Existence of HPMDs with Block Size Five and Index $\lambda \geq 2$

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ABSTRACT. In this paper, it is shown that the necessary condition for the existence of a holey perfect Mendelsohn design (HPMD) with block size 5, type h^n and index λ, namely, $n \geq 5$ and $\lambda n(n-1)h^2 \equiv 0 \pmod 5$, is also sufficient for $\lambda \geq 2$. The result guarantees the analogous existence result for group divisible designs (GDDs) of type h^n having block size 5 and index 4λ.

1 Introduction

Let $\lambda DKn_1, n_2, \ldots, n_h$ be the complete multipartite directed graph with vertex set $X = \cup_{1 \leq i \leq h} X_i$, where X_i $(1 \leq i \leq h)$ are disjoint sets with $|X_i| = n_i$ and where two vertices x and y from different sets X_i and X_j are joined by exactly λ arcs from x to y and λ arcs from y to x. A *holey perfect Mendelsohn design* (briefly denoted by (v, k, λ)-HPMD) is an ordered pair $(X, \mathbf{A})$ where X is a v-set and $\mathbf{A}$ is a set of k-circuits (directed cycles of length k), called *blocks*, which form an arc-disjoint decomposition of $\lambda DKn_1, n_2, \ldots, n_h$ with the property that, for any integer r $(1 \leq r \leq k-1)$ and any two vertices x and y from different sets X_i and X_j, there are exactly λ circuits $c \in \mathbf{A}$ such that the directed distance along c from x to y is r. Each X_i $(1 \leq i \leq h)$ is called a *hole* (or *group*) of the design and the multiset $\{n_1, n_2, \ldots, n_h\}$ is called the *type* of the design. For a (v, k, λ)-HPMD, we use an "exponential" notation to describe its type: a type $1^i 2^r 3^k \ldots$ denotes i occurrences of 1, r occurrences of 2, etc.

A (v, k, λ)-HPMD of type 1^v is referred to as a (v, k, λ)-PMD. If we ignore the cyclic order of the vertices in the circuits, then a (v, k, λ)-HPMD of type h^n becomes a group divisible design with block size k and of group-type h^n having index $(k-1)\lambda$. The concept of HPMDs has played an important role in the discussion of the existence of PMDs. Since a (v, k, λ)-HPMD of type h^n contains $\lambda n(n-1)h^2/k$ blocks, we obtain the following basic necessary condition for existence:

$$\lambda n(n-1)h^2 \equiv 0 \pmod{k}. \tag{1.1}$$

The existence question for HPMDs was posed in [11] and the existence problem has been solved for (v, k, λ)-HPMD of type h^n, where $k = 3$ and 4 (see [7, 8]).

For $k = 3$, the following theorem was established in [8].

Theorem 1.1. *A $(v, 3, \lambda)$-HPMD of type h^n exists if and only if $\lambda n(n-1)h^2 \equiv 0 \pmod 3$ except for the type 1^6 with $\lambda = 1$.*

More recently, the following result was established in [7].

Theorem 1.2. *A $(v, 4, \lambda)$-HPMD of type h^n exists if and only if $\lambda n(n-1)h^2 \equiv 0 \pmod 4$ except for the types 2^4 and 1^8 with $\lambda = 1$, and the type h^4 where h is odd and λ is odd.*

For the existence of $(v, 5, \lambda)$-HPMDs of type h^n, the results have not been very conclusive to date. Only the case where $\lambda = 1$ has been thoroughly investigated and the following results were obtained in [4].

Theorem 1.3. *The necessary condition for the existence of a $(v, 5, \lambda)$-HPMD of type h^n, namely, $n \geq 5$ and $n(n-1)h^2 \equiv 0 \pmod 5$, is also sufficient, except possibly for the following cases:*

(1) *$h \equiv 1, 3, 7$ or $9 \pmod{10}$, $h \neq 3$, and $n \in \{6, 10, 15, 20, 30\}$;*

(2) *$h = 3$ and $n \in \{6, 30, 56\}$;*

(3) *the pairs $(h, n) \in \{(5, 6), (15, 6), (15, 18), (15, 28)\}$.*

The following results can be found in [5].

Theorem 1.4. *Let λ be an integer greater than one. A $(v, 5, \lambda)$-PMD exists if and only if $v \geq 5$ and $\lambda v(v-1) \equiv 0 \pmod 5$.*

In this paper, we shall investigate the existence of $(v, 5, \lambda)$-HPMDs of type h^n where $\lambda \geq 2$ and show that the necessary condition for the existence of a $(v, 5, \lambda)$-HPMD, namely,

$$n \geq 5 \text{ and } \lambda n(n-1)h^2 \equiv 0 \pmod 5 \tag{1.2}$$

is also sufficient for $\lambda \geq 2$. It is also worth mentioning that the result of this paper guarantees the analogous existence result for group divisible designs (GDDs) having block size 5, type h^n and index 4λ.

2 The Construction of $(v, 5, \lambda)$-HPMDs, $\lambda \equiv 0 \pmod 5$

A *quasigroup* is an ordered pair $(Q, \cdot)$, where Q is a set and $(\cdot)$ is a binary operation on Q such that the equations

$$a \cdot x = b \text{ and } y \cdot a = b$$

are uniquely solvable for every pair of elements a, b in Q. The multiplication table of a quasigroup defines a Latin square, that is, a Latin square can be viewed as the multiplication table of a quasigroup with the headline and sideline removed. For a finite set Q, the *order* of the quasigroup $(Q, \cdot)$ is $|Q|$. A quasigroup $(Q, \cdot)$ is called *idempotent* if the identity

$$x^2 = x$$

holds for all x in Q.

Two quasigroups $(Q, \cdot)$ and $(Q, \otimes)$ defined on the same set Q are said to be *orthogonal* if the pair of equations $x \cdot y = a$ and $x \otimes y = b$, where a and b are any two given elements of Q, are satisfied simultaneously by a unique pair of elements from Q. We remark that when two quasigroups are orthogonal, then their corresponding Latin squares are also orthogonal in the usual sense.

Making use of quasigroups, we have the following construction.

Construction 2.1 If there exist three mutually orthogonal idempotent quasigroups of order n, then there exists a $(3n, 5, 5)$-HPMD of type 3^n.

Proof: Let $(Q, \otimes_j)$, $j = 1, 2, 3$, be the three mutually orthogonal idempotent quasigroups of order n where $Q = \{1, 2, \ldots, n\}$. The required HPMD will be based on $Z_3 \times Q$ having holes $Z_3 \times \{j\}$ $(j \in Q)$. The block set **A** consists of the following 5-circuits:

$$((0,i),(0,j),(0,i\otimes_1 j),(1,i\otimes_2 j),(1,i\otimes_3 j)) \pmod{3, -},$$
$$((0,i),(1,j),(0,i\otimes_1 j),(1,i\otimes_2 j),(0,i\otimes_3 j)) \pmod{3, -},$$
$$((0,i),(2,j),(2,i\otimes_1 j),(0,i\otimes_2 j),(1,i\otimes_3 j)) \pmod{3, -},$$

where $(i, j) \in Q$ and $i \neq j$. □

The next recursive constructions for HPMDs make use of group divisible designs. A *group divisible design* (or GDD) of index λ, is a triple $(X, \mathbf{G}, \mathbf{B})$ which satisfies the following properties:

(1) **G** is a partition of a set X (of *points*) into subsets called *groups*,

(2) $\mathbf{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in exactly λ blocks.

The *group-type* (or *type*) of the GDD is the multiset $\{|G|: G \in \mathbf{G}\}$. As with HPMDs, the group-type of a GDD will be denoted by an "exponential" notation. A GDD $(X, \mathbf{G}, \mathbf{B})$ will be referred to as a (K, λ)-GDD if $|B| \in K$ for every block B in $\mathbf{B}$.

Using our notation, a *transversal design* (TD) of index λ, $TD(k, \lambda, m)$, can be defined to be a $(\{k\}, \lambda)$-GDD of type m^k. In addition, we can define a *pairwise balanced design* (PBD) of index λ, $B(K, \lambda; v)$, to be a (K, λ)-GDD of type 1^v. Furthermore, a PBD $B(\{k\}, \lambda; v)$ is well known as a *balanced incomplete block design* (BIBD) with parameters v, k and index λ. When $K = \{k\}$, we simply write k for K.

In the recursive constructions of GDDs and PBDs, the "weighting" technique and Wilson's Fundamental Construction (see [10]) are quite often used, where we start with a "master" GDD and small input designs to obtain a new GDD. Similar techniques will be applied in our constructions of HPMDs, where we either start with an HPMD and use TDs as input designs or start with a GDD and use some HPMDs as input designs. We shall make use of the following two constructions. For more details of this technique, the reader is referred to [6, 11].

Construction 2.2 Suppose that a $(v, 5, \lambda_1)$-HPMD of type $\{h_1, h_2, \dots, h_n\}$ and a $TD(5, \lambda_2, m)$ exist. Then there exists a $(mv, 5, \lambda_1\lambda_2)$-HPMD of type $\{mh_1, mh_2, \dots, mh_n\}$.

Construction 2.3 Suppose that there is a (K, λ_1)-GDD of type $\{h_1, h_2, \dots, h_n\}$. If for every block size $u \in K$ there is a $(v, 5, \lambda_2)$-HPMD of type m^u, then there exists a $(v', 5, \lambda_1\lambda_2)$-HPMD of type $\{mh_1, mh_2, \dots, mh_n\}$.

As a special case of Construction 2.2, we have

Construction 2.4 Suppose that a $(v, 5, \lambda_1)$-PMD and a $TD(5, \lambda_2, m)$ exist. Then there exists a $(mv, 5, \lambda_1\lambda_2)$-HPMD of type m^v.

The following construction is a special case of Construction 2.3.

Construction 2.5 Suppose that there is a PBD $B(K, \lambda_1; v)$. If for every block size $u \in K$ there is a $(v, 5, \lambda_2)$-HPMD of type m^u, then there exists a $(v', 5, \lambda_1\lambda_2)$-HPMD of type m^v.

We now list some known results for applying the above constructions.

Lemma 2.6. [2] *There exists a $TD(5, 1, n)$ for every positive integer $n \notin \{2, 3, 6, 10\}$.*

Lemma 2.7. [2] *There exists a $TD(6, 1, n)$ for every integer $n \geq 5$ and $n \notin \{6, 10, 14, 18, 22\}$.*

Lemma 2.8. [9] *For all positive integers r, there exsits a $TD(5, \lambda, r)$ where $\lambda \geq 2$ is an integer.*

Lemma 2.9. [9] *Let $v \geq 6$ and λ be positive integers. If $v \equiv 0$ or $1 \pmod 3$ and $\lambda \equiv 0 \pmod 5$, then there exists a $B(6, \lambda; v)$.*

Lemma 2.10. [3] *There exist three mutually orthogonal idempotent quasigroups of order n for every integer $n \geq 5$ and $n \notin \{6, 10\}$.*

Lemma 2.11. *There exists a $(18, 5, 2)$-HPMD of type 3^6.*

Proof: Let the vertex set of the complete multipartite directed graph $2DK_{3,3,\ldots,3}$ be $X = \cup_{0 \leq i \leq 5} X_i$, where $X_i = \{i, i{+}6\, i{+}12\}$ for $i = 0, 1, \ldots, 5$. Then the block set **A** of the $(18, 5, 2)$-HPMD of type 3^6 consists of the following 5-circuits:

$$
\begin{aligned}
(0,1,2,4,3) &\mod 18,\\
(0,2,1,15,10) &\mod 18,\\
(0,3,8,13,10) &\mod 18,\\
(0,3,13,11,4) &\mod 18,\\
(0,4,11,3,7) &\mod 18,\\
(0,7,5,14,9) &\mod 18.
\end{aligned}
$$

□

The following result comes from the construction of a $(54, 5, 1)$-HPMD of type 9^6 due to R.J.R. Abel [1].

Lemma 2.12. *There exists a $(18, 5, 3)$-HPMD of type 3^6.*

Proof: Let the vertex set of the complete multipartite directed graph $3DK_{3,3,\ldots,3}$ be $X = \cup_{0 \leq i \leq 4} X_i$, where $X_i = \{i, i{+}5\, i{+}10\}$ for $i = 0, 1, \ldots, 4$ and $X_5 = \{\infty_j : j = 1, 2, 3\}$. Then the block set **A** of the $(18, 5, 3)$-HPMD of type 3^6 consists of the following 5-circuits:

$$\begin{aligned}
(0,14,3,1,\infty_1) &\mod 15,\\
(0,4,13,6,\infty_1) &\mod 15,\\
(0,3,2,14,\infty_1) &\mod 15,\\
(0,3,7,14,\infty_2) &\mod 15,\\
(0,14,7,3,\infty_2) &\mod 15,\\
(0,7,13,11,\infty_2) &\mod 15,\\
(0,9,6,8,\infty_3) &\mod 15,\\
(0,6,8,4,\infty_3) &\mod 15,\\
(0,11,12,14,\infty_3) &\mod 15,\\
(0,1,9,7,8) &\mod 15,\\
(j,j+u,j+2u,j+3u,j+4u), &
\end{aligned}$$

where $u \in \{3,6,9,12\}$ and $j \in \{0,1,2\}$ and all operations are performed in Z_{15}. □

We are now able to present our main result of this section.

Theorem 2.13. *For all positive integers h, n and λ satisfying $n \geq 5$ and $\lambda \equiv 0 \pmod 5$, there exists a $(nh,5,\lambda)$-HPMD of type h^n.*

Proof: From Lemma 2.9, we have a $B(6,\lambda;v)$ for each given value of λ where $v \geq 6$ and $v \equiv 0$ or $1 \pmod 3$. If $v \geq 5$ and $v \equiv 2 \pmod 3$, we may delete one point from a $B(6,\lambda;v+1)$ to produce a PBD $B(\{5,6\},\lambda;v)$. Consequently, a PBD $B(\{5,6\},\lambda;v)$ exists for each integer $v \geq 5$ where $\lambda \equiv 0 \pmod 5$. We can then apply Construction 2.5, making use of a $(5h,5,1)$-HPMD of type h^5 and a $(6h,5,1)$-HPMD of type h^6 from Theorem 1.3, to obtain the desired result for all stated values of h except where $h \equiv 1,3,7$ or $9 \pmod{10}$ or $h \in \{5,15\}$. From Lemma 2.6, we know that a $TD(5,1,h)$ exists for each of the above outstanding values of h except for $h = 3$. Furthermore, for all integers $n \geq 5$ and $\lambda \equiv 0 \pmod 5$, a $(n,5,\lambda)$-PMD exists by Theorem 1.4. So, for all integers $n \geq 5$ and $\lambda \equiv 0 \pmod 5$, we can apply Construction 2.4 to get an $(nh,5,\lambda)$-HPMD of type h^n where $h \equiv 1,3,7$ or $9 \pmod{10}$ and $h \neq 3$, or $h \in \{5,15\}$. It remains to deal with the case $h = 3$. Construction 2.1 together with Lemma 2.10 provide us with a $(3n,5,\lambda)$-HPMD of type 3^n for all stated values of λ and $n \notin \{6,10\}$. Now, for $n = 6$, we write $\lambda = 2s + 3t$ where s and t are nonnegative integers determined by λ. A $(18,5,\lambda)$-HPMD of type 3^6 can be formed by taking s copies of a $(18,5,2)$-HPMD and t copies of a $(18,5,3)$-HPMD with type 3^6 which exist by Lemmas 2.11 and 2.12. For $n = 10$, we have a $(30,5,1)$-HPMD of type 3^{10} from Theorem 1.3 and hence we have a $(30,5,\lambda)$-HPMD of type 3^{10} for all stated values of λ. This completes the proof. □

3 The Main Result

In this section, we establish our main result. In view of Theorem 2.13, we need only to determine the existence of $(nh, 5, \lambda)$-HPMDs of type h^n where $\lambda \not\equiv 0 \pmod 5$. Furthermore, when $\lambda \geq 4$, it can be written as $\lambda = 2s + 3t$ where s and t are nonnegative integers determined by λ. An $(nh, 5, \lambda)$-HPMD of type h^n can be formed by taking s copies of an $(nh, 5, 2)$-HPMD and t copies of an $(nh, 5, 3)$-HPMD with type h^n. Therefore, it suffices to establish the result for the case where $\lambda = 2$ and 3. It is important to observe that the necessary condition for the existence of an $(nh, 5, \lambda)$-HPMD of type h^n is the same for $\lambda = 1, 2$ and 3. Consequently, for the most part, the results have been established in Theorem 1.3. What we need to do is to tackle the possible exceptions listed in Theorem 1.3; namely,

(1) $h \equiv 1, 3, 7$ or 9 (mod 10), $h \neq 3$, and $n \in \{6, 10, 15, 20, 30\}$;

(2) $h = 3$ and $n \in \{6, 30, 56\}$;

(3) the pairs $(h, n) \in \{(5, 6), (15, 6), (15, 18), (15, 28)\}$.

We first deal with the cases (1) and (3) above.

Lemma 3.1. *Both an $(nh, 5, 2)$-HPMD and an $(nh, 5, 2)$-HPMD with type h^n exist if h and n satisfy each of the following:*

(1) $h \equiv 1, 3, 7$ *or* 9 (mod 10), $h \neq 3$, *and* $n \in \{6, 10, 15, 20, 30\}$;

(2) $(h, n) \in \{(5, 6), (15, 6), (15, 18), (15, 28)\}$.

Proof: From Lemma 2.6, we know that a $TD(5, 1, h)$ exists whenever $h \equiv 1, 3, 7$ or 9 (mod 10) and $h \neq 3$, or $h \in \{5, 15\}$. By Theorem 1.4, we also know that both an $(n, 5, 2)$-PMD and an $(n, 5, 3)$-PMD exist if $n \in \{6, 10, 15, 20, 30\}$. So, Construction 2.4 can be applied to establish the conclusion for $h \equiv 1, 3, 7$ or 9 (mod 10), $h \neq 3$ and $n \in \{6, 10, 15, 20, 30\}$, or $(h, n) \in \{(5, 6), (15, 6)\}$. Now for $(h, n) \in \{(15, 18), (15, 28)\}$, note that a $(90, 5, 1)$-HPMD of type 5^{18} and a $(140, 5, 1)$-HPMD of type 5^{28} were shown to exist in Theorem 1.3. Applying Construction 2.2, using a $TD(5, 2, 3)$ and a $TD(5, 3, 3)$ from Lemma 2.8, gives the result. This completes the proof. □

Now we turn to the case where $h = 3$ and $n \in \{6, 30, 56\}$.

Lemma 3.2. *If $\lambda = 2$ or 3 and $n \in \{6, 30, 56\}$, then a $(3n, 5, \lambda)$-HPMD of type 3^n exists.*

Proof: For $n = 6$, the constructions are provided in Lemmas 2.11 and 2.12. For $n = 30$, we first create a $B(\{5, 6\}, 1; 30)$ from a $TD(6, 1, 5)$. For $n = 56$, we first create a $B(\{5, 6, 11\}, 1; 56)$ by deleting all but one point from a group of a $TD(6,1,11)$. We then apply Construction 2.5, using $(3n,5,\lambda)$-HPMDs of type 3^n for $n \in \{5,6,11\}$, to establish the conclusion. □

The foregoing can be summarized in the following theorem.

Theorem 3.3. *Let $\lambda \geq 2$ and $\lambda \not\equiv 0 \pmod 5$. Then the necessary condition for the existence of an $(nh,5,\lambda)$-HPMD of type h^n, namely, $n \geq 5$ and $\lambda n(n-1)h^2 \equiv 0 \pmod 5$, is also sufficient.*

Theorem 2.13 and Theorem 3.3 together give our main result.

Theorem 3.4. *Let $\lambda \geq 2$ be an integer. Then the necessary condition for the existence of an $(nh,5,\lambda)$-HPMD of type h^n, namely, $n \geq 5$ and $\lambda n(n-1)h^2 \equiv 0 \pmod 5$, is also sufficient.*

As already mentioned, the existence of an $(nh,5,\lambda)$-HPMD of type h^n implies the existence of a GDD having block size 5, type h^n and index 4λ. However, it should be pointed out that the converse is not necessarily true. As a consequence of Theorem 3.4, we have essentially established the following analogous result for GDDs.

Theorem 3.5. *Let $\lambda \geq 2$ be an integer. Then the necessary condition for the existence of a GDD having block size 5, type h^n and index 4λ, namely, $n \geq 5$ and $\lambda n(n-1)h^2 \equiv 0 \pmod 5$, is also sufficient.*

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