Graphs where Star sets are matched to their complements

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Abstract

Consider those graphs G of size $2n$ that have an eigenvalue λ of multiplicity n and where the edges between the star set and its complement is a matching. We show that λ must be either 0 or 1 and completely characterize the corresponding graphs.

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1 Introduction

Let G be a finite simple graph with an eigenvalue λ of multiplicity m. A subgraph X of G is a **star set** for λ if $|V(X)| = m$ and λ is not an eigenvalue for $\bar{X} = G - X$. The subgraph $\bar{X}$ is called the **star complement** for λ. See [3] for the basic properties of star sets where they are called star cells.

A fruitful approach, taken in [4, 5, 6, 7], has been to fix a graph $\bar{X}$ and an eigenvalue λ and determine the graphs which have $\bar{X}$ as a star complement for λ. It was shown in [5] that for each $\bar{X}$ and $\lambda \notin \{-1, 0\}$ there are only finitely many such graphs. Indeed, if $\lambda \notin \{-1, 0\}$ then $\bar{X}$ is a *location dominating* set for X, that is every vertex of X is adjacent to a unique, non-empty subset of $\bar{X}$. If $|V(\bar{X})| = t$ then $|V(X)| < 2^t$. This is strengthened in [6] to $|V(X)| \leq (t-1)(t+4)/2$.

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The central structural result for star complements is:

Theorem 1 **([3] Theorems 7.4.1 and 7.4.4)** *Let G be a graph with $(0,1)$-adjacency matrix* $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$*, where A has size $m \times m$ and λ is not an eigenvalue of C. Then λ is an eigenvalue of G of multiplicity m if and only if*

$$\lambda I - A = B^T(\lambda I - C)^{-1}B.$$

The approach taken in this paper is to fix the edges between the star set and its complement, i.e. fix the matrix B, and determine the corresponding eigenvalues and graphs. To this end, let $\mathcal{F}$ be a family of $0-1$ matrices. Let $E[\mathcal{F}, \lambda]$ be the set of *connected* graphs with the properties that:

1. The adjacency matrix of $G = X \cup \bar{X}$ has the form

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

   where A is the adjacency matrix of X and C that of $\bar{X}$;
2. B is a member of $\mathcal{F}$;
3. G has the eigenvalue λ with multiplicity $|V(X)|$;
4. λ is not an eigenvalue of $\bar{X}$.

Note that the connectivity constraint is no real restriction since if two graphs both have eigenvalue λ of multiplicity k then so does their disjoint union.

In this paper, we take $\mathcal{F} = \{I_n : n = 1, 2, \dots ,\}$, that is the set of identity matrices. If $G \in E[\mathcal{F}, \lambda]$, then the edges between X and $\bar{X}$ form a matching. This matching induces a correspondence between the vertices of X and $\bar{X}$.

Our first result is Theorem 2.

Theorem 2 *The set $E[\{I_n | n = 1, 2, \dots ,\}, \lambda]$ is non-empty only if $\lambda \in \{-1, 0, 1\}$.*

Moreover, for each n we can completely determine the graphs contained in $E[\{I_n\}, \lambda]$.

Theorem 3 *A graph G is in $E[\{I_n\}, \lambda]$ if and only if one of the following holds:*

1. *$\lambda = -1$, $n = 1$ and G is isomorphic to K_2 where both X and $\bar{X}$ are singletons;*

2. *$\lambda = 0$, $n = 2$ and G is isomorphic to C_4 where both X and $\bar{X}$ are isomorphic to K_2;*

3. *$\lambda = 1$ and either $n = 1$ and G is isomorphic to K_2 where both X and $\bar{X}$ are singletons,*

   or $n = 5$ and G is isomorphic to the Petersen graph where both X and $\bar{X}$ are isomorphic to C_5.

These last two results are in contrast to those found in [1] where $\mathcal{F} = \{J_n - I_n | n = 1, 2, \ldots, \}$. It was shown there that for all $n > 0$, each of the sets $E[\{J_n - I_n, \}, 0]$, $E[\{J_{2n} - I_{2n}\}, -1]$, and $E[\{J_{5n} - I_{5n}\}, -2]$ is non-empty. Also, it is shown that λ is an integer, $-1 - \sqrt{8+n}/2 \leq \lambda \leq \sqrt{8+n}/2$ and the cases where at least one of X and $\bar{X}$ is regular is completely solved.

2 Basic Equations and Proof of Theorem 2

We are taking X and $\bar{X}$ to be of size n and $B = I_n$ so that $B^T = B$. The equation in Theorem 1 can be re-written as

$$(A - \lambda I_n)(C - \lambda I_n) = I_n.$$

Let a_{ik} and c_{ik} be the (i,k)-th entries of A and C respectively. For any $1 \leq i, k \leq n$, considering the (i,k)th entry in this matrix equation gives:

$$\delta_i^k = \sum_j a_{ij} c_{jk} - \lambda(a_{ik} + c_{ik}) + \delta_i^k \lambda^2 \tag{1}$$

where $\delta_i^k = 1$ if $i = k$ and $\delta_i^k = 0$ otherwise.

If $i \neq k$ then this equation reduces to

$$\sum_j a_{ij} c_{jk} = \lambda(a_{ik} + c_{ik}). \tag{2}$$

If neither X nor $\bar{X}$ contain an edge then both are singletons with the eigenvalue 0, while G is isomorphic to K_2 and the corresponding eigenvalue for G can be $\lambda = -1$ and also $\lambda = 1$.

We may now assume that there is some edge (i, k) in X or $\bar{X}$. From (2) it follows that λ is a non-negative number that is either an integer or an integer plus 1/2. Also equation (1) reduces to

$$\lambda^2 = 1 - \sum_j a_{ij} c_{ji} \tag{3}$$

from which it follows that $\lambda \in \{0, 1\}$ and so Theorem 2 is proved.

3 Proof of Theorem 3

Part (1), $\lambda = -1$: In this case, we already know from the proof of Theorem 3 that both X and $\bar{X}$ contain no edges, and so are singletons by connectedness, and that $G = K_2 \in E[\{I_1\}, -1]$.

Part (2), $\lambda = 0$: Let both X and $\bar{X}$ be isomorphic to K_2 and let G be isomorphic to C_4. The eigenvalues of G are 0, 2, and -2 with multiplicities 2, 1 and 1 respectively. The eigenvalues of $\bar{X}$ are 1 and -1 both with multiplicity 1 so that $G \in E[\{I_2\}, 0]$.

Suppose now that $G \in E[\{I_n | n = 1, 2, \dots\}, 0]$ and that G is connected. Then from the equations (2) and (3) we obtain for all $i \neq k$

$$\sum_j a_{ij} c_{jk} = 0 \tag{4}$$

and for $i = k$

$$\sum_j a_{ij} c_{ji} = 1. \tag{5}$$

Equation (5) implies that for every i there is exactly one vertex j where i is adjacent to j in both X and $\bar{X}$. Equation (4) tells us i and j are not adjacent to any other vertex of either X or $\bar{X}$. The matching given by B only connects i and j and since G is connected it follows that X and $\bar{X}$ are both isomorphic to K_2 and that G is isomorphic C_4.

Part (3), $\lambda = 1$: From the proof of Theorem 3 we already know that $G = K_2 \in E[\{I_1\}, 1]$.

Let both X and $\bar{X}$ be isomorphic to C_5 and G isomorphic to the Petersen graph. See Figure 1 in which the dashed edges form the matching corresponding to the matrix B. The eigenvalues of $\bar{X}$ are 0, $(-1 - \sqrt{5})/2$ and $(-1 + \sqrt{5})/2$ with multiplicities 1, 2 and 2 respectively, and those of G

are -1, 1, 3 and -2 with multiplicities 1, 5, 1 and 4 respectively. Thus $G \in E[\{I_5\}, 1]$.

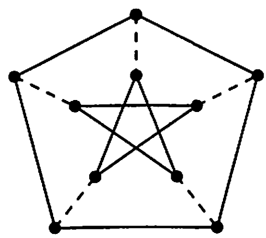

Figure 1: Petersen Graph

Suppose now that $G \in E[\{I_n | n = 1, 2, \dots, \}, 1]$ and that G is connected. Equations (2) and (3) become:

for $i \neq k$

$$\sum_j a_{ij} c_{jk} = (a_{ik} + c_{ik}) \tag{6}$$

and for $i = k$ then equation (1) reduces to

$$\sum_j a_{ij} c_{ji} = 0. \tag{7}$$

Since equations (6) and (7) are symmetric in A and C and because of the matching between X and $\bar{X}$ we can translate the problem to one of colouring the edges of the complete graph K_n in the following fashion. Let $|V(G)| = 2n$ then $|V(X)| = |V(\bar{X})| = n$. Consider now the complete graph K_n where vertex i corresponds to both vertices i of X and $\bar{X}$. An edge of K_n is coloured green if it is in neither $E(X)$ nor $E(\bar{X})$; it is coloured red in if it is $E(X)$; it is coloured blue if it is in $E(\bar{X})$. Note that the roles of red and blue could be interchanged because of the symmetry of equations (6) and (7). Because we are only interested in G being a connected graph then it follows that between any two vertices of K_n there is a path which contains no green edges.

Clearly, no edge can be assigned the colour green and another colour. If an edge is assigned both red and blue colours then the left-hand side of (7) is at least 1 which is impossible. Hence, an edge can be assigned only one colour.

Note that this also implies that the right-hand side of equation (6) is either 0 or 1.

We use $(i,j,k) = XY$ to indicate that the edge (i,j) is coloured X and that (j,k) is coloured Y where $X, Y \in \{R, B, G\}$, and R is for red, B for blue and G for green.

Claim 1: If (i,k) is not a green edge then there exactly two vertices j and l such that $(i,j,k) = RB$ and $(i,l,k) = BR$.

Proof of Claim 1: If (i,k) is not a green edge then $a_{ik} + c_{ki} = 1$, so from (6), $\sum_p a_{ip}c_{pk} = 1$ and therefore, there is exactly one j for which $a_{ij} = c_{jk} = 1$. Hence, $(i,j,k) = RB$.

Since the matrices are symmetric, we also have $a_{ki} + c_{ik} = 1$, so $\sum_p a_{kp}c_{pi} = 1$ and therefore there is exactly one l for which $a_{kl} = c_{li} = 1$. Hence, $(i,l,k) = BR$.

Claim 2: If $(i,j,k) = RB$ or $(i,j,k) = BR$ then (i,k) is coloured blue or red.

Proof of Claim 2: If $(i,j,k) = BR$ or $(i,j,k) = RB$ then from equation (6) $a_{ik} + c_{ik} = \sum_p a_{ip}c_{pk}$. This sum includes the term $a_{ij}c_{jk} = 1$ and, since an edge cannot be assigned both red and blue, then $a_{ik} + c_{ik} = 1$. Therefore either $a_{ij} = 1$ or $c_{ij} = 1$ but not both.

We are now ready to prove Theorem 3. Let $V(G) = \{1, 2, 3, \ldots, n\}$.

If $n = 1$ then there are no edges to colour and so equations (5) and (6) are automatically satisfied. The graph G is then a K_2 with X and $\bar{X}$ both being singletons.

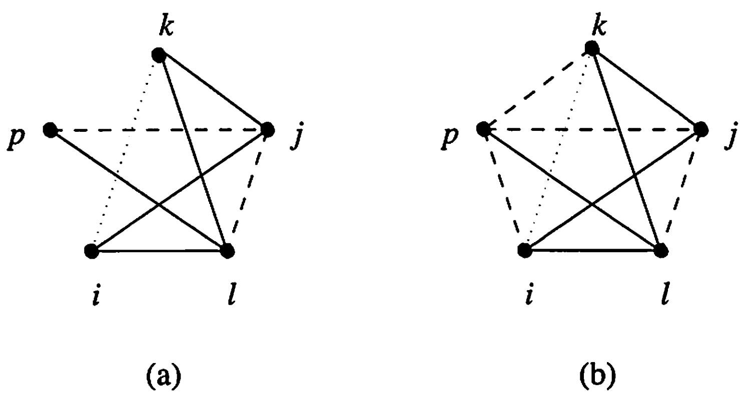

Figure 2: The Edge Colourings for $n \geq 5$.

If $n > 1$ then without loss of generality, let $(1,2)$ be red. Then by Claim 1 there are vertices 3 and 4 such that $(1,3,2) = BR$ and $(1,4,2) = RB$ and thus $n \geq 4$. Now if there is a vertex i, $i \geq 5$ then either $(1, i, 2) = RR$ or $(1, i, 2) = BB$ because of Claim 1.

Suppose now that $n \geq 5$. We first show that there are no green edges. Suppose that there is a green edge. To each green edge associate a shortest path that connects the endpoints of the edge, but that contains no green edges. Since G is connected such a path exists. Let (i, k) be a green edge with the shortest associated path and call one such path P. For $j \in V(P) - \{i, k\}$, if (i, j) is green then the edge (i, j) together with the subpath of P from i to j would have been chosen rather than (i, k). If (i, j) is not green then there would be a path shorter than P with no green edges connecting i and k unless $V(P) = \{i, j, k\}$. Now, by Claim 2 (i, j) and (j, k) have the same colour. We assume that $(i, j, k) = RR$. The case $(i, j, k) = BB$ is similar and is ommitted. By Claim 1 there is a vertex l with $(i, l, j) = RB$. Since $(k, j, l) = RB$ then (k, l) is not green by Claim 2 and it must be red because (i, l, k) cannot be RB by Claim 2 again. From Claim 1, with (l, j), then there is a vertex p with $(l, p, j) = RB$ and thus p is distinct from both i and k. (See Figure 2a.) The edge (i, p) cannot be green by Claim 1 since $(i, j, p) = RB$. If (i, p) is red then $(i, p, j) = RB = (i, l, j)$ contrary to Claim 2. Thus (i, p) is blue. Also, since (i, k) is green, Claim 2 gives that (p, k) is blue. But now $(j, i, p) = (j, k, p) = RB$, see Figure 2b, which contradicts Claim 1. Therefore, there can be no green edges.

Recall that $(1, 2) = R$, $(1, 3, 2) = BR$ and $(1, 4, 2) = RB$.

If $(3, 4) = R$, then by Claim 1 applied to $(1, 3)$ there is a vertex 5 such that $(1, 5, 3) = RB$ and since $(2, 4, 1) = BR$ then $(2, 5)$ must be coloured red. But now $(213) = (253) = RB$ contradicting Claim 1, see Figure 3a.

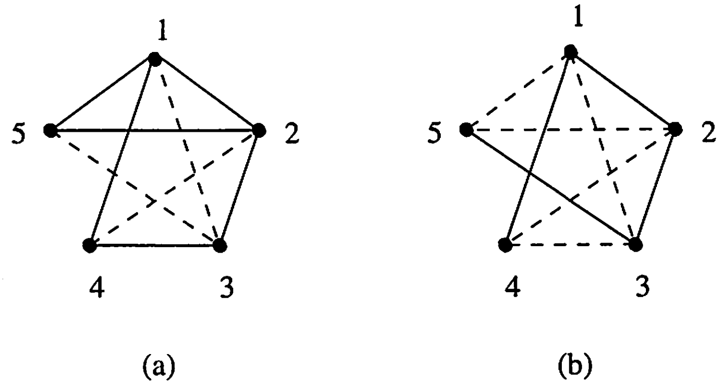

Figure 3: More Edge Colourings for $n \geq 5$.

Thus $(3, 4) = B$. Again, by Claim 1, there must be a vertex 5 such that

$(1,5,3) = BR$ and also $(2,5) = B$, see Figure 3b. If $(4,5) = B$ then $(3,5,4) = (3,2,4) = RB$ contradicting Claim 1 therefore $(4,5) = R$.

We have just shown that if $n = 5$ then X and $\bar{X}$ are 5-cycles with G being the Petersen graph.

For $n > 5$ we may assume that the edges between the vertices $\{1,2,3,4,5\}$ are coloured as in Figure 3a (see Figure 4). Recall that $(1,i,2) = XX$ for all $i \geq 5$. Consider vertex 6. If $(1,6) = (2,6) = R$ then $(5,1,6) = (5,2,6) = BR$ which contradicts Claim 1. Thus $(1,6) = (2,6) = B$. Now $(2,1,6) = RB$ and Claim 1 forces $(2,3,6) = RR$ but now $(3,6,1) = (3,5,1) = RB$ which contradicts Claim 1.

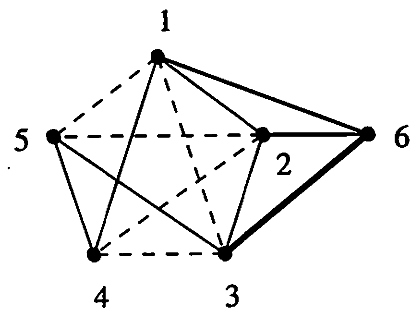

Figure 4: The Edge Colourings for $n = 6$.

Therefore it follows that $n = 1$ or $n = 5$ and the proof is complete.

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