Graph Partitions II

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Abstract

The following partition problem was first introduced by R.C. Entringer and has subsequently been studied by the first author and more recently by Bollobás and Scott, who consider the hypergraph version as well, using a probabilistic technique. The partition problem is that of coloring the vertex set of a graph with s colors so that the number of induced edges is bounded for each color class. The techniques employed are non-constructive and non-probabilistic and improve the known bounds in the previous papers.

1. Introduction

We split this paper into two parts. In Part I we restrict our attention to the case $s = 2$ and improve the first result [6] that verified the original conjecture of Erdös. Some useful sets employed in [6] are modified. In Part II we use our result from Part I and improve the bound given in [7]. The reader who is familiar with the previous work may skip the following paragraphs; we make every attempt to be consistent with our previous notation.

For our purposes, graphs are finite and simple. We use the standard notation as in [3]. For a given graph G, let $(U_1, \ldots, U_s)$ denote a partition of $V(G)$ into s-classes; we also refer to $(U_1, \ldots, U_s)$ as an s-coloring of $V(G)$, where U_j denotes the vertices colored j. Let $e(U_i)$ denote the number of edges in the induced subgraph $G[U_i]$ and let $\gamma_s(U_1, \ldots, U_s) = \max_{1 \leq i \leq s} \{e(U_i)\}$. The problem is to minimize γ_s over all partitions $(U_1, \ldots, U_s)$; define $\gamma_s(G) = \min_{(U_1,\ldots,U_s)} \gamma_s(U_1, \ldots, U_s)$.

Paul Erdös conjectured [5] $\gamma_2(G) \leq \frac{e(G)}{4} + O(\sqrt{e(G)})$, where $e(G)$ denotes the number of edges in G. In [6], the first author verified the conjecture and showed it was best possible. Roger Entringer [4] posed the problem to find $\gamma_2(G)$ and proposed a related matrix discrepancy problem. The solution of the matrix problem, Porter and Székely [9], gives a bound asymptotic to $\gamma_s(G)$, however it did not lead to a solution of the partition problem. In [7], the first author gives an upper bound for $\gamma_s(G)$ when s is a power of 2, i.e., $s = 2^p$. In [1], Bollobás and Scott, using a probabilistic technique, give various upper bounds on $\gamma_s(G)$ for any s. In [2], they extend the problem and study the analogous hypergraph model. In this paper we use a non-constructive, non-probabilistic technique that gives an upper bound for $\gamma_s(G)$ that depends on the size of G. In [10], Shahrokhi and Székely show that the computation of γ_s is NP-hard.

Let $e[U_1, \ldots, U_s] = |\{x_i x_j | x_i x_j \in E(G),\ x_i \in U_i,\ x_j \in U_j\}|$, where $i \neq j$, and define $M_s(U_1, \ldots, U_s) = e[U_1, \ldots, U_s]$, and

$$M_s(G) = \max_{(U_1,\ldots,U_s)} M_s(U_1, \ldots, U_s).$$

We refer to $M_s(G)$ as the max s-cut of G.

2. Part I

Given a finite and simple graph G, let (U, V) be a bipartition of $V(G)$. Then, $U \cup V = V(G)$, $U \cap V = \emptyset$. Define $\gamma(U, V) = \max\{e(U), e(V)\}$, $\gamma_2(G) = \min_{(U,V)} \{\gamma(U, V)\}$. Let $e(U, V) = |\{uv : uv \in E(G),\quad u \in U,\ v \in V\}|$, and $d_U(x)$ to be number of vertices in U adjacent to x ($x \in V(G)$). Clearly, $e(U, V) = \sum_{x \in U} d_V(x)$.

Theorem 1.

$$\frac{\gamma_2(G)}{e(G)} \leq \frac{1}{4}\left(1 + \sqrt{\frac{8}{9e(G)}}\right).$$

A series of lemmas will give the proof of the theorem. Define $\Omega = \max_{(U,V)}\{e(U, V)\}$, Ω is known as the max cut of G. Let $S = \{(U, V) : e(U, V) = \Omega\}$, note $S \neq \emptyset$.

Lemma 1. *For any $(U, V) \in S$, we have $e(U, V) \geq 2\gamma(U, V)$.*

Proof. Suppose $e(U) = \max\{e(U), e(V)\} = \gamma(U, V)$. Then for any $x \in U$, $d_V(x) \geq d_U(x)$ (otherwise, $\exists x_0 \in U$ with $d_U(x_0) > d_V(x_0)$. Then $(U - x_0,\ V + x_0)$ is a bipartition of $V(G)$ with $e(U - x_0,\ V + x_0) > e(U, V) = \Omega$, contradicting the definition of Ω). Hence,

$$e(U, V) = \sum_{x \in U} d_V(x) \geq \sum_{x \in U} d_U(x) = 2e(U) = 2\gamma(U, V).$$

□

Define $T = \{(U, V) : e(U, V) \geq 2\gamma(U, V)\}$, since $S \subseteq T$, $T \neq \emptyset$. Let $\alpha = \min_T\{\gamma(U, V)\}$. Let $(A, B) \in T$ with $e(A) = \gamma(A, B) = \alpha$ and $\Delta =$

$e(A) - e(B) \geq 0$. If $\Delta = 0$, $e(A) = e(B)$. Then

$$\frac{e(A)}{e(G)} = \frac{e(A)}{2e(A) + e(A,B)} \leq \frac{e(A)}{4e(A)} = \frac{1}{4}$$

as $e(A,B) \geq 2e(A)$. So, $\frac{\gamma_2(G)}{e(G)} \leq \frac{1}{4}$ since $\gamma_2(G) \leq e(A)$.

Following we consider the case of $\Delta > 0$.
Define

$$X = \left\{ x \in A \mid d_B(x) > \frac{3}{4}\Delta \right\},$$

$$Y = \left\{ y \in A \mid d_A(y) \neq 0,\ d_B(y) \leq \frac{3}{4}\Delta \right\}.$$

Note $e(A) = e(X) + e(Y) + e(X,Y)$.

Lemma 2. *For any vertex $y \in Y$, $d_B(y) > 3d_A(y)$.*

Proof. Otherwise $\exists\, y_0 \in Y$, $d_B(y_0) \leq 3d_A(y_0)$. As $e(A - y_0) = e(A) - d_A(y_0)$; $e(B + y_0) = e(B) + d_B(y_0)$; $e(A - y_0,\ B + y_0) = e(A,B) + d_A(y_0) - d_B(y_0)$, then:

$$\begin{aligned}
e(A - y_0, B + y_0) &= e(A,B) + d_A(y_0) - d_B(y_0) \\
&\geq e(A,B) + d_A(y_0) - 3d_A(y_0) \\
&\geq e(A,B) - 2d_A(y_0) \geq 2(e(A) - d_A(y_0)) \\
&= 2e(A - y_0); \\
e(A - y_0, B + y_0) &= e(A,B) + d_A(y_0) - d_B(y_0) \\
&\geq 2e(A) + d_A(y_0) - d_B(y_0) \\
&\geq 2e(B) + 2\Delta + \frac{1}{3}d_B(y_0) - d_B(y_0) \\
&\geq 2e(B) + 2 \cdot \frac{4}{3}d_B(y_0) + \frac{1}{3}d_B(y_0) - d_B(y_0) \\
&= 2e(B) + 2d_B(y_0) = 2e(B + y_0),
\end{aligned}$$

since $d_B(y_0) \leq \frac{3}{4}\Delta$. Therefore $e(A - y_0, B + y_0) \geq 2\gamma(A - y_0, B + y_0)$. So $(A - y_0, B + y_0) \in T$. But as $d_A(y_0) \neq 0$, $e(A - y_0) < e(A)$, and $e(B + y_0) = e(B) + d_B(y_0) \leq e(B) + \frac{3}{4}\Delta < e(A)$, hence $\gamma(A - y_0, B + y_0) < e(A) = \alpha$, which contradicts the definition of α. Therefore, $\forall y \in Y$, $d_B(y) > 3d_A(y)$. ☐

Define ξ by $\xi \sum_{y \in Y} d_A(y) = e(Y) + e(X,Y)$. Since $\sum_{y \in Y} d_A(y) = e(X,Y) + 2e(Y)$, $\xi \leq 1$.

Lemma 3. *If $e(X) = 0$, then $\frac{e(A)}{e(G)} < \frac{1}{4}$.*

Proof. $e(A,B) = \sum_{y\in A} d_B(y) \geq \sum_{y\in Y} d_B(y) > 3\sum_{y\in Y} d_A(y)$ by Lemma 2, and, $e(A) = e(X) + e(Y) + e(X,Y) = e(Y) + e(X,Y) = \xi \sum_{y\in Y} d_A(y)$. So, since $\xi \leq 1$,

$$\frac{e(A)}{e(G)} = \frac{e(A)}{e(A)+e(A,B)+e(B)}$$
$$\leq \frac{e(A)}{e(A)+e(A,B)} < \frac{\xi\sum_{y\in Y} d_A(y)}{\xi\sum_{y\in Y} d_A(y) + 3\sum_{y\in Y} d_A(y)} \leq \frac{1}{4}.$$

□

For $e(X) \neq 0$, let $k = |X|$. Then $e(X) = \frac{ck(k-1)}{2}$ for some $c \leq 1$. Clearly, $k \geq 2$.

Case I: $\Delta \leq \frac{4}{3}c(k-1)$:

Since $e(A,B) \geq 2e(A)$, $e(G) = e(A) + e(B) + e(A,B) \geq e(A) + e(A) - \Delta + 2e(A) = 4e(A) - \Delta$.

So,

$$\frac{e(A)}{e(G)} = \frac{e(A)}{e(A)+e(B)+e(A,B)} \leq \frac{e(A)}{4e(A)-\Delta}$$
$$= \frac{e(X) + \xi\sum_{y\in Y} d_A(y)}{4e(X) - \Delta + 4\xi\sum_{y\in Y} d_A(y)}.$$

Lemma 4.

$$e(X) \leq \frac{1}{4}\left(4e(X) - \Delta + \sqrt{\frac{8}{9}(4e(X)-\Delta)}\right).$$

Proof. It is sufficient to show that $e(X) \leq \frac{1}{4}\left(4e(X) - \frac{4}{3}c(k-1) + \sqrt{\frac{8}{9}(4e(X) - \frac{4}{3}c(k-1))}\right)$, since $\Delta \leq \frac{4}{3}c(k-1)$. That is, $\frac{4}{3}c(k-1) \leq \sqrt{\frac{8}{9}} \cdot \sqrt{4e(X) - \frac{4}{3}c(k-1)} = \sqrt{\frac{8}{9}} \cdot \sqrt{c(k-1)(2k-\frac{4}{3})}$, because $4e(X) - \frac{4}{3}c(k-1) = 2ck(k-1) - \frac{4}{3}c(k-1) = c(k-1)(2k-\frac{4}{3})$.

That is, $\frac{4}{3}\sqrt{c(k-1)} \leq \frac{4}{3}\sqrt{k-\frac{2}{3}}$. Since $c \leq 1$. □

Hence, Theorem 1 holds under Case I by the following:

$$\begin{aligned}
e(A) &= e(X) + \xi \sum_{y \in Y} d_A(y) \\
&\le \frac{1}{4}\left(4e(X) - \Delta + \sqrt{\frac{8}{9}(4e(X) - \Delta)}\right) + \xi \sum_{y \in Y} d_A(y) \\
&\le \frac{1}{4}\left(4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y)\right. \\
&\qquad \left. + \sqrt{\frac{8}{9}(4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y))}\right) \\
&\le \frac{1}{4}\left(e(G) + \sqrt{\frac{8}{9}e(G)}\right),
\end{aligned}$$

since

$$e(G) \ge 4e(A) - \Delta = 4e(X) - \Delta + 4\xi \sum_{y \in Y} d_A(y).$$

□

Case II: $\Delta > \frac{4}{3}c(k-1)$

Notice $e(A) = e(X) + e(Y) + e(X, Y) = e(X) + \xi \sum_{y \in Y} d_A(y)$, and $e(G) = 2e(A) + e(A, B) - \Delta = 2e(X) + 2\xi \sum_{y \in Y} d_A(y) + e(X, B) + e(A \backslash X, B) - \Delta$, hence

$$\frac{e(A)}{e(G)} = \frac{e(X) + \xi \sum_{y \in Y} d_A(y)}{2e(X) + e(X, B) - \Delta + 2\xi \sum_{y \in Y} d_A(y) + e(A \backslash X, B)}.$$

Lemma 5.

$$e(X) \le \frac{1}{4}\left(2e(X) + e(X, B) - \Delta + \sqrt{\frac{8}{9}(2e(X) + e(X, B) - \Delta)}\right).$$

Proof. Since $e(X, B) > \frac{3}{4}\Delta k$, and $e(X) = \frac{ck(k-1)}{2}$, then $2e(X) + e(X, B) - \Delta > ck(k-1) + \frac{3}{4}\Delta k - \Delta = ck(k-1) + \Delta(\frac{3}{4}k - 1)$. Since $k \ge 2$, $\frac{3}{4}k - 1 > 0$, as $\Delta > \frac{4}{3}c(k-1)$, $2e(X) + e(X, B) - \Delta > ck(k-1) + \frac{4}{3}c(k-1)(\frac{3}{4}k - 1) = c(k-1)(2k - \frac{4}{3})$. Hence it is sufficient to show $e(X) = \frac{ck(k-1)}{2} \le \frac{1}{4}\big(c(k-1)(2k - \frac{4}{3}) + \sqrt{\frac{8}{9}c(k-1)(2k - \frac{4}{3})}\big)$.

We need $\frac{4}{3}c(k-1) \le \sqrt{\frac{8}{9}c(k-1)(2k-\frac{4}{3})}$; that is $\sqrt{c(k-1)}\cdot\frac{4}{3} \le \sqrt{\frac{8}{9}\cdot 2\cdot}$ $\sqrt{k-\frac{2}{3}}$, i.e., $\sqrt{c(k-1)} \le \sqrt{k-\frac{2}{3}}$, since $c \le 1$ the last inequality holds.

Since $e(A\backslash X, B) \ge e(Y,B) = \sum_{y\in Y} d_B(y) > 3\sum_{y\in Y} d_A(y)$ by Lemma 2,

$$\frac{\xi \sum_{y\in Y} d_A(y)}{2\xi \sum_{y\in Y} d_A(y) + e(A\backslash X, B)} < \frac{\xi \sum_{y\in Y} d_A(y)}{2\xi \sum_{y\in Y} d_A(y) + 3\sum_{y\in Y} d_A(y)} \le \frac{1}{5} < \frac{1}{4}.$$

Hence,

$$\begin{aligned}
e(A) &= e(X) + \xi \sum_{y\in Y} d_A(y) \\
&< \frac{1}{4}\left\{2e(X) + e(X,B) - \Delta + \sqrt{\frac{8}{9}(2e(X) + e(X,B) - \Delta)}\right\} \\
&+ \frac{1}{4}\left\{2\xi \sum_{y\in Y} d_A(y) + e(A\backslash X, B)\right\} \\
&\le \frac{1}{4}\left\{\left(2e(X) + e(X,B) - \Delta + 2\xi \sum_{y\in Y} d_A(y) + e(A\backslash X, B)\right)\right. \\
&+ \left.\sqrt{\frac{8}{9}(2e(X) + e(X,B) - \Delta + 2\xi \sum_{y\in Y} d_A(y) + e(A\backslash X, B))}\right\} \\
&= \frac{1}{4}\left(e(G) + \sqrt{\frac{8}{9}e(G)}\right).
\end{aligned}$$

Combining Case I and Case II we have Theorem 1, i.e., since

$$\gamma_2(G) \le \gamma(A,B) = e(A),\ \frac{\gamma_2(G)}{e(G)} \le \frac{e(A)}{e(G)} \le \frac{1}{4}\left(1 + \sqrt{\frac{8}{9e(G)}}\right).$$

□

Note 1: Comparing with [6], this paper changes the definition of X, Y from

$$X = \left\{x \in A \mid d_B(x) > \frac{1}{2}\Delta\right\},$$

$$Y = \left\{y \in A \mid d_A(y) \neq 0,\ d_B(y) \leq \frac{1}{2}\Delta\right\} \text{ to}$$

$$X = \left\{x \in A \mid d_B(x) > \frac{3}{4}\Delta\right\},$$

$$Y = \left\{y \in A \mid d_A(y) \neq 0,\ d_B(y) \leq \frac{3}{4}\Delta\right\}.$$

Then we sharpen the bound of $\frac{\gamma_2(G)}{e(G)}$ from $\frac{1}{4}\left(1+\sqrt{\frac{2}{e(G)}}\right)$ to $\frac{1}{4}\left(1+\sqrt{\frac{8}{9e(G)}}\right)$, which then gives us the following inductive argument that produces the asymptotically best general result.

3. Part II

Given a graph G, take a partition $(U_1, U_2, \ldots, U_s)$ of $V(G)$ into s classes with $\bigcup U_i = V(G)$, $U_i \cap U_j = \emptyset$ for $i \neq j$. Let $\gamma_s(U_1, \ldots, U_s) = \max\{e(U_1), e(U_2), \ldots, e(U_s)\}$, where $e(U_i)$ denotes the number of edges in the induced subgraph $G[U_i]$, and $\gamma_s(G) = \min\limits_{(U_1,\ldots,U_s)} \gamma_s(U_1, \ldots, U_s)$.

Theorem 2. *If $s = 2^p$, then $\gamma_s(G) \leq \frac{e(G)}{s^2} + \frac{k}{s}\sqrt{e(G)}$, where $k = \frac{4}{9}\left(\sqrt{5} + \frac{\sqrt{2}}{2}\right) \cong 1.3081$.*

If $p = 1$, then $s = 2$. From Theorem 1, $\gamma_2(G) \leq \frac{1}{4}\left(e(G) + \sqrt{\frac{8}{9}e(G)}\right) = \frac{1}{4}e(G) + \frac{\sqrt{2}}{6}\sqrt{3(G)}$. As $\frac{\sqrt{2}}{6} < \frac{k}{2}\left(k = \frac{4}{9}\left(\sqrt{5} + \frac{\sqrt{2}}{2}\right)\right)$, $\gamma_2(G) \leq \frac{e(G)}{s^2} + \frac{k}{s}\sqrt{e(G)}$ for $s = 2$.

Now we consider $p \geq 2$, then $s \geq 4$. We consider the following two cases.

Case 1: $e(G) < \frac{s^2}{2}$.

Lemma 6. *If $(U_1, U_2, \ldots, U_s)$ is a max s-cut of G, then* $\max\{e(U_1), \ldots, e(U_s)\} = 0$.

Proof. The proof is by contradiction, i.e., assume $e(U_1) = \max\{e(U_1, e(U_2), \ldots, e(U_s)\} > 0$. Suppose $ab \in e(U_1)$ where $a, b \in V(U_1)$. As $(U_1, U_2, \ldots, U_s)$ is a max s-cut, $\forall U_i,\ U_j \in \{U_1, \ldots, U_s\}$ $(i \neq j)$, we have $d_{U_j}(x) \geq d_{U_i}(x)$,

$\forall x \in U_i$ (please see the proof in Lemma 1). Thus, $e(U_i, U_j) = \sum_{x \in U_i} d_{U_j}(x) \geq \sum_{x \in U_i} d_{U_i}(x) = 2e(U_i)$. Similarly, $e(U_i, U_j) \geq 2e(U_j)$. So $\sum_{i=2}^{s} e(U_1, U_i) \geq 2(s-1)e(U_1) \geq 2(s-1)$. We first show $\forall i, j \neq 1$, $i \neq j$, $e(U_i, U_j) \geq 1$:

Contrarily, assume there exists such $i, j \neq 1$, $i \neq j$, with $e(U_i, U_j) = 0$. Then let $U_i' = U_i \cup U_j$, $U_j' = a$, $U_1' = U_1 - a$, $U_k' = U_k$ ($k \neq 1, i, j$). Thus $(U_1', \ldots, U_s')$ is another s-cut. But, obviously, $\sum_{i \neq j} e(U_i', U_j') \geq \sum_{i \neq j} e(U_i, U_j) + 1$ (because at least ab is a new edge in the cut (U_i', U_j')) and $M_s(U_1', \ldots, U_s') > M_s(U_1, \ldots, U_s)$, contradicting that $(U_1, \ldots, U_s)$ is a max s-cut. Hence $\forall i, j \neq 1$, $i \neq j$, $e(U_i, U_j) \geq 1$.

Thus,

$$\begin{aligned}
e(G) &= \sum_{i=1}^{s} e(U_i) + \sum_{i \neq j} e(U_i, U_j) \\
&\geq e(U_1) + \sum_{i=2}^{s} e(U_1, U_i) + \sum_{i,j \neq 1, i \neq j} e(U_i, U_j) \\
&\geq 1 + 2(s-1) + \binom{s-1}{2} \\
&= 1 + 2s - 2 + \frac{s^2 - 3s + 2}{2} \\
&= \frac{s^2 + s}{2} > \frac{s^2}{2}.
\end{aligned}$$

But, since $e(G) < \frac{s^2}{2}$, we have our contradiction.

Therefore, $\max\{e(U_1), e(U_2), \ldots, e(U_s)\} = 0$.

So, $\gamma_s(U_1, \ldots, U_s) = 0$ if $(U_1, U_2, \ldots, U_s)$ is a max s-cut. Thus, $\gamma_s(G) = 0$ because $\gamma_s(G) = \min_{(U_1, \ldots, U_s)} \gamma_s(U_1, \ldots, U_s)$, then, of course, $\gamma_s(G) \leq \frac{e(G)}{s^2} + \frac{k}{s}\sqrt{e(G)}$. □

Case 2: $e(G) \geq \frac{s^2}{2}$.

The proof is by induction on p. From the former proof we have $\gamma_s(G) \leq \frac{e(G)}{s^2} + \frac{k}{2}\sqrt{e(G)}$ for $p = 1$. Suppose when $s = 2^{p-1}$, $\gamma_s(G) \leq \frac{e(G)}{s^2} + \frac{k}{s}\sqrt{e(G)}$ holds. Now we consider $s = 2^p$.

Let $(U_1, \ldots, U_{2^{p-1}})$ be a 2^{p-1}-partition, which satisfies:

$$\gamma_{2^{p-1}}(U_1, \ldots, U_{2^{p-1}}) \leq \frac{e(G)}{2^{2p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)}.$$

For each U_i, let (A_i, A_i') be a bipartition satisfying

$$\gamma_2(A_i, A_i') \le \frac{1}{4}\left(e(U_i) + \sqrt{\frac{8}{9}e(U_i)}\right) \quad \text{(by Theorem 1).}$$

Thus, $(A_1, A_1', \ldots, A_{2^{p-1}}, A_{2^{p-1}}')$ is a 2^p-partition. Then,

$$\begin{aligned}
\gamma_{2^p}(G) &\le \gamma_{2^p}(A_1, A_1', \ldots, A_{2^{p-1}}, A_{2^{p-1}}') = \max_{1\le i\le 2^{p-1}} \gamma_2(A_i, A_i') \\
&\le \max_{1\le i\le 2^{p-1}} \left\{\frac{1}{4}\left(e(U_i) + \sqrt{\frac{8}{9}e(U_i)}\right)\right\} \\
&\le \frac{1}{4}\left\{\frac{e(G)}{2^{2p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)}\right. \\
&\qquad \left. + \sqrt{\frac{8}{9}\left(\frac{e(G)}{2^{p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)}\right)}\right\}.
\end{aligned}$$

Hence, it is sufficient to prove

$$\frac{1}{4}\left\{\frac{e(G)}{2^{2p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)} + \sqrt{\frac{8}{9}\left(\frac{e(G)}{2^{p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)}\right)}\right\} \le \frac{e(G)}{2^{2p}} + \frac{k}{2^p}\sqrt{e(G)}.$$

That is, $\sqrt{\frac{8}{9}\left(\frac{e(G)}{2^{2p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)}\right)} \le \frac{k}{2^{p-1}}\sqrt{e(G)}$.

That is, $\frac{e(G)}{2^{2p-2}} + \frac{k}{2^{p-1}}\sqrt{e(G)} \le \frac{9}{8}\cdot\frac{k^2}{2^{2p-2}}\cdot e(G)$; that is, $\frac{k}{2^{p-1}} \le (\frac{9}{8}k^2 - 1)\cdot \frac{1}{2^{2p-2}}\cdot\sqrt{e(G)}$.

As $e(G) \ge \frac{s^2}{2} = \frac{2^{2p}}{2}$, $\sqrt{e(G)} \ge \frac{2^p}{\sqrt{2}}$, it is sufficient to show $\frac{k}{2^{p-1}} \le (\frac{9}{8}k^2 - 1)\cdot\frac{1}{2^{2p-2}}\cdot\frac{2^p}{\sqrt{2}}$.

That is, $k \le \sqrt{2}(\frac{9}{8}k^2 - 1)$, i.e., $k \ge \frac{4}{9}(\sqrt{5} + \frac{1}{\sqrt{2}})$. As $k = \frac{4}{9}(\sqrt{5} + \frac{1}{\sqrt{2}})$, $k \le \sqrt{2}(\frac{9}{8}k^2 - 1)$. So $\gamma_s(G) \le \frac{e(G)}{s^2} + \frac{k}{s}\sqrt{e(G)}$.

Therefore, for any G, when $s = 2^p$, $\gamma_s(G) \le \frac{e(G)}{s^2} + \frac{k}{s}\sqrt{e(G)}$, $k = \frac{4}{9}(\sqrt{5} + \frac{\sqrt{2}}{2})$.

Note 2: Let $G = K_{sn+1}$, then $\gamma_s(G) \cong \frac{e(G)}{s^2} + \frac{1}{\sqrt{2}s}\cdot\sqrt{e(G)}$.

Hence, our result, $\gamma_s(G) = \frac{e(G)}{s^2} + O\left(\frac{\sqrt{e(G)}}{s}\right)$, is the best possible.

Conclusions

We summarize the known results. For any graph G, there is a partition $(U_1, \ldots, U_s)$ of $V(G)$ so that $e(U_i) \le \frac{e(G)}{s^2} + R$, $1 \le i \le s$.

For $s = 2^p$:

Theorem 3. $R = O\left(\frac{\sqrt{e(G)}}{s}\right)$.

For general s:

(Bollobás, Scott [1]) $R = \min\{(\Delta e(G) \log s)^{\frac{1}{2}},\ (4e(G))^{\frac{4}{5}} (\log s)^{\frac{2}{5}}\}$ where Δ denotes the largest degree in G.

[8]: $R = 4s\sqrt{e(G)}$.

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