Self-Orthogonal Diagonal Latin Square with Missing Subsquare

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ABSTRACT. It is proved in this paper that for any integer $n \geq 136$, a SODLS(v, n) (self-orthogonal diagonal Latin square with missing subsquare) exists if and only if $v \geq 3n + 2$ and $v - n$ even.

1 Introduction

A *Latin square* of order n is $n \times n$ array such that every row and every column is a permutation of a n-set. A *transversal* in a Latin square is a set of positions, one per row and per column among which the symbols occur precisely once each. A transversal T is *symmetric* if $(i, j) \in T$ if and only if $(j, i) \in T$. A pair of transversals T and S are symmetric if $(i, j) \in T$ if and only if $(j, i) \in S$. A *transversal Latin square* is a Latin square whose main diagonal is a transversal. It is easy to see that the existence of a transversal Latin square is equivalent to the existence of a idempotent square. A *diagonal Latin square* is a transversal Latin square whose back diagonal also forms a transversal.

Two Latin squares of order n are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square is *self-orthogonal* if it is orthogonal to its transpose. Orthogonal (transversal) Latin squares of order v are denoted briefly by OLS(v) (OILS(v)). Self-orthogonal (diagonal) Latin sanare of order v is denoted briefly by SOLS(v) (SODLS(v)).

For the spectra of SOLS and SODLS, we have

Theorem 1.1. ([2]) *A SOLS of order v exists for all integers v, with the exception of $v \in \{2, 3, 6\}$.*

Theorem 1.2. ([8]) *A SODLS of order v exists for all integers v, with the exception of $v \in \{2, 3, 6\}$ and the possible exception of $v \in \{10, 14\}$.*

The problem we study in this paper is the self-orthogonal diagonal Latin squares analogue of the Doyen-Wilson theorem [5]. We begin with some definitions. If a self-orthogonal diagonal Latin square has a subsquare occupying the central position, the subsquare itself must be orthogonal diagonal Latin square. We refer to it as ***self-orthogonal diagonal subsquare.*** We denote by SODLS(v, n) a self-orthogonal diagonal Latin square of order v with self-orthogonal diagonal subsquare of order n. It is easy to see that the existence of a SODLS(v, n) requires that $v - n$ is even. In particular, any SODLS(v) is a SODLS(v, 1) when v is odd. In view of Theorem 1.2, no self-orthogonal diagonal Latin square can contain orthogonal diagonal subsquare of order 2, 3 or 6. However, we can construct self-orthogonal diagonal Latin square missing subsquare of these orders. We also let SODLS(v, n) denote a self-orthogonal diagonal Latin square of order v with missing subsquare of order n occupying the central position. It is not necessary for such self-orthogonal diagonal subsquare of order n to exist. We refer to the subsquare as the hole.

Some simple computation shows

Theorem 1.3. *If there exists a SODLS(v, n), then $v \geq 3n + 2$ and $v - n$ even*

SODLS with missing subsquare have been studied by several researchers. Some applications to the construction of other types of designs are as follows: orthogonal diagonal Latin squares, incomplete self-orthogonal Latin square and magic square with magic subsquare.

In this paper, we prove the necessary condition is also sufficient for $n \geq 136$.

Theorem 1.4. *For any positive integer $n \geq 136$, there exists a SODLS(v, n) if and only if $v \geq 3n + 2$ and $v - n$ even.*

In the remainder of this paper we shall assume that the reader is familiar with the concepts of incomplete orthogonal Latin squares (briefly IOLS) and incomplete self-orthogonal Latin square (briefly ISOLS), and the various methods of constructing (v, n)-ISOLS starting with a SOLS(n) (see, for example, [4,10]), and starting with a (n, k)-ISOLS (see, for example, [7,10]). We shall also assume that the reader is familiar with the various techniques of constructing SODLS(v) from SOLS(v) by permuting rows and columns (see, for example, [8,11,12]).

For the spectra of IOLS and ISOLS, we have

Theorem 1.5. ([9]) *If $v \geq 3n$ and $v \neq 6$, then there exists a (v, n)-IOLS.*

Theorem 1.6. ([10]) *If $v \geq 3n+1$ and $v \neq 6$, then there exists a (v,n)-ISOLS except possible for $(v,n) \in \{(6m+i, 2m): i = 2 \text{ or } 6\}$.*

2 Some special cases

First we state a starter-adder type construction for (v,n)-ISOLS. The main idea is to generate each square under a cyclic group of order $v-n$, from its first row and from the last n elements of the first column. Let $X = \{0,1,\ldots,v-n-1\} \cup Y$, where $Y = \{x_1, x_2, \ldots, x_n\}$. Suppose L is a square based on X with a hole indexed by Y. We shall denote by $e_L(i,j)$ the entry in the cell (i,j) of the array L. The first row is given by the vectors $\underline{e} = (e_L(0,0), \ldots, e_L(0, v-n-1))$ and $\underline{f} = (e_L(0, v-n), \ldots, e_L(0, v-1))$, and the last n elements of the first column are given by the vector $\underline{g} = (e_L(v-n, 0), \ldots, e_L(v-1, 0)$. The L is constructed modulo $v-n$ in the range $\{0,1,\ldots,v-n-1\}$, where the x_i's act as "infinity" elements as follows:

(1) $e_L(s+1, t+1) = e_L(s,t)$ if $e_L(s,t) = x_i$, and $e_L(s+1,t+1) \equiv e_L(s,t) + 1 (mod v-n)$ otherwise, where $0 \leq s,\ t < v-n-1$.

(2) $e_L(s+1, v-n-1+t) \equiv e_L(s, v-n-1+t) + 1 (mod v-n)$, where $1 \leq t \leq n$, $0 \leq s < v-n-1$.

(3) $e_L(v-n-1+t, s+1) \equiv e_L(v-n-1+t, s) + 1 (mod v-n)$, where $1 \leq t \leq n$, $0 \leq s \leq v-n-1$.

We remark that there are obviously conditions which the vectors $\underline{e}$, $\underline{f}$, $\underline{g}$ must satisfy in order to produce the (v,n)-IOILS, but we shall not concern ourselves with that, the reader may see [10].

Lemma 2.1. *Suppose there exists a (v,n)-ISOLS constructed by the starter-adder method, $v-n$ is even and the $(1+(v-n)/2)$-st element in the starter set $\underline{e}$ is not infinity element. Then there exists a SODLS(v,n).*

Proof: We begin with the (v,n)-ISOLS and permute rows and columns with permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & (v-n)/2 & (v-n)/2+1 & (v-n)/2+2 & \cdots & v-n \\ 1 & 2 & \cdots & (v-n)/2 & v-n & v-n-1 & \cdots & (v-n)/2+1 \end{pmatrix}$$

Then we obtain the required design.

From [10] we have

Lemma 2.2. *For $(v,n) \in F$, there exists a (v,n)-ISOLS constructed by the starter-adder method such that the $(1+(v-n)/2)$-st element in the starter set is not an infinity element, where*

$$F = \{(19,5), (t,2), (s,3): t = 12, 16, 20; s = 11, 13\}$$

Combining Lemmas 2.1 and 2.2 we have

Lemma 2.3. *There exists a SODLS(v, n) for $(v, n) \in F$.*

We also need some other small designs.

Lemma 2.4. *There exists a SODLS(38,12).*

Proof: We begin with (4,1)-ISOLS. From this ISOLS we fill the main diagonal with (11;3,2)-ISOLS by modifying the array constructed by Zhu in [16], which we show below, and the others with one of a (9,3)-IOLS and its transpose. We obtain the required design by filling the size 11 hole with SODLS(11,3) for which existence comes from Lemma 2.3, and permute rows and columns as Wallis and Zhu did in [12], in which the size 12 hole consists of size 3 hole in SODLS(11,3) and the central position of filling arrays in (4,1)-ISOLS.

1	7	5	9	10	2	6	11	3	4	8
4	2	8	10	3	7	1	5	11	6	9
9	6	3	1	8	10	11	2	4	5	7
11	8	6	4	1	9	10	3	5	7	2
7	4	11	8	5	3	2	6	10	9	1
5	11	9	2	7	6	4	10	1	8	3
3	10	4	5	11	1				2	6
6	1	10	11	2	4				3	5
10	5	2	3	6	11				1	4
8	9	7	6	4	5	3	1	2		
2	3	1	7	9	8	5	4	6		

(11;3,2)-ISOLS

Lemma 2.5. *There exists a SODLS(49,15).*

Proof: We begin with SODLS(5). From this SODLS, we fill the main diagonal with (11;3,2)-ISOLS and the back diagonal with modified (11;3,2)-ISOLS, that is, by permuting the first 6 columns so that the main diagonal of the upper left part in the (11;3,2)-ISOLS becomes its back diagonal, but leave the central cell. Fill all other cells with one of a (9,3)-IOLS and its transpose. We obtain the required design by filling the size 13 hole with SODLS(13,3) for which existence comes from Lemma 2.3, in which the size 15 hole consists of the central. position of filling arrays in SODLS(5).

Lemma 2.6. *There exists a SODLS(v, n) for $(v, n) \in \{(3t + k, t) \colon t = 20, 22, 30; k = 6, 10, 14\}$.*

Proof: We begin with the SODLS$(t, 2)$ with 14 disjoint common transversals including the main diagonal and the back diagonal which consist of the elements which is not in the subarray for which existence comes from [10] and the decompositions $22 = 5 \times 4 + (1 + 1)$ and $30 = 7 \times 4 + (1 + 1)$. We fill the k transversals including the main diagonal and the back diagonal with (4,1)-ISOLS or modified (4,1)-ISOLS and the others with one of an OLS(3) and its transpose. We obtain the required design by filling the size $k + 6$ hole with SODLS$(k + 6, 2)$ for which existence comes from Lemma 2.3, in which the size t hole consists of size 2 hole in SODLS$(k + 6, 2)$ and the central cells of filling 3×3 arrays in SODLS$(t, 2)$.

3 A general bound

Let $P = \{S_1, S_2, \ldots, S_n\}$ be a partition of a set S, where $n \geq 2$. A *partitioned incomplete Latin square* (or PILS) having partition P is an $|p| \times |p|$ array L, indexed by S, which satisfies the following properties:

(1) a cell of L either contains a symbol from S or is empty

(2) the subarray indexed by $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are called *holes*)

(3) the elements occurring in row (or column) s of L are precisely these in $S \backslash S_i$, where $s \in S_i$.

The *type* of L is the multiset $\{|S_1|, |S_2|, \ldots, |S_n|\}$. We use the notation $1^{u_1}2^{u_2}\ldots$ to describe a type, where there are precisely u_i occurrences of i, for $i = 1, 2, \ldots$.

Suppose L and M are PILS having the same partition P. We say that L and M are *orthogonal* if their superposition yields every ordered pair in $S^2 \backslash (\cup S_i^2)$. The term "self-orthogonal PILS" is abbreviated to SOPILS.

We shall assume that the reader is familiar with the standard terminology of group-divisible designs (GDDs) and Wilson's "Fundamental Construction" (see, for example, [14]). Of course, a GD$[k, 1, n; kn]$ is equivalent to $k - 2$ POLS(n), $k - 2$ pairwise orthogonal Latin squares of order n.

Lemma 3.1. ([3]) *If prime power $q > 3$, then there exists a SOPILS(1^q) with transversal back diagonal.*

We need the following recursive construction for SOPILS.

Lemma 3.2. ([4]) *Suppose that $(X, \mathbb{G}, \mathbb{A})$ is a GDD, w is a weighting, and let $k \geq 1$. Further, suppose that, for every block $A \in \mathbb{A}$, there are SOPILS of type $w(A)$. Then there are SOPILS of type $\{\sum_{x \in G} w(x) : G \in \mathbb{G}\}$.*

We now state the main construction.

Lemma 3.3. *If n, m and k are positive integers, m odd, $2 \leq n \leq 3m - 3$, $1 \leq k \leq 2m$ and $k \neq 3, 4$, such that there exists a GD$[10, 1, m; 10m]$.*

Then there exists a SODLS($7m+n+k, n$) for even $7m+k$. Further, for $7m+n+5 \le v \le 9m+n$ and $v-n$ even, there exists a SODLS(v, n).

Proof: In all but three groups of the GD$[10, 1, m; 10m]$, we give the points weight 1. In the third last group, we give s (s odd and $s \ge 1$) points weight 1 and give the remaining points weight 0. In the second last group, we give t points weight 1 and give the remaining points weight 0. We observe that if $s+t = k \ge 1$ and $k \ne 3, 4$, then we can choose s and t such that both SODLS(s) and SODLS(t) exist. In the last group, we give weight 0, 2 or 3, such that total weight is n. We can apply Lemma 3.2 with the necessary input designs from Theorem 1.6 in which one size 8 block input is a SOPILS(1^8) from Lemma 3.1 (or when $s+t = 2m$, one size 9 block input is an SOPILS(1^9) from Lemma 3.1), to obtain a SOPILS($m^7 s^1 t^1 n^1$). We then fill the size m holes with SODLS($m, 1$), the size s hole with SODLS($s, 1$) and the size t hole with SODLS(t), and obtain the required design by permuting rows and columns as Wallis and Zhu did in [12].

Lemma 3.4. ([1]) *There is a series of positive integers*

$$M = (m_i : i = 1, 2, 3, \dots) = (17, 19, 23, 25, 27, 29, 31, 37, 41, \dots),$$

such that $m_{i+1} - m_i \le 8$, $7m_{i+1} + 4 \le 9m_i$, and there exist GD$[10, 1, m_i; 10m_i]$ for all $i \ge 1$.

We are now in a position to prove

Theorem A. *For any positive integer $n > 48$, if $v \ge 10n/3 + 66$ and $v - n$ even, then there exists a SODLS(v, n).*

Proof: Our proof relies heavily on Lemmas 3.3 and 3.4. First of all, for any fixed $n > 48$, there exists an $i \ge 1$ such that $3m_i - 3 < n \le 3m_{i+1} - 3$. Thus we have $3m_{i+1} - n < 3(m_{i+1} - m_i) + 3 \le 27$ and $m_{i+1} \le (n + 26)/3$. Applying Lemmas 3.3 and 3.4 recursively, we know that there exist SODLS(v, n) whenever $v \ge 7m_{i+1} + n + 5$. Therefore there exist SODLS(v, n) whenever $v \ge 7(n + 26)/3 + n + 5$, that is, whenever $v \ge 10n/3 + 66$.

We also obtain

Lemma 3.5. *If v is even and $v \ge 136$, then there exists a SODLS($v, 12$).*

4 The main result

A Latin square is symmetric if it is equal to its transpose. We denote by SOLSSOM(v) a self-orthogonal Latin square of order v with a symmetric orthogonal mate, and USOLSSOM(v) a self-orthogonal Latin square of order v with a constant main diagonal symmetric orthogonal mate. It is easy to see that the existence of an USOLSSOM(v) required that v is even.

Lemma 4.1. ([6,13,15])

(1) *If n is odd and $n > 3$, then there exists a SOLSSOM(n);*

(2) *If n is even and $n \notin E$, then there exists an USOLSSOM(n), where*

$$E = \{2, 6, 10, 14, 46, 54, 58, 62, 66, 70\}$$

Lemma 4.2.

(1) *If there exists a SOLSSOM(n) and n odd, then there exists a SODLS(n) with n disjoint common symmetric transversals including the back diagonal, each posesses one position of the main diagonal;*

(2) *If there exists an USOLSSOM(n), then there exists a SODLS(n) which possesses n disjoint common transversals including the main diagonal and the back diagonal.*

Proof: Suppose C is a symmetric orthogonal mate. By applying a permutation simultaneously to the rows and columns, as Wallis did in [11], we can produce a Latin square with constant back diagonal. We do the same permutation to a self-orthogonal Latin square A and its transpose A', and obtain the required SODLS(n).

We first consider the case $v = 3n + k$, n even.

Lemma 4.3. *For even n, if there exist a SODLS(n) with k disjoint common symmetric transversals including the main diagonal and the back diagonal, then there exists a SODLS($3n + k, n$), $2 \le k \le n$, k even and $k = 2, 6, 10$ or 14.*

Proof: We begin with the SODLS(n), and fill the k disjoint common transversals with (4,1)-lSOLS and the others with one of an OLS(3) and its transpose, hut the back diagonal with modified (4,1)-ISOLS, that is, by permuting the first 3 columns so that the main diagonal of the upper left part in the (4,1)-ISOLS becomes its back diagonal. Note that there exist SODLS(k) from Theorem 1.2, we obtain the required design by permuting rows and columns, in which the size n hole consists of the central cells of filling 3×3 arrays in SODLS(n).

Combining Lemmas 4.1 - 4.3 we have

Lemma 4.4. *If n is even and $n \notin E$, then there exists a SODLS($3n+k, n$), for $2 \le k \le n$, k even and $k = 2, 6, 10$ or 14.*

Combining Lemma 3.5 and Lemma 2.4 we have

Lemma 4.5. *If n is even and $n \ge 136$, then there exists a SODLS($3n + 2, n$).*

Proof: We begin with the SODLS(n, 12) for which existence comes from Lemma 3.5. We fill the diagonals in the upper left part with (4,1)-ISOLS but the back diagonal with modified (4,1)-ISOLS, and the others with one of an OLS(3) and its transpose. We then have a SODLS($3n+2, 38$). Note that there exists a SODLS(38,12) from Lemma 2.4, so the result follows. The size n hole consists of size 12 hole in SODLS(38,12) and the central cells of filling 3×3 arrays in SODLS(n, 12).

For the cases $k = 6, 10$ and 14, we need

Lemma 4.6. *If n is even and $n \geq 108$, then there exists a SOLSSOM(k) such that $n = 4k + t$ $t \in E_1 = \{20, 22\}$, or an USOLSSOM(k) such that $n = 3k + t$, $t \in E_1 \cup \{30\}$.*

Proof: From Lemma 4.1, it is not difficult to check that the assertion is true.

Lemma 4.7.

(1) *If there exists a SOLSSOM(k), k odd, then for $t \leq k$ there exists a SODLS($4k + t, t$) with t disjoint common symmetric transversals including the main diagonal and the back diagonal which consist of the elements which are not in the subarray;*

(2) *If there exists an USOLSSOM(k), then there exist SODLS($4k + t, t$) (for $t \leq k$) and SODLS($3k + t, t$) (for $2 \leq t \leq k$) with t disjoint common symmetric transversals including the main diagonal and the back diagonal which consist of the elements which are not in the subarray.*

Proof:

(1) We begin with the SOLSSOM(k). Then we have a SODLS(k) with k disjoint common symmetric transversals including the back diagonal. From this SODLS(k), we fill t disjoint symmetric transversals including the back diagonal with (5,1)-ISOLS or modified (5,1)-ISOLS, and the others with SOLS(4). The result follows.

(2) We begin with the USOLSSOM(k), then we have a SODLS(k) with k disjoint common symmetric transversals including the main diagonal and the back diagonal. From this SODLS(k), we fill t disjoint symmetric transversals including the main diagonal and the back diagonal with (5,1)-ISOLS, (4,1)-ISOLS, or modified (5,1)-ISOLS, (4,1)-ISOLS, and the others with one of an OLS(3) and its transpose. The result follows.

Combining Lemmas 4.6 and 4.7 we have

Lemma 4.8. *If n is even and $n \geq 108$, then there exists a SODLS(n, t), $t = 20, 22$ or 30, with t disjoint common symmetric transversals including the main diagonal and the back diagonal which consist of the elements which are not in the subarray.*

Lemma 4.9. *If n is even and $n \geq 108$, then there exists a SODLS($3n+k, n$) for $k = 6, 10$ and 14..*

Proof: We begin with the SODLS(n, t) as in Lemma 4.8, and fill k disjoint common symmetric transversals including the main diagonal and the back diagonal with (4,1)-ISOLS or modified (4,1)-ISOLS, and the others with one of an OLS(3) and its transpose. Then we obtain the required desian by filling the size $3t + k$ hole with SODLS($3t + k, t$)for which existence comes from Lemma 2.6, and permuting rows and columns.

Up to now, we have obtained

Theorem B. *If n is even and $n \geq 136$, then there exists a SODLS($3n + k, n$), $2 \leq k \leq n$ and k even.*

We then consider the case $v = 3n + k$, n odd.

Lemma 4.10. *If n is odd and $n > 3$, then there exists a SODLS($3n+k, n$), $2 \leq k \leq n$, k even and $k \neq 4$.*

Proof: We begin with the SOLSSOM(n), then we have a SODLS(n) with n disjoint common symmetric transversals including the back diagonal, each possesses one position of the main diagonal. From this SODLS(n), we fill the k disjoint symmetric transversals including the back diagonal with (4,1)-ISOLS, or modified (4,1)-ISOLS, but leave the cells in the main diagonal, fill the main diagonal with (4,1)-ISOLS or (5;1,1)-ISOLS but leave the central cell. Fill all other cells with one of an OLS(3) and its transpose. Finally, fill the size 5 hole with SODLS(5) and fill the size $k - 1$ hole with SODLS($k - 1, 1$), and permute rows and columns.

For the case $k = 4$, we need the following result.

Lemma 4.11. ([3]) *If q is odd prime power and $q > 3$, then there exists a set of $q - 1$ POLS(q) which consists of a constant main diagonal square P_0, a symmetric square P_1 and $(q - 3)/2$ self-orthogonal Latin squares $P, P_2, \ldots, P_{(q-3)/2}$ and their transposes.*

Note that if $n = q_1^{\alpha} q_2^{\beta} \ldots q_k^{\gamma}$, where the q_i are distinct odd primes, and if $r = q_1^{\alpha} < q_2^{\beta} < \cdots < q_k^{\gamma}$, then there are squares $P, P_0, P_1, P_2, \ldots, P_{(r-3)/2}$ of order n. We then have

Lemma 4.12. *If $n = q_1^{\alpha} a_2^{\beta} \ldots q_k^{\gamma}$, where the q_i are distinct odd primes, and if $7 \leq q_1^{\alpha} < a_2^{\beta} < \cdots < q_k^{\gamma}$, then there is a SODLS($n$) with a pair of symmetrically placed transversals meeting in the central cell.*

Proof: Suppose P_1 is a symmetric square. By applying a permutation simultaneously to the rows and columns, we can produce a Latin square with constant back diagonal. We do the same permutation to P, P_2 and their transposes to obtain self-orthogonal diagonal Latin squares P and P_2. From the element in the central cell of P_2 and its transpose we obtain a pair of symmetrically placed transversals in P and its transpose.

Lemma 4.13. *Suppose k and m are odd and there exists a SODLS(k) with a pair of symmetrically placed transversals meeting in the central cell. Then there exists a SODLS($3km+4, km$) if a $(3m, m)$-IOLS, a $(3m+1, m)$-ISOLS and a SODLS($3m+4, m$) all exist.*

Proof: We begin with the SODLS(k) and fill the 4 transversals with $(3m+1, m)$-ISOLS or modified $(3m+1, m)$-ISOLS and let the central cell be empty, the others with one of a $(3m, m)$-IOLS and its transpose. Then we obtained the required result by filling the size $3m+4$ hole with SODLS($3m+4, m$) and permuting rows and columns.

Lemma 4.14. *If n is odd and $n \geq 15$, then there exists a SODLS($3n+4, n$).*

Proof: Write $n = 3^\alpha 5^\beta K$, where $\gcd(K, 30) = 1$. There are four cases to consider, and, in each, Lemma 4.13 is applied. If both $\alpha \neq 1$ and $\beta \neq 1$, put $k = n$ and $m = 1$; if $\alpha \neq 1$ but $\beta = 1$, put $k = n/5$ and $m = 5$; if $\alpha = 1$ but $\beta \neq 1$, put $k = n/3$ and $m = 3$; and if $\alpha = \beta = 1$, put $k = n/15$ and $m = 15$. The necessary input designs come from Theorems 1.2, 1.5 and 1,6 and Lemmas 2.3 and 2.5.

Up to now we have obtained

Theorem C. *If n is odd and $n \geq 15$, then there exists a SODLS($3n+k, n$), $2 \leq k < n$ and k even.*

Proof of Theorem 1.4: Since $10n/3 + 66 \leq 4n$ whenever $n \geq 136$, the result is an immediate consequence of Theorems A, B and C.

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