Smallest critical sets for the latin squares of orders six and seven

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ABSTRACT: A critical set in a latin square of order n is a set of entries in a latin square which can be embedded in precisely one latin square of order n. Also, if any element of the critical set is deleted, the remaining set can be embedded in more than one latin square of order n. A smallest critical set in a latin square is a critical set of minimum cardinality. In this paper we find smallest critical sets for all the latin squares of orders six and seven. We also find smallest critical sets of orders six and seven which are also weak critical sets. In particular, we find a weak critical set of size twelve for the dihedral group of order six.

1 Introduction

A *latin square* L of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs precisely once in each row and column. For convenience, a latin square will sometimes be represented as a set of ordered triples $(i, j; k)$, this is read to mean that element k occurs in cell (i, j) of the latin square L. For a latin square of order n we shall use $1, 2, 3, \ldots, n$ as the entries, and rows and columns will also be labelled from 1 to n. If a latin square L contains an $s \times s$ subarray S and if S is a

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latin square of order s, then S is said to be a *latin subsquare* of L. A *partial latin square* P of order n is an $n \times n$ array with entries chosen from a set N, of size n, such that each element of N occurs at most once in each row and column. Thus P may contain a number of empty cells. We sometimes denote P by the set $\{(i,j;k) | i,j,k \in N\}$. A *critical set* in a latin square L of order n, is a set $C = \{(i,j;k) \mid i,j,k \in N\}$, such that

(1) L is the only latin square of order n which has element k in cell (i,j) for each $(i,j;k) \in C$; and

(2) no proper subset of C satisfies (1).

A *smallest critical set* in a latin square L is a critical set of minimum cardinality.

Let $P = \{(i,j;k) \mid i,j,k \in \{1,2,\ldots,n\}\}$ be a partial latin square of order n. Then $|P|$ is said to be the *size* of the partial latin square and the set of cells $\{(i,j) \mid (i,j;k) \in P\}$ is said to determine the *shape* of P. Let P and P' be two partial latin squares of order n, with the same shape. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no cell in P contains the same entry as the corresponding cell in P'. Given two partial latin squares P and P' of order n, of the same size and shape, with the property that P and P' are disjoint and mutually balanced, then P is said to be a *latin interchange* and P' is said to be a *disjoint mate* of P. Note that, given a latin square L, any latin subsquare in L, with order greater than one is a latin interchange.

The following lemma states the relationship between critical sets and latin interchanges in a latin square.

Lemma 1 *Let L be a latin square and C a partial latin square contained in L. Then C is a critical set if and only if the following hold:*

1. *for any latin interchange I in L, $|C \cap I| \geq 1$; and*

2. *for each $(i,j;k) \in C$, there exists a latin interchange I in L such that $I \cap C = \{(i,j;k)\}$.*

Let L be a latin square of order n and let $\{a,b,c\} = \{1,2,3\}$. The (a,b,c)-*conjugate* of L, $L_{(a,b,c)}$, is defined as follows:

$$L_{(a,b,c)} = \{(x_a, x_b; x_c) \mid (x_1, x_2; x_3) \in L\}.$$

Two latin squares L and L' of order n are *isotopic* if there are three bijections from the rows, columns, and symbols of L to the rows, columns, and symbols, respectively, of L', that map L to L'. Two latin squares L and L'

of order n are *main class isotopic* if L is isotopic to any conjugate of L'. Table 1 shows the number of main and isotopic classes for latin squares of order $1 \leq n \leq 7$ (see Denes and Keedwell [7]).

$n =$	1	2	3	4	5	6	7
Main classes	1	1	1	2	2	12	147
Isotopic classes	1	1	1	2	2	22	563

Table 1

In the process of completing the critical set C to the latin square L of order n which it characterizes, we say that adjunction of a triple $t = (r, c; s)$ is *forced* (see [11]) in the process of completion of a set T of triples ($|T| < n^2$, $C \subseteq T \subset L$) to the complete set of triples which represents L, if either

(i) $\forall r' \neq r$, $\exists z \neq c$ such that $(r', z; s) \in T$ or $\exists z \neq s$ such that $(r', c; z) \in T$ (that is, in the partial completion F of L, each cell of column c except that in row r is either in a row of F which already contains the symbol s or else is already filled with an element z distinct from s), or

(ii) $\forall c' \neq c$, $\exists z \neq r$ such that $(z, c'; s) \in T$ or $\exists z \neq s$ such that $(r, c'; z) \in T$ (that is, in the partial completion F of L, each cell of row r except that in column c is either in a column of F which already contains the symbol s or else is already filled with an element z distinct from s), or

(iii) $\forall s' \neq s$, $\exists z \neq r$ such that $(z, c; s') \in T$ or $\exists z \neq c$ such that $(r, z; s') \in T$ (that is, in the partial completion F of L, every symbol except s already occurs either in column c or else in row r of F).

The critical set C is called *strong* if we can define a sequence of sets of triples $C = F_1 \subset F_2 \subset F_3 \subset \ldots \subset F_r = L$ such that each triple $t \in F_{i+1} \setminus F_i$ is forced in F_i for $1 \leq i \leq r - 1$. A critical set which is not strong is called *weak*.

This paper mainly deals with smallest critical sets for the latin squares of orders six and seven. The sizes of smallest critical sets for the latin squares of orders four and five have already been determined (see [8, 6]). Recently, Howse in [10] finds smallest critical sets for all the latin squares of order six. She also finds smallest critical sets of orders seven and nine which are based on the groups $\mathbb{Z}_7$ and $\mathbb{Z}_9$. Donovan and Howse in [9] provide a general method for finding a critical set for any latin square of order n. Using this method they produce a critical set for each latin square of order seven. However, this method does not necessarily generate a smallest critical set for a given latin square of order n. In this paper we find a smallest critical set for each main class of latin squares of order seven. Using the fact that smallest critical sets for the latin squares in the same main class have the same size (see [8]), the size of smallest critical sets for all the latin squares of order seven are determined.

Keedwell in [11] proves that the smallest order for which there exists a latin square which has a weak critical set is order five. Burgess in his PhD Thesis (see [3]) finds a weakly completable set of order n for each $n \geq 5$. Sittampalam and Keedwell find a weak critical set of size twelve in the dihedral group of order six (see [15]). In this paper we also find smallest critical sets of orders six and seven which are also weak critical sets. Given that the weak critical sets are also smallest critical sets, these results are "best possible".

Define $\text{scs}(n)$ to be the cardinality of smallest critical sets in any latin square of order n. Mahmoodian in [14] and Bate and van Rees in [2] independently conjecture that $\text{scs}(n) = \lfloor n^2/4 \rfloor$. This conjecture has been shown to be true for $1 \leq n \leq 6$ (see [2, 8, 6, 10]). We show that the conjecture is also true for $n = 7$.

2 Technique

In this section we describe how to get "close" to a smallest critical set for a given latin square. This method was first used by Khodkar in [12, 13] and then by Adams and Khodkar in [1].

Let L be a latin square of order n with the entries on $\{1, 2, \ldots, n\}$, and let $\{P_i\}_{i \in I}$ be a family of latin interchanges of L. Let

$$T_i = \{(r_i, s_i) \mid (r_i, s_i; t_i) \in P_i, \ \exists t_i \in \{1, 2 \ldots, n\}\}$$

be the shape of P_i. We form the inequality

$$\sum_{(r_i, s_i) \in T_i} x_{n(r_i - 1) + s_i} \geq 1$$

for each T_i.

Now consider the following integer programming problem.

$$\text{Minimize} \quad \sum_{j \in J} x_j, \quad \text{where } J = \{1, 2, \ldots, n^2\}$$

subject to:

$$\sum_{(r_i, s_i) \in T_i} x_{n(r_i - 1) + s_i} \geq 1 \ \text{ for all } i \in I; \text{ and}$$

$$x_j = 0 \text{ or } 1 \text{ for all } j \in J.$$

Let m be the integer optimal solution for the above integer programming problem. Then, by Lemma 1, the size of a smallest critical set for L cannot be less than m. Moreover, if $S = \{x_i\}_{i \in K}$ is a feasible solution for the above system such that $S \setminus \{x_i\}$ is not feasible for all $i \in K$, then the corresponding partial latin square with S is a critical set in L if it can uniquely be completed to L (see Example 2.1 in [1]).

Now we describe a procedure for finding smallest critical sets for a given latin square.

Procedure: Let L be a latin square of order n. The procedure for finding a critical set with smallest size in L is as follows.

(1) Find all the latin subsquares of sizes $2, 3, 4, \ldots, n$ in L.

(2) Form the corresponding integer programming problem with these latin interchanges.

(3) Find the optimal integer solution for the system. Then form the corresponding partial latin square, P say, in L.

(4) Stop if P has only one completion, otherwise there is at least one latin interchange in L which does not intersect P (see Lemma 1).

(5) Add the corresponding constraints with the latin interchanges to the system and return to Step (3).

By means of this method we have determined a smallest critical set in one latin square of order six from each of the 12 main classes and a smallest critical set in one latin square of order seven from each of the 147 main classes. Here, we state our main result.

Theorem 1 *(I)* (See also [10]) *Of the* 12 *main classes for latin squares of order six, there is one class with smallest critical set of size* 9, *five classes with smallest critical sets of size* 10, *five classes with smallest critical sets of size* 11 *and one class with smallest critical set of size* 12.
(II) Of the 147 *main classes for latin squares of order seven, there is one class with smallest critical set of size* 12, *two classes with smallest critical sets of size* 13, 116 *classes with smallest critical sets of size* 14, *twenty six classes with smallest critical sets of size* 15, *one class with smallest critical set of size* 16 *and one class with smallest critical set of size* 17.

Corollary 2 $\mathrm{scs}(7) = 12$.

Remark 3 The critical sets 6.3, 6.7 and 6.8 are weak critical sets of order six. Note that the latin square 6.7 is based on the dihedral group of order six and has a weak critical set of size twelve. The critical sets 7.3 and 7.42 are weak critical sets of order seven.

We conclude this section with the following open problem.

Open problem. For a given latin square of order $n \geq 6$, does there exist a weak critical set among the smallest critical sets in this latin square?

Computational details. The integer programming was carried out using CPLEX [5] on the Silicon Graphics Origin 2000 supercomputer at the University of Queensland, Australia.

3 Table of results

In this section we give a smallest critical set for each main class of latin squares of order $4 \leq n \leq 7$. Here we use the numbering system used in [4] but the entries in a latin square of order n are $1, 2, 3 \ldots, n$. Denes and Keedwell in [7, Chapter 4] exhibit a classification list for latin squares of order 2 to 6 with different numbering system. When the order is less than 6 it is easy to match their numbering system with the numbering system in this paper. The latin squares of order six in [7, Chapter 4] are matched with the latin squares of order six in this paper as follows: 6.1 and 6.1.1.1, 6.2 and 6.4.1.1, 6.3 and 6.8.1.1, 6.4 and 6.11.1.1, 6.5 and 6.12.1.1, 6.6 and 6.9.1.1, 6.7 and 6.2.1.1, 6.8 and 6.5.1.1, 6.9 and 6.10.1.1, 6.10 and 6.7.1.1, 6.11 and 6.6.1.1, and 6.12 and 6.3.1.1.

For each critical set, an identifying number is given, followed by four integers. The first three integers (see also [4]) give the number of transversals, the number of 2×2 subsquares, and the number of 3×3 subsquares, respectively, in the completed latin square. The fourth integer gives the size of the smallest critical set for the corresponding latin square. In the following critical sets, $*$ indicates an empty cell.

1	*	*	*
*	*	4	*
*	*	*	2
*	3	*	1

4.1:8,12,0,5

1	*	3	*
*	*	*	*
3	*	*	*
*	*	*	2

4.2:0,4,0,4

1	*	*	*	*
*	*	*	*	*
*	*	*	*	2
*	*	*	2	3
5	1	*	*	*

5.1:15,0,0,6

*	*	*	4	*
2	1	*	*	*
*	*	*	*	*
*	5	*	3	*
*	*	1	*	4

5.2:3,4,0,7

*	2	*	*	5	*
*	*	4	*	*	*
*	*	*	*	*	*
4	*	6	*	*	1
*	*	1	*	*	4
*	5	*	*	*	*

6.1:0,9,4,9

*	*	3	*	*	6
2	*	*	*	*	*
*	*	5	*	*	*
4	*	*	5	2	*
*	6	1	*	*	3
*	*	*	*	3	*

6.2:32,9,0,11

1	*	3	*	*	*
*	1	*	*	*	3
*	*	*	6	2	*
*	*	*	1	*	*
*	6	*	*	*	4
*	*	5	*	4	*

6.3:24,15,0,11

1	2	*	*	*	*
2	1	*	*	6	*
*	*	1	*	*	5
*	*	*	*	*	*
*	6	*	*	*	*
*	*	5	*	*	4

6.4:8,7,0,10

*	*	*	*	5	*
2	*	4	*	*	*
*	*	*	*	1	5
4	*	6	2	*	*
*	*	*	*	*	2
6	*	*	*	*	*

6.5:8,5,0,10

*	2	*	*	*	*
*	1	4	*	6	*
3	*	*	*	*	2
*	*	6	*	*	*
*	6	1	2	*	*
*	*	*	*	*	5

6.6:0,9,4,11

*	2	3	*	*	*
2	*	*	*	*	5
*	*	*	6	*	*
4	*	*	5	1	*
*	3	*	*	4	*
*	*	5	*	*	1

6.7:0,27,4,12

*	*	3	4	*	*
2	*	*	*	6	*
*	5	*	*	*	*
*	*	2	*	*	3
*	3	*	*	4	1
6	*	*	*	*	*

6.8:0,19,0,11

*	*	3	*	*	*
2	*	*	*	6	*
*	*	*	6	2	4
*	*	*	*	*	*
*	*	*	2	1	*
*	3	5	*	*	*

6.9:0,15,0,10

1	*	3	4	*	*
*	*	*	*	6	*
3	*	1	*	*	*
*	6	*	*	2	*
5	*	*	*	*	*
*	*	*	*	*	1

6.10:8,11,0,10

1	*	3	*	*	*
*	*	*	5	*	*
3	*	*	*	1	*
*	6	5	2	*	*
*	*	*	*	*	*
*	5	*	*	*	2

6.11:8,4,0,10

1	*	*	4	*	*
*	3	*	*	*	4
*	1	*	6	*	*
*	6	5	*	1	*
*	*	*	*	*	*
*	*	4	*	*	2

6.12:0,0,4,11

*	*	3	4	*	*	*
*	3	1	*	4	7	*
*	*	*	*	*	*	5
*	*	6	*	1	*	*
5	*	*	*	*	*	2
*	7	4	*	3	*	*
*	*	*	3	*	*	*

7.1:3,18,1,15

*	*	3	*	*	6	*
2	*	*	5	4	*	*
*	*	*	6	*	*	5
*	*	6	*	1	3	*
5	*	*	*	*	*	3
*	*	4	*	*	*	*
7	*	*	2	*	*	*

7.2:23,26,3,15

*	*	*	*	5	*	*
*	3	*	*	*	7	*
*	*	2	6	7	*	*
4	*	6	*	*	*	1
5	4	*	*	*	*	3
6	7	4	*	*	*	*
*	*	*	1	*	2	*

7.3:63,42,7,17

*	2	*	*	5	*	*
*	*	*	*	*	*	6
3	*	*	6	*	*	*
*	5	*	*	1	2	*
*	*	*	*	*	*	2
6	*	*	*	*	4	*
7	*	4	3	*	*	*

7.4:23,14,1,14

1	2	*	*	*	*	7
2	*	1	*	*	*	*
*	1	*	*	*	*	*
*	*	*	*	3	*	*
*	*	*	3	*	4	*
*	*	*	*	4	5	3
7	*	*	*	*	3	*

7.5:19,6,0,14

*	2	*	*	*	*	*
2	3	*	*	6	*	*
3	4	*	*	*	*	2
*	*	6	7	*	5	*
*	6	7	*	*	*	*
*	*	1	5	*	*	*
*	*	*	*	*	*	*

7.6:25,0,0,14

1	2	3	*	*	*	*
2	3	*	*	*	*	*
3	*	*	*	*	*	*
*	*	*	*	*	*	*
*	*	*	*	*	*	4
*	*	*	*	*	4	5
*	*	*	*	4	5	6

7.7:133,0,0,12

1	*	*	*	*	6	*
*	*	4	*	*	*	*
*	4	*	2	7	*	*
*	*	*	*	*	2	*
5	*	*	6	*	1	*
*	7	*	*	4	*	*
*	*	5	*	3	*	*

7.8:21,18,1,14

*	*	*	4	*	*	*
*	1	*	*	*	7	*
3	*	*	*	*	*	6
4	*	*	*	*	*	3
*	*	7	*	*	*	*
*	7	5	1	*	*	*
*	*	2	*	3	*	4

7.9:30,16,1,14

*	*	*	4	5	6	*
2	1	*	*	*	7	*
3	4	*	*	*	*	*
*	*	6	7	*	*	*
*	3	*	*	*	*	1
6	*	*	*	*	1	3
*	*	5	*	*	*	*

7.10:43,30,3,16

*	*	*	*	*	*	*
2	*	*	3	*	*	5
*	4	1	*	7	*	*
*	*	*	*	*	3	2
*	*	7	*	*	4	*
*	7	5	*	*	*	*
*	*	*	5	*	*	3

7.11:43,18,3,14

1	*	*	*	*	6	*
*	*	4	3	*	*	*
*	4	*	*	7	*	*
*	*	*	*	*	3	*
*	*	7	*	2	*	*
6	*	*	1	*	*	3
*	*	2	*	*	*	4

7.12:55,22,3,14

*	2	*	*	*	6	*
*	*	*	3	*	*	5
*	*	1	*	*	5	6
4	*	*	*	*	3	*
*	6	7	*	2	*	*
*	7	2	*	*	*	*
7	*	*	*	*	*	*

7.13:13,18,1,15

*	2	*	*	*	6	*
*	*	*	3	*	*	5
3	4	*	*	7	*	*
4	*	*	*	*	*	*
*	*	*	1	*	*	*
*	7	*	*	4	1	*
*	*	5	6	*	*	1

7.14:33,22,1,15

1	*	3	*	*	6	*
2	1	*	*	*	*	*
*	*	*	2	7	*	*
*	*	*	7	*	3	*
*	6	*	*	4	*	*
6	*	*	*	*	1	*
*	*	5	*	2	*	*

7.15:15,22,1,15

*	*	3	*	*	*	7
2	*	*	*	6	*	*
*	*	1	5	*	*	*
*	3	*	*	*	*	*
*	*	7	*	*	3	4
6	*	*	2	*	*	1
*	*	*	6	*	*	*

7.16:30,8,0,14

*	*	*	*	*	6	*
*	1	4	*	*	*	5
*	4	1	*	*	*	*
*	*	*	7	*	*	*
*	*	*	*	*	*	4
6	7	*	2	*	*	*
7	*	*	6	3	*	*

7.17:14,15,0,14

1	*	*	*	*	6	*
*	*	*	3	6	*	*
*	4	*	*	*	*	*
4	*	*	*	*	5	*
5	*	7	*	*	4	*
*	*	*	2	*	*	*
*	*	2	6	3	*	*

7.18:20,12,0,14

*	*	3	*	*	*	*
2	*	*	*	6	7	*
*	*	1	5	*	2	*
4	*	*	*	*	*	*
*	*	*	2	*	*	1
*	*	5	1	*	*	*
7	*	*	*	4	*	*

7.19:22,11,0,14

*	*	3	*	*	6	*
2	*	*	3	*	*	5
*	*	1	*	*	*	*
4	*	*	*	*	*	2
*	6	7	*	*	1	*
*	*	*	*	2	*	4
7	*	*	*	*	*	*

7.20:15,14,0,14

```
* * * * 5 * *
* 1 * 3 * 7 *
3 * * * * * *
* 3 * 7 * * *
5 * * * 4 * *
* * 5 * 2 4 *
* * 2 6 * * *
```
7.21:24,11,0,14

```
* * * * * * 7
* 1 4 * 6 * *
* 4 1 * * * *
* * * 7 * * 2
* * * * * 4 *
* 5 * * * * *
7 * * 2 * * 3
```
7.22:18,11,0,13

```
* * 3 * 5 * 7
* 1 * * * * *
3 * * * 7 * *
* * * * * * *
* * * 6 * 1 *
* * 7 2 3 * *
* 6 * 1 * * 4
```
7.23:22,13,0,14

```
1 2 * * 5 * *
* * * 3 6 7 *
* * 1 * * * *
4 * * 7 * * *
5 6 * * 1 * *
* * 5 * * 4 *
* 5 * * * * *
```
7.24:19,14,0,15

```
1 * 3 * * * *
* * * * 6 7 *
3 * 1 * * * 6
* * * 7 2 * 1
5 * * * * 4 *
* 7 * * * * 2
* 5 * * * * *
```
7.25:13,18,0,15

```
* * 3 4 * * *
2 * * * 6 * *
* 4 * 5 * * *
* 5 * 7 * * *
* * * * 2 * *
6 * 2 * * * 3
7 * * * * 4 *
```
7.26:11,16,0,14

```
* * * * * 6 *
2 * * 3 * 7 *
* 4 1 * * * 6
* * * 7 * 3 *
* * * * * * 4
* * 5 * * * 1
7 * * * 4 * *
```
7.27:13,10,0,14

```
* 2 3 * * * 7
* * 4 * 6 * *
* * * * * 2 *
* 5 * 7 * * 2
* * * * * * 3
* 7 * 2 * 1 *
* 3 * * 4 * *
```
7.28:16,16,0,15

```
* * 3 * * 6 *
* 1 * * * 7 5
3 * * * * * *
* * * * 1 * 2
* 6 * * * 1 *
* * 2 * 4 * 3
7 * * 6 * * *
```
7.29:21,16,0,15

```
* * 3 * * * *
2 * * * 6 * *
* 4 * * * * *
4 5 * * * 3 *
* * * 2 * * *
6 * * 1 2 * *
* 3 * * * 5 1
```
7.30:32,14,0,14

```
* * * * 5 6 *
* 1 * 3 * 7 *
3 * * * * * *
* * * 7 * 3 *
* * * 2 * * *
* * 2 * 4 * *
* * 5 * 2 * 4
```
7.31:15,12,0,14

```
* * * * * * *
* 1 4 * 6 * *
3 * * * * 2 *
4 * * * * 3 2
* 7 2 6 * * *
* * * * * 5 *
* 6 * 1 * * *
```
7.32:24,14,0,14

```
* * 3 4 * * 7
* 1 * * 6 * *
* 4 * * * * *
* 5 * * 1 3 *
* * 2 * * * *
* * 7 * * * 4
* 6 * * * 1 *
```
7.33:15,12,0,14

```
* 2 * 4 * 6 *
* * * * * * 5
3 4 * * * 2 *
4 * * * * * *
5 * 7 * 1 * *
* * 5 * * * 1
* 6 * * * * *
```
7.34:19,16,1,14

```
* 2 * * * * 7
* * 4 * 6 * *
* * 1 * * * *
4 * 6 * * 1 *
* * * * * * 2
* 7 * 2 * 3 *
* 3 * * 4 * *
```
7.35:24,12,0,14

```
* 2 * 4 * * *
* * * * 6 * 5
3 * 1 * * * 6
* * 6 * * * 3
* * * 1 * 4 *
* 7 * 2 4 * *
* * * * * * *
```
7.36:29,14,1,14

```
1 * 3 * 5 * *
* * * 3 * * *
* 4 * * * 2 6
* * * * 2 * *
* 6 * * * * 2
* * 5 * 3 * *
* * * 6 * * 4
```
7.37:14,10,0,14

```
1 2 * * 5 * *
* * 4 * * 7 *
* * * 5 * * *
* 5 * * 2 * *
* 6 7 * * 3 4
* * * * * 4 *
* 3 * * * * *
```
7.38:13,16,0,14

```
* * * * 5 6 *
* 1 4 3 * * *
* 4 1 * * * *
* * * * * * 3
5 * * * * * 1
6 * * * * 5 *
7 3 * * * * *
```
7.39:9,8,0,14

```
1 2 * * 5 * *
* * 4 * * 7 *
* * * 5 * * *
* 5 * * 2 * *
* * 7 * * 3 *
* * * 1 * 4 *
* 3 * * * * 4
```
7.40:15,14,0,14

```
* 2 * * * 6 *
* * * 3 * * 5
* 4 1 * * 2 *
4 * * * * * 3
5 * * * 3 * 1
* * * 2 * * *
* 6 * * * * *
```
7.41:20,12,0,14

```
1 * * * * 6 *
* 1 4 * * * *
* * * * * 2 6
* * * * 2 3 *
5 7 * * * * *
* * 7 * * * *
* * 5 * 3 4 2
```
7.42:18,20,0,15

```
1 * * 4 * * *
* * * * * 7 *
* 4 * * * * 6
* * * 7 * 1 *
* 6 * * 2 * 4
* * * * * * 3
7 * 2 * * 5 *
```
7.43:16,10,0,14

```
1 * * 4 5 * *
* 1 * * * * *
* * * * * 2 6
* * 6 7 * * 2
5 * * * 1 * *
* * 2 * 4 * 3
* * * * * * *
```
7.44:19,14,0,14

```
* 2 * * * * 7
* * * 3 * * 5
3 * 1 5 * * *
* * * * * * 2
* 6 * * 1 * *
* 7 * * 2 4 *
7 * * 6 * * *
```
7.45:22,18,0,15

```
* * 3 * * 6 *
* 1 * * * * *
* 4 * * 7 * *
* * 6 * 3 * *
* 7 * * 1 * 4
6 * * * * 5 *
* * 5 1 * * *
```
7.46:19,12,0,14

```
1 * * * 5 * 7
* * 4 3 * * *
3 * * * 7 * *
* * 6 * * * *
* * 2 6 * 4 *
* * * * 2 * *
7 * * * * * 1
```
7.47:14,10,0,14

```
1 2 * * * * 7
* * 4 3 6 * *
* * * 5 * * *
* * 6 * 3 * *
* 7 * 6 * * *
* * * * * * 1
7 * * * * * 4
```
7.48:15,12,0,14

```
* 2 * 4 * * 7
* 1 * * 6 * *
* 4 1 * * * *
4 * * 7 * * 2
* * * * * * *
* * * * * * 4
* 6 5 * 1 * *
```
7.49:23,10,0,14

```
1 * * 4 * * 7
* * * 3 * * *
* * 1 * * 2 *
4 5 * * * * *
* * * 7 * * 4
* * 5 * * * *
* 6 2 * * 5 *
```
7.50:19,12,0,14

* * * 4 * 6 7	* * 3 * 5 * *	* * * * * 6 *	1 2 * * 5 * *	1 * 3 * 5 * *
* 1 * 3 * * *	* 1 4 * * * *	2 * 4 * * * *	* * 4 * * 7 *	* * 4 * * * *
* * * * * * 6	* 4 * * * * 6	* * * 5 * 1 6	3 * * * * * 6	* * * * * * 6
* 3 * * 1 * *	* * * * * 5 *	4 * * * * * *	* * * * 1 * *	* 3 * * * 5 *
* * * * * * *	* 6 * 2 * * 1	* * * 6 * * 3	* * * 6 * 4 3	* 6 * 2 * * *
6 * * * * 4 *	* * 5 * 3 2 *	* * 7 1 * * *	* 5 * 2 * * *	* * 5 * 4 * *
7 * * 6 * 2 4	* * * 6 * * *	7 * 5 * 4 * *	* * * 1 * * 4	7 * * 6 * * 2
7.81:25,10,0,14	7.82:19,8,0,14	7.83:19,8,0,14	7.84:22,14,0,15	7.85:12,11,0,14

1 2 * * * 6 *	* 2 3 * * * *	* * * * 5 6 7	1 2 * * 5 * *	* * * * * * 7
* * * 3 * * *	2 * * * 6 7 *	2 1 4 * * * *	2 * 4 * * 7 *	2 * * 3 6 * *
* * 2 * * 1 6	* * * * 7 * *	* 4 2 * * * *	* * * 5 * * *	3 4 * * * * *
* * 6 * * 2 *	* 5 * * * * 3	4 * 6 * * * *	* 5 * * 1 * *	* * * * 1 * *
* * * * * * *	* * * * * * *	5 * * * * 3 *	* * 7 * * 4 *	* * 7 * * * 1
* 7 * * 3 * *	* * 5 * * 3 1	* * * * 3 5 *	* * * * * 3 4	6 * * * 3 * *
7 3 * * 4 * *	7 * * 6 2 * *	* * * * * * *	* 3 * * * * *	* * 1 * * 5 4
7.86:19,8,0,14	7.87:19,8,0,14	7.88:18,7,0,14	7.89:17,7,0,14	7.90:23,6,0,14

1 * * * * * *	1 * * * * * *	1 2 * * * 6 *	* 2 * * 5 * *	* 2 * 4 * * 7
* * 4 * 6 7 *	* * 4 * 6 7 *	* * 4 3 * * 5	* * * 3 * 7 *	2 1 4 * * * *
* 4 * * 7 * *	* 4 * * 7 * *	3 * * * * 1 *	3 * * * * 1 *	* 4 2 * * * *
* * * * * 2 3	* * * * * 2 3	* * * * 1 2 *	* * * 7 * 3 2	* * * * 3 * *
* 7 * * * * *	* 7 * * * * *	* * * * * * *	5 * * * * * *	5 * * * * 3 *
* * * 1 * * 2	* * * 2 * * 1	* * 7 * 4 * *	* * * * 4 * *	6 * * * * 5 3
* * 5 2 * * 1	* * 5 1 * * 2	* * 5 * * * 4	* 6 5 * 2 * *	* * * * * * *
7.91:22,12,0,14	7.92:22,6,0,14	7.93:14,9,0,14	7.94:22,9,0,14	7.95:12,10,0,14

1 * * 4 * * *	* * 3 4 * * *	1 * * * * * *	1 * * * 5 * 7	1 * 3 * * * *
* 1 * * * 7 *	* 1 * * 6 * *	* * 4 * 6 * 5	* * * 3 * 7 *	* * * * 6 * *
* * * * * * 6	* * * 5 * * *	3 * * 5 * 1 *	* * * 5 * * *	* 4 * * 7 * 6
* 5 * * * * *	4 * * * * 2 *	* * 7 * * * *	4 * * * 1 * *	* * * 6 * * *
5 7 1 * * * *	* 3 6 * * * 1	* 6 * * * 4 *	5 * 1 * * * *	5 * 1 * * * *
* * * 2 * * 4	* * * 2 * 5 4	* * * 1 * 3 *	* 7 * 2 * 3 *	* * 5 1 * 2 *
* * * * 4 3 2	* 6 * * * * *	* * 6 * 4 * *	* 3 * * * * *	* 3 * * 4 * *
7.96:13,5,0,14	7.97:7,6,0,14	7.98:21,9,0,14	7.99:25,10,0,14	7.100:16,8,0,14

1 * * 4 * * *	1 * * * 5 6 *	1 * 3 * 5 * *	* 2 * * 5 * *	* * 3 * 5 * *
* 1 * * * * 5	* 1 4 3 * * *	* * * * * 7 5	2 * * 3 6 * *	2 * * * * 7 5
* * * * 7 * *	* 4 * * * * *	* 4 * * * * *	* * * * * 1 *	* * * * * * *
* 5 * * * * *	* * * * * * *	* * * * 3 * 1	* * 7 * * * 1	* 5 * * 3 * 1
5 6 1 * * * *	5 * * * * 2 *	* 6 * 7 * 4 *	* * 1 7 * 3 *	* * * * * * 3
* * * * 2 4 3	6 * 5 * * * *	6 7 * * * * *	6 * * * 2 * *	6 * * 7 * * *
* * * 2 4 * *	* 3 * 1 * * 4	* * * * 1 * *	* * * * * 5 *	7 6 * 2 * * *
7.101:25,12,0,14	7.102:26,11,0,14	7.103:22,7,0,14	7.104:21,10,0,14	7.105:30,10,0,14

* 2 * * * 6 *	1 * 3 * 5 * *	* * * * * 6 7	* 2 3 * * * 7	1 * * * * * 7
* 1 * 3 * * *	* * * * * 7 *	2 * 4 3 * * *	* * 4 * 6 * *	* * 4 * * * *
3 * * * * * 6	3 * * 5 * * *	* 4 * * * 1 6	* 4 2 * * * *	3 * * * 7 * 6
4 * * * * * *	* * 1 * * * *	* * * 7 2 * *	* * * * * * 2	* 6 5 * * 2 *
* 7 * 2 1 * *	* 7 * 2 * * 4	5 * * 2 3 * *	5 * * * * * *	* * * * * * *
6 * * * * 5 4	* * * * * 4 2	* * * * * * *	6 * * 1 * * *	* 5 * * * 4 *
* * * 1 * * *	* * 5 * * 2 *	* * * * * * 4	7 * * 6 1 * *	7 * * * * 5 1
7.106:26,9,0,14	7.107:16,10,0,14	7.108:14,8,0,14	7.109:25,16,0,14	7.110:25,10,0,14

```
* * 3 * * * 7
* 1 4 3 * * *
* * * * 7 * 2
* 3 * * 2 * *
5 * 1 * * * *
* 5 * 1 * * *
7 * * * * * 4
```

7.111:14,8,0,15

```
* 2 * * 5 * 7
* * 4 3 * * *
3 * 5 * * * *
4 * * * * * *
* 7 * 1 * * 6
6 * * * * 3 *
* 6 * * * 2 *
```

7.112:18,9,0,15

```
* * * * 5 * 7
2 1 * 3 * * *
* * 5 * 7 * *
* * * * * 3 *
* 6 7 * 4 * *
* * * 5 * * *
* 3 * 2 * * *
```

7.113:28,9,0,14

```
* * * 4 5 6 *
2 1 * * * 7 *
* * * 6 * * *
* * 1 * * * 6
* * 7 * * * *
* 7 2 * * * *
* * * 5 * * 4
```

7.114:19,8,0,14

```
* 2 * 4 * * *
* * * 3 6 * *
* 4 * * * 1 2
* * * * 2 * *
5 * 6 * * * *
6 * 7 * * 5 *
7 * * * * * 1
```

7.115:19,10,0,15

```
* 2 * 4 5 * *
* * * 3 * * 5
* * * * 7 * *
* * 1 * * * *
* 7 6 * * * *
* * * 5 * * 4
7 6 2 * * * *
```

7.116:23,6,0,14

```
1 * 3 * * * 7
* * 4 * * * *
* * * 6 * * *
* 5 * * * 2 *
* * 7 * * * 3
* 7 * 5 * * *
* * * 2 1 5 *
```

7.117:21,10,0,14

```
1 2 * * * 6 *
* * 4 3 * * 5
* * 5 * * * 2
* * * * * * *
* 6 * * * * *
6 7 * * 1 * *
* * * * * 5 4
```

7.118:17,5,0,14

```
* * * * * 6 *
2 * * 3 * * 5
* 4 * * 7 1 *
4 * * * * * *
* 7 * * * 4 *
* * * 2 * * *
* * 2 5 * * 1
```

7.119:17,7,0,14

```
* * * * * * 7
2 * 4 * 6 * *
* * * * * 1 *
4 * * * * 2 *
* 7 * 1 * * 3
* 3 * * * * 1
7 * 2 * * * *
```

7.120:21,5,0,14

```
* * * 4 * * 7
* 1 * * * * *
3 * 5 * * 1 *
4 * * 7 * * 6
* 7 * * * * *
* * * 2 * 5 *
* * 1 * * 2 *
```

7.121:18,10,0,14

```
* 2 * * 5 * 7
* * * 3 * * *
3 * * 6 * 1 *
* 5 2 * * * *
* 7 * 1 * * *
6 * * * * * 1
* * * * 2 * *
```

7.122:18,8,0,14

```
* 2 * * * * *
* * 4 * 6 * *
* * * * * * *
* 5 * 7 * 3 *
5 * 6 * * * 4
* 3 * 5 * * *
* * 1 * 4 * 3
```

7.123:17,8,0,14

```
* * * * * * *
2 * * 3 * * 5
* 4 * * 7 1 *
* * * * 1 * *
5 * * 2 * * *
6 * * * * 5 *
* 6 1 * 4 * *
```

7.124:23,12,0,14

```
* * 3 4 5 * *
2 1 * * * 7 *
* * 5 6 * * *
* * * * * 2 *
* * * * 4 * *
6 7 * * * * *
7 * * 5 * * *
```

7.125:13,2,0,14

```
1 2 * * * * 7
* * 4 3 6 * *
* * * * * 1 2
* 5 6 7 * * *
* * * * 3 4 *
* * * * * * *
* * * * 4 2 *
```

7.126:25,12,0,15

```
* 2 * 4 * * *
* * * * * 7 5
* 4 * * * * *
* * 6 * 3 * 1
* * * 1 2 4 *
6 7 * * * * *
* * 1 * * * 3
```

7.127:21,10,0,15

```
* * 3 * * * *
* 1 * 3 6 * *
* 4 * * * 1 2
* * 6 7 3 * *
* * * 1 * * *
* * * * * 5 4
* * * * * 4 *
```

7.128:11,6,0,14

```
* * 3 4 * 6 *
2 1 * * * * 5
* * * 6 * * *
* 5 * * * * *
* * * 7 * 4 *
* 7 * * * * 1
* * 6 * 1 * *
```

7.129:21,7,0,14

```
* 2 * * 5 6 *
* * 4 * 6 * 5
* * * * * * *
* * * * 3 * 6
* * * 7 * * *
* * 7 2 * * *
7 * 2 1 * * *
```

7.130:23,4,0,14

```
* 2 3 * * * 7
* * * 3 * * *
* * * * * 1 *
* 7 * * * * *
5 * * * 4 * 1
6 3 7 * * * *
* * * * 1 4 *
```

7.131:32,8,0,14

```
* 2 * * * 6 *
* * 4 3 * * 5
* * 5 * * * 1
* * * * * * *
* 6 * * 4 * *
6 7 * * 2 * *
* * * * * 5 4
```

7.132:28,7,0,14

```
* 2 * * * * *
* 1 * 3 * * 5
* * * 6 7 * *
4 * 7 * 2 * *
* * 2 * 4 * *
* * * 5 * * *
* 3 * * * 5 *
```

7.133:26,8,0,14

```
* * * 4 * * 7
2 1 * * 6 * *
3 * * * * * *
* * * 5 2 * *
* * * 7 * 1 4
6 * * * * * *
* * 6 * * 4 *
```

7.134:16,5,0,14

```
* 2 * * * * 7
2 * * 3 6 * *
3 * 1 6 * * *
* * * * * * *
* 7 * * * 4 2
6 * * * * * *
* 4 * 5 * * *
```

7.135:31,11,0,14

```
* 2 * * 5 * *
* 1 4 * * 7 *
* * 1 * * * 4
* * * * 2 1 *
5 * * 2 3 * *
* * 7 * * * *
* * * 5 * * *
```

7.136:22,6,0,14

```
* 2 3 * 5 * *
* * * 3 * 7 *
* 5 * * * * *
* * * 7 * 3 1
5 4 * * * * *
* * * * * * 3
* * * 1 4 * *
```

7.137:23,10,0,14

```
* 2 * * 5 * 7
* * 4 3 * * *
3 * 1 6 * * *
* * * * * * 2
* * * * 1 * *
* 7 * * 2 4 *
* * 6 * * * *
```

7.138:34,11,0,14

```
1 * 3 * * * 7
* * 4 * * * *
* 5 * 6 * 2 *
* * * 5 * * *
* * 7 * * * 3
* * * * 3 * *
* * * 2 * 5 *
```

7.139:36,13,0,13

```
* 2 * * * 6 *
* * * 3 * * 5
* * 1 * * 4 *
* * 5 7 * * 3
* 4 * * * 2 *
* * * 5 * * *
* * * * 4 * 1
```

7.140:31,8,0,14

*	2	*	*	5	*	7
*	*	4	3	*	*	*
3	*	1	6	*	*	*
*	6	*	*	*	2	*
*	7	6	*	*	*	*
*	*	*	*	2	5	*
*	*	*	*	4	*	*

7.141:45,16,0,15

*	*	3	4	*	*	7
2	1	*	*	6	*	*
3	*	*	*	*	4	*
*	6	*	*	2	*	*
*	*	*	7	*	*	*
6	*	*	*	*	*	*
*	*	*	5	*	*	4

7.142:31,10,0,14

*	2	*	4	*	*	7
*	*	*	*	*	*	*
*	*	*	6	*	1	4
*	*	1	*	2	5	*
*	4	*	*	*	*	6
*	*	5	*	*	3	*
*	*	*	*	1	*	*

7.143:20,4,0,14

*	*	*	4	*	*	7
2	*	4	5	*	*	*
*	*	*	*	7	*	6
4	*	*	*	*	*	*
5	*	*	6	*	*	*
*	7	*	*	3	*	1
*	*	*	*	1	*	*

7.144:15,4,0,14

1	*	*	*	5	*	7
2	*	*	*	*	*	*
*	4	*	*	*	*	*
*	6	*	7	*	3	*
5	*	*	*	*	*	1
*	*	*	3	1	*	*
*	3	*	6	*	*	*

7.145:11,2,0,14

*	*	*	*	*	*	*
*	1	4	*	*	7	*
*	4	5	*	*	*	*
*	*	*	3	2	*	*
*	*	*	*	*	*	2
6	*	*	2	3	*	*
*	5	*	*	1	*	4

7.146:7,3,0,14

*	*	3	*	*	6	*
*	1	4	*	*	*	3
*	*	*	7	*	2	*
*	*	*	6	*	*	*
*	*	*	*	*	*	1
6	7	*	2	*	*	*
*	3	*	*	*	*	4

7.147:15,1,0,14

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