AN AUTOMORPHISM-FREE 4-(15,5,5) DESIGN

G.B. KHOSROVSHAHI*
Department of Mathematics, University of Tehran, and
Institute for Studies in Theoretical Physics and Mathematics (IPM),
Tehran, Iran
rezagbk@karun.ipm.ac.ir

H.R. MAIMANI
Institute for Studies in Theoretical Physics and Mathematics (IPM), and
Department of Mathematics, Shahid Rajaee University,
Tehran, Iran

R. TORABI
Department of Mathematics, University of Tehran, and
Institute for Studies in Theoretical Physics and Mathematics (IPM),
Tehran, Iran

Abstract
Employing trading signed design algorithm, we construct an automorphism-free 4-(15, 5, 5) design.

1. Introduction

The family of 4-$(15, 5, \lambda)$ designs offers some challenging problems to design theorists. A summary of information on the family is as follows:

- for $\lambda = 1$, the design does not exist [7];
- for $\lambda = 2$, the existence is unknown;
- for $\lambda = 3$, there are designs as the derivatives of 5-(16, 6, 3) design constructed by Brouwer [2]. Also, Brouwer, using a group of order 42, has constructed a 4-(15, 5, 3) design directly [2].

*Author for correspondence; Mailing address: IPM, P.O. Box 19395-5746, Tehran, Iran.

- for $\lambda = 4$, employing a group of size 60, Brouwer again has constructed a design [2].
- for $\lambda = 5$, there is a design as a derivative of 5-(16, 6, 5) by Brouwer [2], and another one by Kreher, using a group of order 39 [5].

In this paper after introducing some notations and a brief description of the underlying algorithm, we present a design and demonstrate that it is in fact automorphism-free.

2. Definitions and Preliminaries

Suppose that t, k, v are positive integers such that $v > k > t$, and λ is a nonnegative integer. Let X be a v-set. The set of all i-subsets of X is denoted by $P_i(X)$. The elements of $P_k(X)$ are called blocks. We impose some ordering on $P_t(X)$ and $P_k(X)$. Let $P_{tk}^v = [p_{AB}]$ be a $(0,1)$ matrix of size $\binom{v}{t} \times \binom{v}{k}$ whose rows and columns are indexed by the elements of $P_t(X)$ and $P_k(X)$, respectively, and for every $A \in P_t(X)$ and $B \in P_k(X)$, p_{AB} is defined as follows:

$$p_{AB} = \begin{cases} 1 & A \subset B, \\ 0 & \textit{otherwise}. \end{cases}$$

Now, we consider the following system of nonhomogeneous linear equations:

$$P_{tk}^v F = \lambda e, \qquad (*)$$

where $e = (1, 1, \ldots, 1)^t$. The solutions of $(*)$ have different names:

- every integral solution of $(*)$, F, is called a t-(v, k, λ) ***signed design***;
- every nonnegative integral solution of $(*)$, F, is called a t-(v, k, λ) ***design***; and
- every t-$(v, k, 0)$ signed design is called a t-(v, k) ***trade***.

We note that the necessary and sufficient conditions for the existence of a t-(v, k, λ) signed design is $\lambda \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}$, for $i = 0, \ldots, t-1$, to be integers.

From the definition of t-(v, k, λ) signed design F, and t-(v, k) trade T, it is clear that $F + T$ is again a signed design. This is the essence of a method called "trading signed design" algorithm. For more on this algorithm one can consult [3,4].

Let $B \in P_k(X)$ be a block of a t-(v,k,λ) design. Then, for $0 \le i \le k$, let x_i be the number of blocks which intersect B in i points. The following relations among x_i's are known as the block intersection equations [1]:

$$\sum_{i=j}^{k} \binom{i}{j} x_i = (\lambda_j - 1)\binom{k}{j}, \qquad 0 \le j \le t.$$

In the case at hand, namely 4-(15,5,5) design, the intersection equations have a unique solution:

$$(x_0, x_1, \cdots, x_5) = (114, 480, 540, 210, 20, 0).$$

Since the elements of $P_k(x)$ are ordered, every component of F corresponds to the frequency of a block, and hence every t-(v,k,λ) design could be considered as an incidence structure $(X, \mathcal{B})$ in which $\mathcal{B}$ is a collection of blocks with nonzero frequencies.

A mappings φ between two designs $D = (X, \mathcal{B})$ and $D' = (X', \mathcal{B}')$ is an isomorphism if $\varphi : X \to X'$ is a one-to-one correspondence and $\varphi(\mathcal{B}) = \mathcal{B}'$. Every isomorphism of a design D to itself is called an automorphism and the set of all the automorphisms of a design with the natural composition rule among mappings form the automorphism group of the design, and is denoted by Aut(D). Let f be an automorphism of a design $D = (X, B)$, then fix$(f) = \{x \in X | f(x) = x\}$.

Let $D = (X, \mathcal{B})$ be a t-(v,k,λ) design, and $x \in X$. If $\mathcal{B}^{(x)} = \{B \backslash \{x\} | x \in B \in \mathcal{B}\}$, then $D_x = (X \backslash \{x\}, \mathcal{B}^{(x)})$ is $(t-1)$-$(v-1, k-1, \lambda)$ design. D_{xy} will mean $(D_x)_y$.

3. A Design

Employing the trading signed design algorithm, we have constructed a $D =$ 4-$(15,5,5)$ design. Below we represent D in hexadecimal representation (high order bit first) as a (0,1) vector of size 3004 with 1365 ones. Note that all the 3003 blocks are ordered lexicographically and a 0 has been added to the end of the vector to make it divisible by 4. Here 10-15 are represented by A-F.

60F360A56B70B2D071A30937C5625C44A85CBA3496B0BF44A38CCE4D4724
00EA1C1FA71578784C1535C49956955A41A5BA991445B520E3B89B80D9FC
017851D582798B32949E2F016CBC8DE1151066CE50B29CD5A918CD8B3232
565A49BC2E90DB023C46289E5E701A0A9F59E80AAD35C630C713252CB8DC
C885AF88953A88770539325CD105B4AA075BA0DB616A7309895B21650BD0
D3E16165525C0F81E7591DA4487F032A4D9304537321AE42F52AD593A17C
630324AD6C48D8C5B133243740CD8A20D73C5A441E51A9A576A249CF5F49
4D8718813C7410955EC4F0822EBD4441B27F6D3CC75050891C8CC38F5915
4F268E58C746036C4ED41C5142BEAEA2A2D82364C947967B24B41A0A846B
8A470597EB498CA72389A8CEC64E2B4E22C2E2C439AB128D29DB1432E445
E12FC124D1AA78ACC88139BA568E5D19A522BD0D49235C84D5C4183B8782
B832B1C71979A8164CB06ED1DA507C5A170D3175C48054D57B3CC6A63089
F88D51256C05D25ABA1F32062B2605E

4. Aut(D)

In this section, we determine the automorphism group of our design. Clearly, if $f \in$ Aut(D) is of prime order, then $o(f) \in \{2, 3, 5, 7, 11, 13\}$.

Let $f \in$ Aut(D), and $o(f) = 13$, so $|\text{fix}(f)| = 2$. Suppose that $\text{fix}(f) = \{x, y\}$. Thus D_{xy} has f as an automorphism. By Nauty [6], we have found that the automorphism groups of all the D_{xy}'s, are trivial. Therefore, $13 \not| \, |\text{Aut}(D)|$.

Now, let $o(f) = 2$, so $|\text{fix}(f)| \geq 1$. If $x \in \text{fix}(f)$, then $f \in \text{Aut}(D_x)$. Again by Nauty [6], we have found that the order of the automorphism group of all the D_x's, are not even. So $2 \not| \, |\text{Aut}(D)|$.

If $f \in$ Aut(D), and $o(f) = 3$, then $f = \sigma_1\sigma_2\cdots$, where σ_i's are cycles of lenght 3. Clearly $|\text{fix}(f)| \neq 12$. Let the points in cycle σ_1 be the set $\{x, y, z\}$. So $f\sigma_1^{-1} \in \text{Aut}(D_{xyz})$. We have $o(f\sigma^{-1}) = 1$ or 3. If $o(f\sigma^{-1}) = 1$, then $f = \sigma_1$, and $|\text{fix}(f)| = 12$. This is a contradiction. Hence $o(f\sigma^{-1}) = 3$. But for all x, y,and z, $3 \not| \, |\text{Aut}(D_{xyz})|$, and so $3 \not| \, |\text{Aut}(D)|$.

Now suppose that $f \in$ Aut(D), and $o(f) = 7$. So $|\text{fix}(f)| = 1$ or 8. First assume that $|\text{fix}(f)| = 8$. Let $x, y, z, w \in \text{fix}(f)$. So the five blocks $B_1, B_2, \cdots, B_5 \in \mathcal{B}$ which contain x, y, z, w must be fixed by f. Since $|B_1 \cup B_2 \cdots \cup B_5| = 9$, so there is an element $x \notin \text{fix}(f)$, $x \in B_i$, for some $1 \leq i \leq 5$. Therefore $|B_i| \geq 11$, and this is a contradiction. So $|\text{fix}(f)| = 1$. Thus f is in the automorphism group of the derived 4-(15, 5, 5) design, through the fixed point of f. But all the derivatives of D have trivial automorphism group. Therefore $7 \not| \ |\mathrm{Aut}(D)|$.

Now we show that for any 4-(15,5,5) design D, $5 \not| |\mathrm{Aut}(D)|$. Let $f \in \mathrm{Aut}(D)$ and $o(f) = 5$. Hence $|\mathrm{fix}(f)| = 0$, 5 or 10. To show this, first we observe that f must move all the blocks of D. Now suppose that $|\mathrm{fix}(f)| = 0$ and hence $f = \sigma_1\sigma_2\sigma_3$ where σ_i's are cycles of lenght 5. We denote the points in the cycle σ_i by point(σ_i), and for any $B \in \mathcal{B}$, $|\mathrm{point}(\sigma_i) \cap B| \leq 4$. Now from the following intersection equations, it follows that there are 115 blocks $B \in \mathcal{B}$ such that $\mathrm{point}(\sigma_i) \cap B = \varnothing$, for each $1 \leq i \leq 3$.

$$\begin{cases} n_0+n_1+n_2+n_3+n_4 = 1365, \\ n_1+2n_2+3n_3+4n_4 = 5 \times 455, \\ n_2+3n_3+6n_4 = 10 \times 130, \\ n_3+4n_4 = 10 \times 30, \\ n_4 = 5 \times 5, \end{cases}$$

where $n_i = |\{B \in \mathcal{B} | |B \cap \mathrm{point}(\sigma_3)| = i\}|$, and we obtain $n_0 = 115$. Now let m_i be the number of these blocks (out of 115 blocks) which intersect the point(σ_1) in i points. Then again from the intersection equations the following relations are easily obtained.

$$m_1 + m_2 + m_3 + m_4 = 115,$$
$$40 \leq m_2 + 3m_3 + 6m_4 \leq 60,$$
$$30 \leq m_3 + 4m_4 \leq 50.$$

It follows that $m_2 = m_3 = 0$, $m_1 = 13 \times 5$, and $m_4 = 5 \times 10$. But $m_1 + m_4$ is at most $\binom{5}{1}\binom{5}{4} + \binom{5}{4}\binom{5}{1} = 50$. The cases $|\mathrm{fix}(f)| = 5$ or 10 are similarly ruled out. Therefore, for any 4-(15,5,5) design $D, 5 \not| \mathrm{Aut}(D)$.

If $f \in \mathrm{Aut}(D)$, and $o(f) = 11$, then there is a block $B \in \mathcal{B}$ such that $f(B) = B$. So $|B| \geq 11$, and this is a contradiction.

By the above argument, our design is an automorphism-free design.

Remark. For any 4-(15,5,5) design D, $|\mathrm{Aut}(D)| = 2^\alpha 3^\beta 7^\gamma 13^\sigma$.

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