There are 270,474,142 Nonisomorphic 2-(9, 4, 6) Designs

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Abstract

We enumerate the 2-(9, 4, 6) designs and find 270,474,142 nonisomorphic such designs in a backtrack search. The sizes of their automorphism groups vary between 1 and 360. Out of these designs, 19,489,464 are simple and 2,148,676 are decomposable.

1. Introduction

A t-(v, k, λ) design is a family of k-subsets, called blocks, out of a v-set of points, such that each t-subset of the v-set is contained in exactly λ blocks. A design with $t = 2$ is called a *balanced incomplete block design* (BIBD). The number of blocks of a design, denoted by b, and the number of blocks in which any point occurs, denoted by r, can be determined from the values of the other parameters:

$$vr = bk, r(k-1) = \lambda(v-1).$$

A 2-(9, 4, 6) design can be constructed easily by taking a 2-(9, 4, 3) design twice. It is much more difficult to enumerate all nonisomorphic 2-(9, 4, 6) designs. Two designs are said to be isomorphic if there is a permutation of the points and a permutation of the blocks that map one design onto the other. If there are such permutations that map a design onto itself, then all permutations of the points constitute a group, the (full) automorphism group of the design.

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We shall now briefly discuss our enumeration of the nonisomorphic 2-(9, 4, 6) designs.

2. The Search and Results

Orderly algorithms [7] have the property that they enumerate all nonisomorphic structures without having to test these pairwise for isomorphism. To enumerate the 2-(9, 4, 6) designs, we use such an orderly algorithm to obtain all nonisomorphic incidence matrices of the designs. An incidence matrix is a (0,1)-matrix of size $v \times b$ with the rows indexed by the points and the columns by the blocks, and where the ones express incidence.

Algorithms for enumerating designs have been considered in several studies in the literature; published results include [1, 2, 3, 6, 8]. We use the common approach of constructing the incidence matrices row-by-row and only considering one representative from each isomorphism class of partial incidence matrices on each level of the search tree. For example, on level 1 of the search tree there is only one isomorphism class consisting of all $1 \times b$ matrices with r ones and $b - r$ zeros.

The matrices must fulfill the following requirements: each row must contain r ones, each column must contain at most k ones, and every pair of rows must have exactly λ ones in common (that is, have inner product λ). To reject isomorphs, we require that every partial incidence matrix to be considered further must be in canonical form (cf. [1, 7, 8]).

The labelling of a partial incidence matrix is defined to be the binary number obtained by concatenating the rows of the matrix. A partial incidence matrix is in *canonical form* if every permutation of the rows and the columns gives a smaller labelling (by normal comparison of binary numbers) than the present one. The algorithm used to check whether a partial incidence matrix is in canonical form is taken from [1].

In appending new rows to the partial incidence matrix, we make use of the observation that for the leftmost 1, we may restrict its placement to a single position: the first possible column (that has less than k ones) [1].

By using orderly algorithms, the enumeration is not restricted by the memory size—unless one wants to save all the designs. With the amount of CPU time available being the only restriction, enumerations of hundreds of millions and even billions of nonisomorphic designs are possible today.

Using the above mentioned approach, it took about three months of CPU time on 233–500 MHz PC computers to enumerate the 2-(9, 4, 6) designs; there are 270,474,142 nonisomorphic designs, 19,489,464 of which are simple. This settles an open question for design number 150 in [4]; previously, it had been shown that there are at least 1.25×10^8 nonisomorphic designs. The search was completed in about one week by distributing it

Group size	Nd	Ns	Nc
1	270340858	19477172	2137845
2	128851	11234	10355
3	1673	697	32
4	2239	220	331
6	254	95	21
8	174	15	59
9	9	6	3
10	1	1	0
12	27	7	4
16	26	5	15
18	4	1	1
24	8	3	0
27	1	1	0
32	6	0	5
40	1	1	0
54	1	1	0
60	1	1	0
64	3	1	2
120	1	1	0
128	1	0	1
144	2	1	2
360	1	1	0
Total	270474142	19489464	2148676

Table 1: Properties of 2-$(9, 4, 6)$ designs.

over a network of computers using the program *autoson* [5].

For each design encountered, we registered the size of the automorphism group and checked whether it is simple or not. We also counted the number of 2-$(9, 4, 6)$ designs that are decomposable into two 2-$(9, 4, 3)$ designs. This was done by combining the 11 nonisomorphic 2-$(9, 4, 3)$ designs in all possible ways, where the automorphism groups of these designs can be used to reduce the number of combinations to be considered. This data of the designs is presented in Table 1, where Nd is the number of designs, Ns is the number of simple designs, and Nc is the number of decomposable designs.

Finally, in Figure 1, we give the canonical incidence matrix of the unique 2-$(9, 4, 6)$ design whose automorphism group has the largest size, 360. As can be seen, it is formed around three copies of the 2-$(6, 3, 2)$ design.

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111111111111111100000000000000000000
111111000000000011111111110000000000
111111000000000000000000001111111111
100000111110000011111000001111100000
010000110001110011000111001100011100
001000101001001110100100111010010011
000100010100101101010010110101001011
000010001010111000101011100010101110
000001000111010100011101010001110101
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Figure 1: The 2-(9, 4, 6) design with the largest automorphism group.

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