On the Number of Multiplicative Partitions of a Multi-partite Number

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Abstract

A *t-partite* number is a t-tuple $\vec{n} = (n_1, \ldots, n_t)$, where $n_1, \ldots, n_t$ are positive integers. For a t-partite number $\vec{n}$, let $f_t(\vec{n})$ be the number of different ways to write $\vec{n}$ as a product of t-partite numbers, where the multiplication is performed coordinate-wise, $(1, 1, \ldots, 1)$ is not used as a factor of $\vec{n}$, and two factorizations are considered the same if they differ only in the order of the factors. This paper gives the following explicit upper bound for the multiplicative partition function $f_t(\vec{n})$: $f_t(n_1, \ldots, n_t) \leq M^{w(t)}$, where $M = \Pi_{i=1}^{t} n_i$ and $w(t) = \frac{\log((t+1)!)}{t \log 2}$.

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1 Introduction

A *t-partite* number is an ordered t-tuple $(n_1, \ldots, n_t)$, where each n_i is a positive integer. A *multiplicative partition* of the t-partite number $\vec{n} \neq (1, 1, \ldots, 1)$ is a representation of $\vec{n}$ as a product of t-partite numbers

$$(a_{1,1}, \ldots, a_{1,t})(a_{2,1}, \ldots, a_{2,t}) \ldots (a_{r,1}, \ldots, a_{r,t}),$$

where $n_i = \Pi_{j=1}^{r} a_{j,i}$ for each i, $1 \leq i \leq t$ (i.e., the multiplication is performed coordinate-wise), and $(1, 1, \ldots, 1)$ is not used as a factor. Let $f_t(n_1, \ldots, n_t)$ denote the number of different multiplicative partitions of $(n_1, \ldots, n_t)$. Two multiplicative partitions are considered the same if they differ only in the order of the factors. Thus, $(2,2)(2,1)(1,2)$ and $(1,2)(2,2)(2,1)$ are considered to be the same multiplicative partitions of $(4,4)$, while $(2,1)(2,1)(1,4)$ and $(1,2)(1,2)(4,1)$ are considered different. For example, $f_3(4,3,2) = 11$, since the eleven multiplicative partitions of $(4,3,2)$ are:

$$\begin{aligned}(4,3,2) &= (4,3,1)(1,1,2) = (4,1,2)(1,3,1) = (4,1,1)(1,3,2) \\ &= (4,1,1)(1,3,1)(1,1,2) = (2,3,2)(2,1,1) = (2,3,1)(2,1,2) \\ &= (2,3,1)(2,1,1)(1,1,2) = (2,1,2)(2,1,1)(1,3,1) \\ &= (2,1,1)(2,1,1)(1,3,2) = (2,1,1)(2,1,1)(1,3,1)(1,1,2).\end{aligned}$$

It is clear that for all $n > 1$ and $t > 1$, $f_1(n) = f_t(n, 1, 1, \ldots, 1)$ and, more generally, that if $r < t$ then $f_t(n_1, \ldots, n_r, 1, 1, \ldots, 1) = f_r(n_1, \ldots, n_r)$.

The multiplicative partition function f_1 was introduced by MacMahon [9]. Hughes and Shallit [7] proved that $f_1(n) \leq 2n^{\sqrt{n}}$ for all n. Dodd and Mattics [4] improved the bound to $f_1(n) \leq n$; and later they lowered it further, showing that

$$f_1(n) \leq \frac{n}{\log n}$$

for all $n > 1$, $n \neq 144$ (see [5]). Asymptotic results involving $f_1(n)$ were studied by Oppenheim [10] and Canfield, Erdős, and Pomerance [3].

Landman and Greenwell [8] generalized the notion of multiplicative partitions to bipartite numbers and proved

$$f_2(m,n) \leq \frac{(mn)^{1.516}}{\log(mn)}.$$

This was improved by Hahn and Kim [6] who showed that

$$f_2(m,n) \leq (2160)^2(mn)^{1.143}.$$

In this paper we obtain a general upper bound for the multiplicative partition function $f_t(\vec{n})$. We prove that

$$f_t(n_1,\ldots,n_t) \leq M^{w(t)},$$

where $M = \Pi_{i=1}^{t} n_i$ and $w(t) = \frac{\log((t+1)!)}{t\log 2}$ (note that for $t=1$ this reduces to the result of Dodd and Mattics that $f_1(n) \leq n$). We also show that the function $w(t)$ is essentially the best possible such that $M^{w(t)}$ is an upper bound.

Throughout the paper we use the following notation. Let N denote the set of all positive integers, and let $p_1 = 2$, $p_2 = 3$, ... be the sequence of primes. If $\vec{r} = \{r_i\}_{i=1}^{\infty}$ is a nondecreasing sequence of real numbers with $r_i > 1$, the arithmetic function F is defined by

$$F(\vec{r};n) = \frac{1}{2}\prod_{j=1}^{n}\frac{r_j}{r_j-1} + \sqrt{(\frac{1}{2}\prod_{j=1}^{n}\frac{r_j}{r_j-1})^2+1}$$

for each nonnegative integer n. For a sequence of real numbers $\vec{r} = \{r_i\}_{i=1}^{\infty}$, the completely multiplicative function $h(\vec{r},\cdot)$, whose domain is N, is defined as follows:

$$h(\vec{r},1) = 1;\ h(\vec{r},p_i) = r_i;\ h(\vec{r},ab) = h(\vec{r},a)h(\vec{r},b) \text{ for all } a,b \in N.$$

Let $B_n = f_1(p_1p_2\ldots p_n)$ be the n^{th} Bell number, that is, the number of partitions of an n-element set (see, for example, [2], page 277).

2 Bounds on the Number of Multiplicative Partitions

To get an upper bound for $f_t(\vec{n})$, we first estimate, in the next theorem, the upper bound for $f_1(m)$. We will need the following two lemmas.

Lemma 1 *Let p be a prime and assume $p^2|s$. Let a be the greatest integer such that $p^a|s$. Then*

$$\sum_{d|\frac{s}{p}} f_1\left(\frac{s}{dp}\right) - \sum_{d|\frac{s}{p^2}, d>1} f_1\left(\frac{s}{dp^2}\right) = f\left(\frac{s}{p^2}\right) + \sum_{d|\frac{s}{p^a}} f\left(\frac{s}{dp}\right).$$

Proof. Note that if $d|\frac{s}{p}$, but $d \not| \frac{s}{p^a}$, then $d = rp$ where $r|\frac{s}{p^2}$. Thus, for such a d, $f_1\left(\frac{s}{dp}\right) = f_1\left(\frac{s}{rp^2}\right)$. Thus

$$\sum_{d|\frac{s}{p}} f_1\left(\frac{s}{dp}\right) = \sum_{d|\frac{s}{p^2}} f_1\left(\frac{s}{dp^2}\right) + \sum_{d|\frac{s}{p^a}} f_1\left(\frac{s}{dp}\right). \tag{1}$$

Substituting the identity

$$\sum_{d|\frac{s}{p^2}} f_1\left(\frac{s}{dp^2}\right) = \sum_{d|\frac{s}{p^2}, d>1} f_1\left(\frac{s}{dp^2}\right) + f_1\left(\frac{s}{p^2}\right)$$

into (1) yields the result. ∎

Lemma 2 *Let $a \geq 2$ be a positive integer, and let $\vec{r} = \{r_i\}$ be a sequence of real numbers with $r_i > 1$. Let $n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$ and $y = \prod_i^m \frac{r_i}{r_i - 1}$. Then*

$$\sum_{d|n} h(\vec{r}; \frac{np_{m+1}^{a-1}}{d}) \leq h(\vec{r}; np_{m+1}^a) \frac{y}{r_{m+1}}.$$

Proof. Since

$$h(\vec{r}; \frac{np_m^{a-1}}{d}) = \frac{h(\vec{r}; np_{m+1}^a)}{r_{m+1} h(\vec{r}; d)},$$

it suffices to show that $\sum_{d|n} \frac{1}{h(\vec{r};d)} \leq y$. Now,

$$\sum_{d|n} \frac{1}{h(\vec{r}; d)} = \prod_{i=1}^m (1 + \frac{1}{r_i} + \frac{1}{r_i^2} + \dots + \frac{1}{r_i^{a_i}}) = \prod_{i=1}^m \frac{r_i - r_i^{-a_i}}{r_i - 1} \leq y,$$

as desired. ∎

In [2], Dodd and Mattics use the fact that, for p a prime not dividing n, $f_1(np) = \sum_{d|n} f_1(d)$. We will make use of this same fact in the proof of the following theorem.

Theorem 3 *Let $\vec{r} = \{r_i\}_{i=1}^{\infty}$ satisfy*

$$r_{i+1} \geq F(\vec{r}; i) = \frac{y_i}{2} + \sqrt{(\frac{y_i}{2})^2 + 1} \tag{2}$$

for all $i \geq 0$, where $y_k = \prod_{j=1}^{k} \frac{r_j}{r_j - 1}$ for $k \geq 0$. Then $f_1(x) \leq h(\vec{r}, x)$ for each positive integer x.

Proof. The desired inequality obviously holds for $x = 1$. Let $n = \prod_{i=1}^{m} p_i^{a_i}$ and $n' = np_{m+1}^{a}$. By inductive assumption, we may say $f_1(x) \leq h(\vec{r}; x)$ for all x, $1 \leq x < n'$. We may assume $a \geq 2$ since

$$\begin{aligned} f_1(np_{m+1}) &= \sum_{d|n} f_1(d) \leq \sum_{d|n} h(\vec{r}; d) = \prod_{i=1}^{m}(1 + r_i + \cdots + r_i^{a_i}) = h(\vec{r}; n)y_m \\ &< h(\vec{r}; n)r_{m+1} = h(\vec{r}; np_{m+1}). \end{aligned}$$

For x a positive integer, denote by $M(x)$ the number of multiplicative partitions of x which have p_{m+1} as a part, and let $N(x)$ denote the number of multiplicative partitions of x which do not have p_{m+1} as a part. We then have

$$\begin{aligned} f_1(n') &= M(n') + N(n') \\ &\leq M(n') + \sum_{d|\frac{n'}{p_{m+1}}, d>1} N(\frac{n'}{dp_{m+1}}) \\ &= f_1(\frac{n'}{p_{m+1}}) + \left[\sum_{d|\frac{n'}{p_{m+1}}, d>1} f_1(\frac{n'}{dp_{m+1}}) - \sum_{d|\frac{n'}{p_{m+1}}, d>1} M(\frac{n'}{dp_{m+1}}) \right] \\ &\leq \sum_{d|\frac{n'}{p_{m+1}}} f_1(\frac{n'}{dp_{m+1}}) - \sum_{d|\frac{n'}{p^2_{m+1}}, d>1} M(\frac{n'}{dp_{m+1}}) \end{aligned}$$

$$= \sum_{d|\frac{n'}{p_{m+1}}} f_1(\frac{n'}{dp_{m+1}}) - \sum_{d|\frac{n'}{p^2_{m+1}}, d>1} f_1(\frac{n'}{dp^2_{m+1}}).$$

Hence, by Lemma 1,

$$f_1(n') \leq f_1(\frac{n'}{p^2_{m+1}}) + \sum_{d|\frac{n'}{p^a_{m+1}}} f_1(\frac{n'}{dp_{m+1}}). \quad (3)$$

By (3) and the induction hypotheses, we have

$$f_1(n') \leq \frac{h(\vec{r}; n')}{r^2_{m+1}} + \sum_{d|n} h(\vec{r}; \frac{np^{a-1}_{m+1}}{d}). \quad (4)$$

Note that (2) implies that $1 + r_m y_m \leq r^2_{m+1}$. Thus, by (4) and Lemma 2, we have

$$f_1(n') \leq h(\vec{r}; n')(\frac{1}{r^2_{m+1}} + \frac{y_m}{r_{m+1}}) \leq h(\vec{r}; n'),$$

completing the proof. ∎

We now present three examples which result from Theorem 3. These will be used later in the paper.

Example 1. Let $r_1 = r_2 = r_3 = 2.88 < 24^{\frac{1}{3}}$, $r_{i+1} = \frac{\lfloor 10F(\vec{r};i) \rfloor + 1}{10}$ for $3 \leq i \leq 15$, where $\lfloor\ \rfloor$ is the floor function, and $r_i = i + 2$ for $i \geq 17$. Then since $F(\vec{r}; i) \leq r_{i+1} \leq i + 2$ for $i \leq 15$ and, for $i \geq 16$,

$$\begin{aligned} F(\vec{r}; i) &= \frac{i+2}{2 \cdot 18} \prod_{j=1}^{16} \frac{r_j}{r_j - 1} + \sqrt{\left(\frac{i+2}{2 \cdot 18} \prod_{j=1}^{16} \frac{r_j}{r_j - 1}\right)^2 + 1} \\ &< \frac{i+1}{2} \frac{17.99}{18} + \sqrt{\left(\frac{i+1}{2} \frac{17.99}{18}\right)^2 + 1} \\ &< i + 2 = r_{i+1}, \end{aligned}$$

we have $f_1(n) \leq h(\vec{r}, n)$.

Example 2. We have that $f_1(n) \leq n$, since

$$F(\{j+1\}_{j=1}^{\infty}; i) = \frac{i+1}{2} + \sqrt{\frac{i+1}{2}} < i + 2 \leq p_{i+1} \text{ for } i \geq 0.$$

Example 3. Let $r_1 = \sqrt{5.5}$, $r_2 = \sqrt{5.5}$, $r_3 = 3.34$, $r_4 = 4.56$, and $r_i = i + 1.6$ for $i \geq 5$. Then we notice that $f_1(n) \leq h(\vec{r}, n)$, since $F(\vec{r}; i) < r_{i+1}$ for $i \leq 4$ and

$$\begin{aligned} F(\vec{r}; i) &= \frac{5.5 \cdot 3.34 \cdot 4.56 \cdot (i+1.6)}{2(\sqrt{5.5}-1)^2 \cdot 2.34 \cdot 3.56 \cdot 5.6} + \sqrt{\left(\frac{174515 \cdot (i+1.6)}{194676(\sqrt{5.5}-1)^2}\right)^2 + 1} \\ &< \frac{i+1.6}{2} + \sqrt{\left(\frac{i+1.6}{2}\right)^2 + 1} \\ &< i + 2.6 = r_{i+1} \end{aligned}$$

for $i \geq 5$.

The next three lemmas show how to obtain upper bounds for f_t from the upper bounds for f_1.

Lemma 4 *For each $i = 1, \ldots, t$, let $\{q_{i,1}, \ldots, q_{i,m_i}\}$ be a set of distinct primes. For each i, let*

$$x_i = \prod_{j=1}^{m_i} q_{i,j}^{a_{i,j}}, \quad s_i = \sum_{k=1}^{i} m_k, \text{ and}$$

$$y = \left(\prod_{j=1}^{m_1} p_j^{a_{1,j}}\right)\left(\prod_{j=1}^{m_2} p_{j+s_1}^{a_{2,j}}\right)\left(\prod_{j=1}^{m_3} p_{j+s_2}^{a_{3,j}}\right)\cdots\left(\prod_{j=1}^{m_t} p_{j+s_{t-1}}^{a_{t,j}}\right),$$

where all of the $a_{i,j}$'s are positive integers. Then $f_t(x_1, \ldots, x_t) = f_1(y)$.

Proof. With each factorization of y:

$$y = (p_1^{c_{1,1}} p_2^{c_{1,2}} \cdots p_{s_t}^{c_{1,s_t}})(p_1^{c_{2,1}} p_2^{c_{2,2}} \cdots p_{s_t}^{c_{2,s_t}}) \cdots (p_1^{c_{r,1}} p_2^{c_{r,2}} \cdots p_{s_t}^{c_{r,s_t}}),$$

we associate the following factorization of $(x_1, \ldots, x_t)$:

$$(x_1, \ldots, x_t) = (b_{1,1}, \ldots, b_{1,t}) \cdots (b_{r,1}, \ldots, b_{r,t}),$$

where $b_{i,1} = q_{1,1}^{c_{i,1}} \cdots q_{1,m_1}^{c_{i,m_1}}$ and, for $2 \leq j \leq t$, $b_{i,j} = \prod_{k=1}^{m_j} q_{j,k}^{c_{i,s_{j-1}+k}}$. This association gives a one-to-one correspondence between the set of multiplicative partitions of y and those of $(x_1, \ldots, x_t)$. ∎

As an example, according to Lemma 4,

$$f_4(6,4,2,2) = f_1(2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) = f_2(60,6) = f_5(4,2,2,2,2).$$

Lemma 5 *If $f_1(m) \le h(\vec{r}, m)$ for all positive integers m where $\vec{r} = \{r_i\}_{i=1}^{\infty}$ is a non-decreasing sequence of positive real numbers, then for all t-partite numbers $(n_1, \ldots, n_t)$,*

$$f_t(n_1, \ldots, n_t) \le \prod_{i=1}^{t} h(\vec{s}, n_i), \text{ where } s_j = \prod_{i=(j-1)t+1}^{jt} r_i^{1/t} \text{ for } j \ge 1.$$

Proof. By Lemma 4 we may assume that for each $i = 1, 2, \ldots, t$, $n_i = \prod_{i=1}^{k_i} p_j^{a_{i,j}}$, where $a_{i,1} > 0$ and $\{a_{i,j}\}_{j=1}^{k_i}$ is a non-increasing sequence of positive integers. Let $k = \max_{1 \le i \le t}\{k_i\}$, and let

$$c_{i,j} = \begin{cases} a_{i,j} & \text{if } 1 \le j \le k_i \\ 0 & \text{if } k_i < j \le k. \end{cases}$$

For fixed j, let $\{d_{i,j}\}_{1 \le i \le t}$ be a rearrangement of $\{c_{i,j}\}_{1 \le i \le t}$, where $\{d_{i,j}\}_{1 \le i \le t}$ has non-increasing order. By Lemma 4,

$$\begin{aligned} f_t(n_1, \ldots, n_t) &= f_1((p_1^{c_{1,1}} \cdots p_t^{c_{t,1}})(p_{t+1}^{c_{1,2}} \cdots p_{2t}^{c_{t,2}}) \cdots (p_{(k-1)t+1}^{c_{1,k}} \cdots p_{kt}^{c_{t,k}})) \\ &= f_1((p_1^{d_{1,1}} \cdots p_t^{d_{t,1}})(p_{t+1}^{d_{1,2}} \cdots p_{2t}^{d_{t,2}}) \cdots (p_{(k-1)t+1}^{d_{1,k}} \cdots p_{kt}^{d_{t,k}})) \\ &\le (r_1^{d_{1,1}} \cdots r_t^{d_{t,1}})(r_{t+1}^{d_{1,2}} \cdots r_{2t}^{d_{t,2}}) \cdots (r_{(k-1)t+1}^{d_{1,k}} \cdots r_{kt}^{d_{t,k}}) \\ &\le (s_1^{d_{1,1}} \cdots s_1^{d_{t,1}})(s_2^{d_{1,2}} \cdots s_2^{d_{t,2}}) \cdots (s_k^{d_{1,k}} \cdots s_k^{d_{t,k}}) \\ &\le (s_1^{c_{1,1}+\cdots+c_{t,1}})(s_2^{c_{1,2}+\cdots+c_{t,2}}) \cdots (s_k^{c_{1,k}+\cdots+c_{t,k}}) \\ &= \prod_{i=1}^{t} h(\vec{s}, n_i). \end{aligned}$$

■

Lemma 6 *If $f_1(m) \le h(\vec{r}, m)$ for all positive integers m, where $\vec{r} = \{r_i\}_{i=1}^{\infty}$ is a non-decreasing sequence of positive real numbers, and α is a real number such that $\prod_{j=1}^{ti} r_j \le (\prod_{j=1}^{i} p_j)^{t\alpha}$ for $i \ge 1$, then $f_t(n_1, \ldots, n_t) \le M^{\alpha}$, where $M = \prod_{i=1}^{t} n_i$.*

Proof. Let $m = \prod_{i=1}^{k} p_i^{a_i}$, where $\{a_i\}_{i=1}^{k}$ is a non-increasing sequence of positive integers. Then

$$h(\vec{s}, m) = \prod_{i=1}^{k} s_i^{a_i} \leq \left(\prod_{i=1}^{k} \left(\prod_{j=1}^{i} p_j \right)^{b_i} \right)^{\alpha} = m^{\alpha},$$

where $s_j = \prod_{i=(j-1)t+1}^{jt} r_i^{1/t}$ for $j \geq 1$, $b_k = a_k$, and $b_i = a_i - a_{i+1}$ for $1 \leq i \leq k-1$. The lemma follows from Lemmas 4 and 5. ∎

From the above lemma, we have

Proposition 7 $f_2(m,n) \leq (mn)^{1.235}$.

Proof. Let r_i be defined as in Example (3). Since $r_{2i-1}r_{2i} < (2i+1.6)^2 \leq p_i^2$ for $i \geq 7$ and $\prod_{j=1}^{2i} r_j \leq (\prod_{j=1}^{i} p_j^2)^{1.235}$ for $i \leq 6$, we see that

$$\prod_{j=1}^{2i} r_j \leq (\prod_{j=1}^{i} p_j^2)^{1.235} \text{ for } i \geq 1.$$

The desired conclusion follows from Lemma 6. ∎

We need the following well-known formula (see, for example, [7, page 200]) to prove the next lemma.

Theorem 8 (Stirling's formula)

$$\frac{7}{8} < \log(n!) - (n + \frac{1}{2}) \log n + n < 1$$

for $n \in N$, $n \geq 2$.

From now on, we use $w(t)$ to denote $\frac{\log(t+1)!}{t \log 2}$.

Lemma 9 *Let* $t \geq 11$ *be an integer. Then*

$$w(t) > \frac{\log(2t+1)! - \log(t+1)!}{t \log 3}. \tag{5}$$

That is, $3^{w(t)} > \prod_{i=t+1}^{2t} (i+1)^{1/t}$.

Proof. It is enough to show that

$$\frac{\log 6}{\log 2}\log(t+1)! - \log(2t+1)! > 0.$$

By direct calculation, one can see that (5) holds for all t, $11 \le t \le 42$. By Theorem 8, we have

$$\begin{aligned}
&\tfrac{\log 6}{\log 2}\log(t+1)! - \log(2t+1)! \\
&> 2.5\,[(t+3/2)\log(t+1) - (t+1) + 7/8] - [(2t+3/2)\log(2t+1) - (2t+1) + 1] \\
&> t\left[\tfrac{\log(t+1)-1}{2} - 2\log 2\right] + (\tfrac{15}{4} - \tfrac{3}{2})\log(t+1) - \tfrac{35}{16} - \tfrac{3\log 2}{2} \\
&> 42\left[\tfrac{\log 44-1}{2} - 2\log 2\right] + (\tfrac{15}{4} - \tfrac{3}{2})\log 44 - \tfrac{35}{16} - \tfrac{3\log 2}{2} \\
&> 0
\end{aligned}$$

for $t \ge 43$. ∎

Lemma 10 *Let t and l be positive integers. Then we have*

$$w(t) > \frac{\log(lt+2t+2)! - \log(lt+t+2)!}{t\log(2l+3)} \quad \textit{for } t \ge 3.$$

That is, $(2l+3)^{w(t)} > \prod_{i=lt+t+1}^{lt+2t}(i+2)^{1/t}$ *for* $t \ge 3$.

Proof. Fix $t \ge 3$. Let $H(x) = \frac{\log(xt+2t+2)}{\log(2x+3)}$ for $x \ge 1$. Then $H(x)$ is a decreasing function of x since

$$\log(2x+3)H'(x) = \frac{t}{xt+2t+2} - \frac{2}{2x+3}H(x) < \frac{t}{xt+2t+2} - \frac{2}{2x+3} < 0.$$

Therefore we have

$$w(t) - \frac{\sum_{i=1}^{t}\log(xt+t+2+i)}{t\log(2x+3)} > w(t) - H(x) \ge w(t) - H(1) = w(t) - \frac{\log(3t+2)}{\log 5}.$$

Let $G(t) = w(t) - \frac{\log(3t+2)}{\log 5}$. It is enough to show $G(t) > 0$ for $t \ge 3$. For $t \ge 56$,

$$\begin{aligned}
G(t) &= w(t) - \frac{\log(3t+2)}{\log 5} \\
&> \frac{\log(t+1)-1}{\log 2} - \frac{\log(3t+2)}{\log 5} \\
&> \log(t+1)\left[\frac{1}{\log 2} - \frac{1}{\log 5}\right] - \frac{1}{\log 2} - \frac{3}{\log 5} \\
&> 0.
\end{aligned}$$

Moreover, by direct calculation, one can prove $G(t) > 0$ for t, $3 \le t \le 55$. Hence the lemma is proved. ∎

Theorem 11 *Let $n_1, n_2, \ldots, n_t$ be positive integers. Then*

$$f_t(n_1, \ldots, n_t) \le M^{w(t)}, \tag{6}$$

where $M = \prod_{i=1}^{t} n_i$.

Proof. From Example (2) and Proposition 7, we may assume $t \ge 3$. First, let $t \ge 11$ and $r_i = i + 1$ for $i \ge 1$. From Lemmas 9 and 10 we have

$$\prod_{i=1}^{t} r_i^{1/t} = 2^{w(t)} \text{ and } \prod_{lt+1}^{(l+1)t} r_i^{1/t} \le (2l+1)^{w(t)} \le p_{l+1}^{w(t)} \text{ for all } l \ge 1.$$

By Lemma 6 and Example (2), inequality (6) holds for $t \ge 11$.

Now fix t, $3 \le t \le 10$. Let $\{r_i\}$ be the same sequence defined in Example (1). By direct computation, one can show

$$\prod_{i=1}^{t} r_i \le 2^{tw(t)} \text{ and } \prod_{i=1}^{2t} r_i \le 6^{tw(t)}.$$

By Lemma 10, we have

$$\prod_{i=1}^{lt} r_i \le 6^{tw(t)} \prod_{i=2t+1}^{lt} r_i \le 6^{tw(t)} \prod_{i=2t+1}^{lt} (i+2) \le 6^{tw(t)} \prod_{j=3}^{l} (2j-1)^{tw(t)} \le \prod_{i=1}^{l} p_i^{tw(t)}$$

for $l \ge 3$. Hence, by Lemma 6, inequality (6) holds for t, $3 \le t \le 10$. The theorem is proved. ∎

In the next theorem we show that the bound given by Theorem 11 cannot be significantly improved. We need the following lemma. Recall that B_n denotes the nth Bell number.

Lemma 12 *Let c be any real number such that $0 < c < 1$. Then*

$$\lim_{n \to \infty} \frac{(n+1)!^c}{B_n} = 0.$$

Proof. For a fixed c, $0 < c < 1$, we can choose k such that $c < \frac{k-2}{k+1} < 1$. Then for any positive integer t, with $t \le k$,

$$B_{kn+t} = \sum_{i=0}^{kn+t-1} \binom{kn+t-1}{i} B_i > \binom{kn+t-1}{k-1} B_{kn-k+t} > n^{k-1} B_{kn-k+t}$$

$$\ge (n!)^{k-1} B_t,$$

since the m^{th} Bell number can be expressed as

$$B_m = \sum_{i=0}^{m} \binom{m}{i} B_i$$

(see [1], Chapter 13).

Hence we have

$$\frac{(kn+t)!^c}{B_{kn+t}} < \frac{[k^{(k+1)n}(n+1)!^k]^c}{(n!)^{k-1}} < \frac{[k^n(n+1)!]^{k-2}}{(n!)^{k-1}} < \frac{k^{nk}(n+1)^{k-2}}{n!} \to 0$$

as $n \to \infty$.

∎

Theorem 13 *Let $u(t) \le w(t)$ for all positive integers t, and assume*

$$f_t(n_1, \ldots, n_t) \le M^{u(t)},$$

for all t-partite numbers $(n_1, \ldots, n_t)$, where $M = \prod_{i=1}^{t} n_i$. Then

$$\lim_{t\to\infty} \frac{u(t)}{w(t)} = 1.$$

Proof. From the fact that $B_t = f_t(2, 2, \ldots, 2) \le 2^{tu(t)}$, we have

$$1 \ge \frac{u(t)}{w(t)} \ge \frac{\log B_t}{t \log 2} \frac{t \log 2}{\log(t+1)!} = \frac{\log B_t}{\log(t+1)!}.$$

By Lemma 12, $\lim_{t\to\infty} \frac{\log B_t}{\log(t+1)!} \ge 1$, and the theorem follows. ∎

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