let T be a tree. Then $\Gamma_o(T) = 3$ if and only if $T = K_{1,n}$ for some integer $n \geq 3$.*

Proof. Assume, first, that $T = K_{1,n}$, where $n \geq 3$. Then the assignment of step 1 to an end-vertex of T and step 2 to two other end-vertices of T shows that $1, 2, 2$ is a minimal orbital sequence for T. Since $\operatorname{diam} T = 2$, every term of a orbital sequence for T is 1 or 2. The assignment of step 1 to the central vertex and an end-vertex of T shows that $1, 1$ is a minimal orbital sequence for T. Hence any orbital sequence s for T of length 4 or more contains $1, 1$ or $1, 2, 2$ as a subsequence and thus s is not minimal. Therefore, $\Gamma_o(T) = 3$.

Conversely, assume that T is a tree with $\Gamma_o(T) = 3$. Thus, $\gamma_o(T) = 2$ or $\gamma_o(T) = 3$. Thus, T is a star or a rooted tree of height 2 and diameter 4. If T is a rooted tree of height 2 and diameter 4, then T contains P_4 as an induced subgraph. Since $\rho_2(P_4) = 4$, it follows that $\rho_2(T) \geq 4$, and so $\Gamma_o(T) \geq 4$. Therefore, T is a star. □

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