# The Edge-Integrity of Some Graphs

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**Abstract.** In a communication network, several vulnerability measures are used to determine the resistance of the network to disruption of operation after the failure of certain stations or communication links. If we think of a graph as modeling a network, the **edge-integrity** of a graph is one measure of **graph vulnerability** and it is defined to be the minumum sum of the orders of a set of edges being removed and a largest remaining component. In this paper the edge-integrity of graphs $B_n$, $H_n$ and $E_p^t$ are calculated. Also some results are given about edge-integrity of these graphs.

**Keywords:** Integrity, Edge-Integrity, Vulnerability.

## 1 Introduction

When investigating the vulnerability of a communication network to disruption, one may want to learn the answer of the following questions (there may be others):
(i) What is the number of elements that are not functioning?
(ii) What is the size of the largest remaining group within which mutual communication can still occour?

In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the answers of these questions can be made to be simultaneously small. The concepts of **vertex-integrity** and **edge-integrity** were introduced as measures of graph vulnerability in this sense by Barefoot, Entringer and Swart [5]. Formally the **vertex-integrity** is defined by

$$I(G) = min_{S \subset V(G)}\{|S| + m(G - S)\},$$

and the **edge-integrity** is defined by

$$I'(G) = min_{S \subseteq E(G)}\{|S| + m(G - S)\},$$

where $m(G - S)$ denotes the order of a largest component of $G - S$.

Barefoot, Entringer and Swart [5] obtained several results on the integrity and Goddard [11] developed some results in his Ph.D. thesis. Moreover Bagga et al. [1, 4] prepared a survey on the vertex-integrity and edge-integrity.

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In section 2 known results on edge-integrity and some definitions are given. Sections 3,4 and 5 give some results on the edge-integrity of graphs $B_n$, $H_n$ and $E_p^t$, respectively. The notations used in this paper are same as in [6].

## 2 Basic Results

In this section we will review some of the known results.

**Theorem 1.** *[1] The edge-integrity of*
*(a) the complete graph $K_n$ is $n$,*

*(b) the null graph $\overline{K_n}$ is 1,*

*(c) the star $K_{1,n}$ is $n+1$,*

*(d) the path $P_n$ is $\lceil 2\sqrt{n}\rceil - 1$,*

*(e) the cycle $C_n$ is $\lceil 2\sqrt{n}\rceil$ for $n \geq 4$.*

Let $\Delta$ and $\lambda$ be maximum vertex degree and edge-connectivity of a graph G, respectively.

**Theorem 2.** *[4] For any graph $G$ with order $n$,*
*(a) $I'(G) \leq n$,*

*(b) $I'(G) \geq I(G)$,*

*(c) $I'(G) \geq \lceil 2\sqrt{n}\rceil - 1$ if $G$ is connected,*

*(d) $I'(G) \geq \Delta(G) + 1$,*

*(e) $I'(G) \geq \min\{\lceil\sqrt{2n\lambda}\rceil, n\}$, if $\lambda \geq 2$.*

The following theorem gives the edge-integrity according to maximum vertex degree of a tree.

**Theorem 3.** *[4] Let $T$ be a tree of order $n$.*
*(a)If $\Delta \geq \frac{n}{2}$, then $I'(T) = \Delta + 1$,*

*(b)If $\Delta < \frac{n}{2}$, then $I'(T) \leq \frac{n+3}{2}$.*

Now we will give some definitions:

**Definition 1.**[6] *The (Cartesian)* **product** *$G_1 \times G_2$ of graphs $G_1$ and $G_2$ has $V(G_1) \times V(G_2)$ as its vertex set and $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if either $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ or $u_2 = v_2$ and $u_1$ is adjacent to $v_1$.*

**Definition 2.** *[6] The* **corona** *$G_1 \circ G_2$ was defined as the graph $G$ obtained by* *taking one copy of* $G_1$ *of order* $n$ *and* $n$ *copies of* $G_2$, *and then joining the ith vertex of* $G_1$ *to every vertex in the ith copy of* $G_2$.

**Definition 3.** *[6] The* **kth power** $G^k$ *of a connected graph* $G$, *where* $k \geq 1$, *is the graph with* $V(G^k) = V(G)$ *for which* $uv \in E(G^k)$ *if and only if* $1 \leq d_G(u,v) \leq k$.

Let $\delta$ be minumum vertex degree of a graph.

**Theorem 4.** *[4] For any graphs* $G$ *and* $H$,
(a) $I'(G \times H) \leq min\{|G|I'(H), |H|I'(G)\}$,

(b) $I'(G \times H) \geq max\{(1+\delta(G))I'(H), (1+\delta(H))I'(G)\}$.

**Definition 4.** *[3] A graph is called* **honest** *if its edge-integrity equals its order.*

**Theorem 5.** *[3] If* $|G| \geq 2$ *and* $|H| \geq 3$, *then* $\overline{G \times H}$ *is* honest *unless* $G = K_2$ *and* $H = K_3$.

**Theorem 6.** *[3] If* $G \neq P_4$ *then* $G$ *or* $\overline{G}$ *is honest.*

## 3 Edge-Integrity of Graph $B_n$

In this section we first consider the **binomial tree** [7] $B_n$ (Figure 1). The binomial tree $B_n$ is an ordered tree defined recursively. The binomial tree $B_0$ consists of a single vertex. The binomial tree $B_n$ consists of two binomial trees $B_{n-1}$ that are linked together: the root of one is the leftmost child of the root of the other.

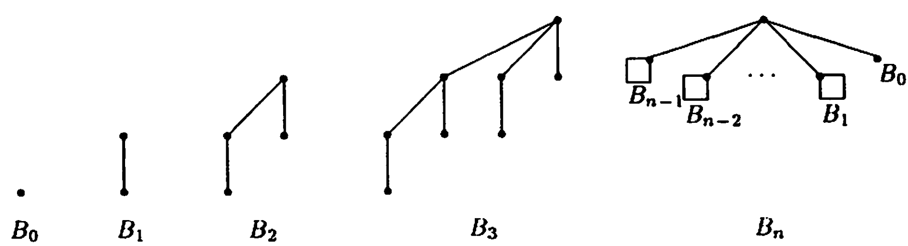


Figure 1.

Now we give the some results on the edge-integrity of binomial tree $B_n$.

**Theorem 7.** *Let* $n \geq 1$ *be a positive integer. Then*

$$I'(B_n) = \begin{cases} 2^{\frac{n+2}{2}} - 1, & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** If we remove $r$ edges where $2^i - 1 \leq r < 2^{i+1} - 1$ and $1 \leq i \leq n-1$, then one of the remaining connected components will have at least $2^{n-i}$ vertices. Moreover, if we remove $r = 2^n - 1$ edges, then the order of largest component is exactly 1. Therefore

$$I'(B_n) = min\{\min_{1 \leq i < n}\{2^i - 1 + 2^{n-i}\}, 2^n\}.$$

The function $2^i - 1 + 2^{n-i}$ takes its minimum value at $i = \frac{n}{2}$ when $n$ is even and $i = \frac{n-1}{2}$ when $n$ is odd. Hence if we substitute the minumum values in the function $2^i - 1 + 2^{n-i}$, then we have

$$2^{\frac{n+2}{2}} - 1 \text{ when } n \text{ is even} \quad \text{and} \quad 2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} - 1 \text{ when } n \text{ is odd.}$$

We easily see that $2^{\frac{n+2}{2}} - 1 \leq 2^n$ and $2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} - 1 \leq 2^n$ for every $n \geq 1$. This proves the theorem.

**Remark** $I'(\overline{B}_n) = 2^n$ for every $n \geq 3$. In fact, by Theorem 7, $I'(B_n)$ is not equal to order of $B_n$ for every $n \geq 3$ (that is, $B_n$ is not honest). Then, $\overline{B}_n$ must be honest by Theorem 6 for every $n \geq 3$. Hence, $I'(\overline{B}_n)$ is equal to $2^n$ by Definition 4.

The following theorems give the edge-integrity of graphs $B_n \times C_m$, $B_n \times P_m$ and $B_n \times K_{1,m}$ and their complements.

**Theorem 8.** *Let m,n be the positive integers. Then*
(a) $I'(B_n \times C_m) = min\{2^n I'(C_m), mI'(B_n)\}$,

(b) $I'(B_n \times P_m) = min\{2^n I'(P_m), mI'(B_n)\}$,

(c) $I'(B_n \times K_{1,m}) = (1+m)I'(B_n)$.

**Proof.(a)** Let S be a subset of $E(B_n \times C_m)$ that achieves the edge integrity of $B_n \times C_m$. Then the number of elements of S must be $m(2^r - 1)$ where $1 \leq r \leq n-1$ or $2^n r$ where $1 \leq r \leq m$.
**Case 1.** If we remove $|S| = m(2^r - 1)$ edges where $1 \leq r \leq n-1$, then $m((B_n \times C_m) - S) \geq m(2^{n-r})$. Hence

$$I'(B_n \times C_m) \geq \min_r\{m(2^r - 1) + m(2^{n-r})\}$$
$$I'(B_n \times C_m) \geq m \min_r\{2^r - 1 + 2^{n-r}\}.$$

By the proof of Theorem 7, we have that $\min_r\{2^r - 1 + 2^{n-r}\}$ is equal to $I'(B_n)$ for every n. Then $I'(B_n \times C_m) \geq mI'(B_n)$.
**Case 2.** If we remove $|S| = 2^n r$ edges where $1 \leq r \leq m$, then $m((B_n \times C_m) - S) \geq \frac{m2^n}{r}$. Hence

$$I'(B_n \times C_m) \geq \min_r\{2^n r + \frac{m2^n}{r}\} \quad \text{and} \quad I'(B_n \times C_m) \geq 2^n \min_r\{r + \frac{m}{r}\}.$$

We easily see that $\min_r \{r + \frac{m}{r}\}$ is equal to $I'(C_m)$ for every m. Then $I'(B_n \times C_m) \geq 2^n I'(C_m)$.

By Case 1 and 2, we have $I'(B_n \times C_m) \geq min\{2^n I'(C_m), mI'(B_n)\}$. Moreover we have $I'(B_n \times C_m) \leq min\{2^n I'(C_m), mI'(B_n)\}$ by theorem 4(a) and this completes the proof.

**(b)** The proof is similar to that of Theorem 8(a).

**(c)** Let S be a subset of $E(B_n \times K_{1,m})$ that achieves $I'(B_n \times K_{1,m})$. Then $|S| = (1+m)(2^r - 1)$ where $1 \leq r \leq n-1$. If we remove $(1+m)(2^r - 1)$ edges where $1 \leq r \leq n-1$, then the largest remaining component has at least $(1+m)(2^{n-r})$ vertices. Hence

$$I'(B_n \times K_{1,m}) \geq \min_r \{(1+m)(2^r - 1) + (1+m)(2^{n-r})\}$$
$$I'(B_n \times K_{1,m}) \geq (1+m) \min_r \{2^r - 1 + 2^{n-r}\}.$$

By the proof of Theorem 7, we have that $\min_r \{2^r - 1 + 2^{n-r}\}$ is equal to $I'(B_n)$ for every n. Then $I'(B_n \times K_{1,m}) \geq (1+m)I'(B_n)$.

By Theorem 4(a), we have $I'(B_n \times K_{1,m}) \leq min\{2^n(1+m), (1+m)I'(B_n)\}$. Since $I'(B_n) \leq 2^n$ by Theorem 2(a), then we have $I'(B_n \times K_{1,m}) \leq (1+m)I'(B_n)$. This completes the proof.

**Corollary 9.** *Let m,n be the positive integers. Then*
(*a*) $I'(\overline{B_n \times C_m}) = m2^n$ *for every* $n > 1$ *and* $m > 3$,

(*b*) $I'(\overline{B_n \times P_m}) = m2^n$,

(*c*) $I'(\overline{B_n \times K_{1,m}}) = (1+m)2^n$.

**Proof.** The proof follows directly from Theorem 5 and Definition 4.

The following theorems give some results on the edge-integrity and corona operation.

**Theorem 10.** *Let $G_m$ be a graph with order m and define* $r = \frac{ln(2^n(1+m))}{ln4}$. *Then $I'(B_n \circ G_m)$ is equal to*

$$\begin{cases} 2^{n-\lfloor r \rfloor}(1+m) + 2^{\lfloor r \rfloor} - 1, & \text{if } \lfloor r \rfloor \leq n \text{ and } \lceil r \rceil > n, \\ 2^{n-\lceil r \rceil}(1+m) + 2^{\lceil r \rceil} - 1, & \text{if } \lceil r \rceil \leq n \text{ and } \lfloor r \rfloor > n, \\ 2^n + m, & \text{if } \lceil r \rceil > n \text{ and } \lfloor r \rfloor > n, \\ min\{2^{n-\lfloor r \rfloor}(1+m) + 2^{\lfloor r \rfloor} - 1, 2^{n-\lceil r \rceil}(1+m) + 2^{\lceil r \rceil} - 1\}, & \text{otherwise.} \end{cases}$$

**Proof.** Let $S$ be a subset of $E(B_n \circ G_m)$ such that $I'(B_n \circ G_m) = |S| + m((B_n \circ G_m) - S)$ and let r be number of elements of S. If S achieves $I'(B_n \circ G_m)$, the edges joining $B_n$ to $G_m$ and the edges of graphs $G_m$ are not in S. Then S must contain only the edges of $B_n$. So if we remove r edges where $2^i - 1 \leq r < 2^{i+1} - 1$ and $1 \leq i \leq n$, then

$$m((B_n \circ G_m) - S) \geq 2^{n-i}m + 2^{n-i}$$

and

$$I'(B_n \circ G_m) = \min_i\{2^i - 1 + 2^{n-i}m + 2^{n-i}\}$$

The function $2^i - 1 + 2^{n-i}m + 2^{n-i}$ takes its minumum value at $r = \dfrac{ln(2^n(1+m))}{ln4}$ and the result follows.

**Theorem 11.** *Let $K_m$ be a complete graph of order m. Then*

$$I'(K_m \circ B_n) = \begin{cases} \binom{m}{2} + 2^n + 1, & if\, m2^n \geq \binom{m}{2}, \\ m(2^n + 1), & if\, m2^n < \binom{m}{2}. \end{cases}$$

**Proof.** Let S be a subset of edges that achives the edge-integrity of graph $K_m \circ B_n$. It is obvious that $|E(K_m)| = \binom{m}{2}$. Then we consider two cases:

**Case 1:** If $m2^n \geq \binom{m}{2}$, then S must contain all of the edges of graph $K_m$. It follows that $|S| = \binom{m}{2}$ and $m((K_m \circ B_n) - S) = 2^n + 1$.

**Case 2:** If $m2^n < \binom{m}{2}$, then S has to contain the edges joining $K_m$ to $B_n$. Hence $|S| = m2^n$ and $m((K_m \circ B_n) - S) = m$.
The proof is completed.

**Theorem 12.** *Let $n \geq 9$ be a positive integer and define $r = \lfloor \frac{3n-9}{8} \rfloor$. Then*

$$I'(B_n^2) \leq 2^{r+1}(2n - 2r + 1) + 2^{n-(r+1)} - (2n + 3).$$

**Proof.** Let S be a subset of $E(B_n^2)$ such that $I'(B_n^2) = |S| + m(B_n^2 - S)$ and let $r$ be number of elements of $S$. If we remove $r$ edges where

$$\sum_{i=0}^{t} 2^i(2(n-i) - 1) \leq r < \sum_{i=0}^{t+1} 2^i(2(n-i) - 1) \quad \text{and} \quad 0 \leq t \leq n - 3,$$

then the order of a largest component must be at most $2^{n-(t+1)}$. Therefore

$$I'(B_n^2) \leq \min_t\{\sum_{i=0}^{t} 2^i(2(n-i) - 1) + 2^{n-(t+1)}\}.$$

Since $\sum_{i=0}^{t} 2^i i = 2^{t+1}(t-1)+2$ and $\sum_{i=0}^{t} 2^i = 2^{t+1}-1$, then we have

$$I'(B_n^2) \le \min_t \{2^{t+1}(2n-2t+1)+2^{n-(t+1)}-(2n+3)\}.$$

By using certain numerical solution method, we easily see that the function $2^{t+1}(2n-2t+1)+2^{n-(t+1)}-(2n+3)$ takes an approximately minumum value at $r = \dfrac{3n-9}{8}$ for $n \ge 9$. This proves the theorem..

Moreover the values of $I'(B_n^2)$ are given for $n \le 8$ in the following Table 1.

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $I'(B_n^2)$ | 2 | 4 | 8 | 15 | 25 | 43 | 67 | 105 |

Table 1.

## 4 Edge-Integrity of graph $H_n$

Now we consider the graph $H_n$ ($n \ge 0$) of a **complete binary tree** [7]. A k-ary tree is a positional tree in which, for every vertex, all children with labels greater than k are missing. Thus, a binary tree is a k-ary tree with k=2. A complete binary tree is a 2-ary tree in which all leaves have the same depth and all internal vertices have degree 2 (Figure 2). The graph $H_n$ has $2^{n+1}-1$ vertices.

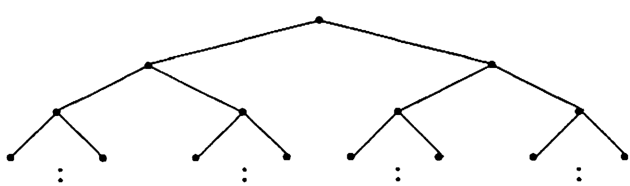

Figure 2.

The following theorems are about the edge-integrity of a complete binary tree. Note that if $n=0$, then $H_0$ is a isolated vertex and $I'(H_0)=1$.

**Theorem 13.** *Let $n \ge 1$ be a positive integer. Then,*

$$I'(H_n) = \begin{cases} 2^{\frac{n}{2}} + 2^{\frac{n+2}{2}} - 1, & \text{if } n \text{ is even,} \\ 2^{\frac{n+3}{2}} - 1. & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** The proof is very similar to that of Theorem 7.

**Corollary 14.** *Let $m \geq 1$ and $n \geq 1$ be positive integers.*
(a) $I'(\overline{H}_n) = 2^{n+1} - 1$,
(b) $I'(H_n \times P_m) = min\{(2^{n+1} - 1)I'(P_m), mI'(H_n)\}$,
(c) $I'(H_n \times C_m) = min\{(2^{n+1} - 1)I'(C_m), mI'(H_n)\}$,
(d) $I'(\overline{H_n \times P_m}) = m(2^{n+1} - 1)$,
(e) $I'(\overline{H_n \times C_m}) = m(2^{n+1} - 1)$.

**Proof.** The proof follows directly Theorem 4,5,6,8,13 and Definition 4.

# 5 Edge-Integrity of graph $E_p^t$

The graph $E_p^t$ has $t$ legs and each leg has $p$ vertices (Figure 3). Thus $E_p^t$ has $n = pt + 2$ vertices.

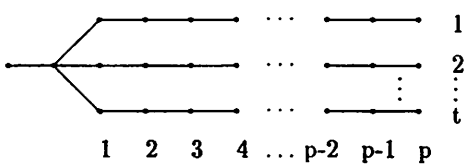


Figure 3.

**Theorem 15.** *Let $k$ be an integer. For $1 \leq k \leq p$,*

$$I'(E_p^t) = \begin{cases} \lceil 2\sqrt{pt+2} \rceil, & \text{if } \lceil kt + \frac{pt+2}{kt+1} \rceil > \lceil 2\sqrt{pt+2} \rceil - 1 \text{ for every } k, \\ \lceil 2\sqrt{pt+2} \rceil - 1, & \text{otherwise.} \end{cases}$$

**Proof.** By Theorem 2(c), we have that $I'(E_p^t) \geq \lceil 2\sqrt{pt+2} \rceil - 1$. To see this, we must remove $kt$ edges where $1 \leq k \leq p$. Then the number of remaining components is exactly $kt + 1$ and one of them must have at least $\frac{pt+2}{kt+1}$ vertices. Therefore

$$I'(E_p^t) \geq \min_{1 \leq k \leq p} \{kt + \frac{pt+2}{kt+1}\} \text{ and so } I'(E_p^t) \geq \lceil 2\sqrt{pt+2} \rceil - 1.$$

Consider $kt + \frac{pt+2}{kt+1}$ for every k. Then we have two cases:

**Case1.** If $\lceil kt + \frac{pt+2}{kt+1} \rceil = \lceil 2\sqrt{pt+2} \rceil - 1$ for at least one k, then $I'(E_p^t) = \lceil 2\sqrt{pt+2} \rceil - 1$.

**Case 2.** If $\lceil kt + \frac{pt+2}{kt+1} \rceil > \lceil 2\sqrt{pt+2} \rceil - 1$ for every k where $1 \leq k \leq p$, then we have $I'(E_p^t) \geq \lceil 2\sqrt{pt+2} \rceil$.

Next we claim that $I'(E_p^t) \leq \lceil 2\sqrt{pt+2} \rceil$. To see that this claim is true, suppose that $I'(E_p^t) \geq \lceil 2\sqrt{pt+2} \rceil + 1$. To obtain this result, we must remove $kt+1$ edges where $1 \leq k \leq p$ and the number of the remaining connected components must be $kt$ where $1 \leq k \leq p$. But, if we remove $kt+1$ edges where $1 \leq k \leq p$ from any graph $E_p^t$, then we have exactly $kt+2$ components. This is a contradiction since we assumed $I'(E_p^t) \geq \lceil 2\sqrt{pt+2} \rceil + 1$. Therefore $I'(E_p^t) \leq \lceil 2\sqrt{pt+2} \rceil$ and so $I'(E_p^t) = \lceil 2\sqrt{pt+2} \rceil$
Hence the proof is completed.

**Theorem 16.** *Let p,t and m be positive integers.*
*(a) If $m \leq p$, then $I'(E_p^t \times P_m) = mI'(E_p^t)$,*

*(b) If $m > p$, then $I'(E_p^t \times P_m) \geq \lceil 2\sqrt{mpt\sqrt{pt+2}} \rceil - pt$.*

**Proof.** Consider the graph $E_p^t \times P_m$ and let S be a subset of edges such that $I'(E_p^t \times P_m) = |S| + m((E_p^t \times P_m) - S)$.
**(a)** If $m \leq p$, then S must contain only the edges of graphs $E_p^t$ in $E_p^t \times P_m$. If we remove $|S| = mkt$ edges where $1 \leq k \leq p$, then the number of remaining components is exactly $kt+1$ and one of them must have at least $\dfrac{m(pt+2)}{kt+1}$ vertices. Therefore

$$I'(E_p^t \times P_m) \geq \min_k \{mkt + \frac{m(pt+2)}{kt+1}\}$$
$$I'(E_p^t \times P_m) \geq m \min_k \{kt + \frac{pt+2}{kt+1}\}.$$

By the proof of Theorem 15, we easily see that $\min_k \{kt + \frac{pt+2}{kt+1}\}$ is equal to $I'(E_p^t)$ for every p,t. Then $I'(E_p^t) \geq mI'(E_p^t)$.

On the other hand, since S contains only the edges of graphs $E_p^t$ in $E_p^t \times P_m$, then $I'(E_p^t) \leq mI'(E_p^t)$. That is, $mI'(E_p^t)$ is an upper bound. Hence the proof is completed.
**(b)** If $m > p$, then S must contain not only edges of graphs $E_p^t$ but also edges of graphs $P_m$ in $E_p^t \times P_m$. First we must remove the edges of graphs $E_p^t$. With an argument given for $I'(E_p^t)$, $|S| \geq m(\sqrt{pt+2} - 1)$ where $m > p$. Next we must remove the edges of graphs $P_m$. Then $|S| \geq m(\sqrt{pt+2} - 1) + kpt$ and $m((E_p^t \times P_m) - S) \geq \dfrac{m(\sqrt{pt+2})}{k+1}$ where $1 \leq k \leq m-1$. Therefore

$$I'(E_p^t \times P_m) \geq m(\sqrt{pt+2} - 1) + \min_k \{kpt + \frac{m(\sqrt{pt+2})}{k+1}\}$$

The function $kpt + \dfrac{m(\sqrt{pt+2})}{k+1}$ takes its minumum value at $k = -1 + \sqrt{\dfrac{m\sqrt{pt+2}}{pt}}$.
If we substitue the value k in the function, then this completes the proof.

**Corollary 17.** *Let p,t and m be positive integers. Then*

$$I'(\overline{E_p^t \times P_m}) = m(pt + 2).$$

**Proof.** The proof follows directly from Theorem 5 and Definition 4.

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