Integrity of Regular Graphs and Integrity Graphs*

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Abstract

The *integrity* of a graph G, $I(G)$, is defined by $I(G) = min_{S \subseteq V(G)}\{|S| + m(G-S)\}$ where $m(G-S)$ is the maximum order of the components of $G-S$. In general the integrity of r-regular graph is not known [8]. We answer the following question for special regular graphs. For any given two integers p and r such that $\frac{pr}{2}$ is an integer, is there a r-regular graph, say G^*, on p vertices having size $q = \frac{pr}{2}$ such that

$$I(G(p, \tfrac{pr}{2})) \leq I(G^*)$$

for all p and r? *Integrity graph* is denoted by $IG(p,n)$. It is a graph with p vertices the integrity n, and has the least number of edges denoted by $q[p,n]$. We compute $q[p.n]$ for some value of p, n.

Keywords: Integrity, Cycle, and Integrity graph.

1 Introduction

Integrity was introduced by Barefoot, Entringer and Swart [3] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. The motivation was that, in some respect, connectivity is oversensitive to local weakness and does not

*Preliminary version of this paper presented at Bas'98 the third symposium on computer networks reflect the overall vulnerability. In an analysis of the vulnerability of a communication network to disruption, two quantities(there may be others) that come to mind are the number of elements that are not functioning and the size of the largest remaining group within which mutual communication can still occur. Formally, the integrity is

$$I(G) = min\ _{S \subseteq V(G)}\{|S| + m(G - S)\}$$

where $m(H)$ is the maximum order of the components of H.

A few further comments on notation are appropriate here. The order and size of G(that is, the number of vertices and edges) will generally be denoted by p and q, respectively. $m(G)$ equals the largest order among the components of G. As usual, V and E will denote respectively the sets of vertices and edges of G. The length of the shortest cycle in a graph G that contains cycles is called the *girth* of G and is denoted by $g(G)$. Graph $G(p, q)$ denotes a graph with $q = |E|$ and $p = |V|$. Let v be a vertex of G and e be an edge of G. Then deleting a vertex or edge, or adding an edge are denoted accordingly by $G - v$, $G - e$, or $G + e$. I-set of graph G is a subset S of $V(G)$ such that $I(G) = |S| + m(G - S)$.

In general the integrity of a graph is not known [8]. But for trees there is a theorem given in [2] that gives an upper bound for the integrity. This upper bound is well-known integrity of path. Therefore what we are interested in is the following question: For any given two integers p and r such that $\frac{pr}{2}$ is an integer, is there a r-regular graph, say G^*, on p vertices having size $q = \frac{pr}{2}$ such that

$$I(G(p, \tfrac{pr}{2})) \leq I(G^*)$$

for all p and r?

For some cases we give an answer to the above question. We also compute the integrity of all $G(6,9)$ and $G(6,12)$ graphs [11] to see whether above statement is true or not.

We organize the rest of the paper as follows. In the next section we review the known theorems on the integrity. In Section 3 we prove that the 2-regular graph C_p has the maximum integrity among all graphs $G(p,p)$. It is shown that $p-2$-regular(if p is even) and $p-3$-regular graphs have the maximum integrity among all graphs

$G(p, \frac{p(p-2)}{2})$ and $G(p, \frac{p(p-3)}{2})$, respectively. Similar case is also shown for some other regular graphs. In Section 4 we first give the definition of integrity graph and determine $q[p, n]$ values for given small p and n.

2 Some of the Results on Integrity

In this section we first state well-known theorems about the integrity of special graphs. In the following theorem, we give the integrity of a variety of families of graphs. These results were first found by Barefoot, Entringer and Swart.

Theorem 2.1 *[2, 3] The integrity of*
(a) the complete graph K_p is p;
(b) the null graph $\overline{K}_p$ is 1;
(c) the star $K_{1,n}$ is 2;
(d) the path P_p is $\lceil 2\sqrt{p+1} \rceil - 2$;
(e) the cycle C_p is $\lceil 2\sqrt{p} \rceil - 1$;
(f) the comet $C_{p-r,r}$ is $I(P_p)$, if $r \leq \sqrt{p+1} - 5/4$; $\lceil 2\sqrt{p-r} \rceil - 1$, otherwise;
(g) the complete bipartite graph $K_{m,n}$ is $1 + min\{m, n\}$;
(h) any complete multipartite graph of order p and largest partite set of order r is $p - r + 1$.

The second theorem tells which graphs have integrity near the extremes of the range; only the case $I(G) = p - 1$ is nontrivial.

Theorem 2.2 *[10] Let G be a graph of order p.*
(a) $I(G) = 1$ if and only if G is null.
(b) $I(G) = 2$ if and only if all nontrivial components of G are edges or the only nontrivial component is a star.
(c) $I(G) = p - 1$ if and only if G is not complete and $\overline{G}$ has girth at least 5.
(d) $I(G) = p$ if and only if G is complete.

The following two theorems are about the integrity of trees. First one gives the maximum integrity among all trees with p vertices.

Theorem 2.3 *[2] Of all trees T_p of order p, the path P_p has maximum integrity.*

Theorem 2.4 *[2] For $2 \leq n \leq \lceil 2\sqrt{p+1} \rceil - 2$ there is a tree T of order p for which $I(T) = n$.*

As we see in Theorem 2.3, among all graphs $G(p, p-1)$, the path P_p has the maximum integrity.

3 Integrity of Regular Graphs

In this section we answer the question asked in Section 1 for special regular graphs. We first state and prove the following theorem.

Theorem 3.1 *Let G be a connected graph with p vertices. If G contains r cycles with no chord, then $I(G) \leq I(C_p)$ if r cycles contain at least one common edge, $I(G) \leq 2\sqrt{p-r+1} \rceil + r - 2$ otherwise.*

Proof. Let C_{n_i}'s be the cycles with no chord in G for $i = 1, 2, ..., r$. Suppose C_{n_i}'s do not have a common edge for all $i = 1, 2, ..., r$. Then pick v_i from each cycle C_{n_i} such that v_i's are all different. Then $G - \{v_1, v_2, ..., v_r\}$ is a tree with $p - r$ vertices. By Theorem 2.3 we have $I(T_{p-r}) \leq \lceil 2\sqrt{p-r+1} \rceil - 2$. Let S be a $I - set$ of T_{p-r}, that is,

$$|S| + m(T_{p-r} - S) \leq \lceil 2\sqrt{p-r+1} \rceil - 2.$$

Define $S' = S \cup \{v_1, v_2, ..., v_r\}$. Hence

$$m(G - S') = m(G - S - \{v_1, v_2, ..., v_r\}) =$$
$$m(G - \{v_1, v_2, ..., v_r\} - S) = m(T_{p-r} - S).$$

Then

$$|S'| + m(G - S') = |S| + r + m(T_{p-r} - S) \leq \lceil 2\sqrt{p-r+1} \rceil + r - 2.$$

Therefore

$$I(G) \leq |S'| + m(G - S') \leq \lceil 2\sqrt{p-r+1} \rceil + r - 2.$$

Now suppose $e = (u, v)$ is the common edge of C_{n_i}'s for $i = 1, 2, ..., r$. Then $G-\{v\}(G-\{u\})$ is a tree with $p-1$ vertices. Define $S' = S \cup \{v\}$ and then we obtain the following.

$$I(G) \leq \lceil 2\sqrt{p} \rceil - 1 = I(C_p).$$

□

Corollary 3.1 *Let C_p be a cycle with $p \geq 4$. Then $I(C_p) = I(C_p + e) = \lceil 2\sqrt{p} \rceil - 1$.*

Proof. Since C_p is a subgraph of $C_p + e$ then $I(C_p) \leq I(C_p + e)$. On the other hand by Theorem 3.1 $I(C_p + e) \leq \lceil 2\sqrt{p} \rceil - 1 = I(C_p)$. □

Observe that for all $p \geq 4$ $G(p, p+1)$ graphs have two cycles with no chord. Therefore we have the following corollary.

Corollary 3.2 *Let $p \geq 4$ be a positive integer, then $I(G(p, p+1)) \leq I(C_p)$ if two cycles with no chord contain at least one common edge, $I(G(p, p+1) \leq \lceil 2\sqrt{p-1} \rceil$ otherwise.*

We now determine lower and upper bounds of the integrity of $G(p, p)$ graphs. A typical $G(p, p)$ graph is depicted in Figure 1.

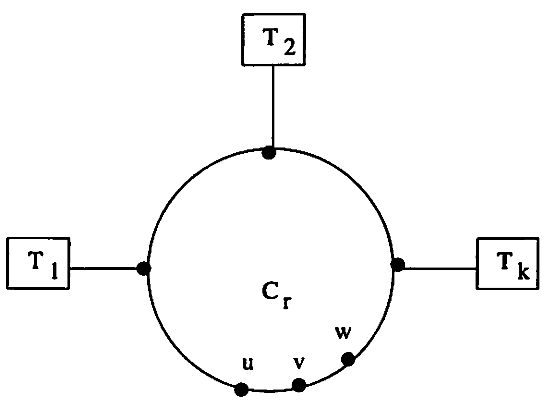

Figure 1

where T_i is either a vertex or a tree for $i = 1, 2, ..., k$ and $r + i_1 + ... + i_k = p$.

Lower bound:

The $G(p,p)$ graph ($p \geq 3$) depicted in Figure 2 has the integrity 3, where T_i's are either vertex or $T_i = P_2$. Obviously this is the minimum integrity for a $G(p,p)$ graph by Theorem 2.2 (a) and (b). Hence $I(G(p,p)) \geq 3$.

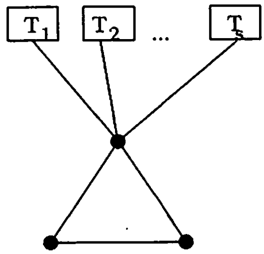

Figure 2

Upper bound:
The following is a corollary to Theorem 3.1 and gives the upper bound for the integrity of all $G(p,p)$ graphs.

Corollary 3.3 *Among all the graphs of $G(p,p)$ where $p \geq 3$, the 2-regular graph C_p has the maximum integrity.*

Proof: First observe that for any given integer $p \geq 3$, $G(p,p)$ graph has a unique cycle as a subgraph. Apply Theorem 3.1 to obtain the result. □

Next we give the integrity of some special regular graphs.

Theorem 3.2 *Let $p \geq 5$ be a integer. Then there exist a $p-3$-regular $G(p, \frac{p(p-3)}{2})$ graph such that*

$$I(G(p, \frac{p(p-3)}{2})) = p - 1.$$

In particular if p is even, then

$$I(G(p, \frac{p(p-2)}{2})) = p - 1.$$

Proof.
If $G(p, \frac{p(p-3)}{2})$ is a $p-3$-regular graph, then it is not complete and $\overline{G}(p, \frac{p(p-3)}{2})$ is union of cycles. In particularly if we take C_p as $\overline{G}(p, \frac{p(p-3)}{2})$, then girth of C_p is $p \geq 5$ and by Theorem 2.2 part (c) there exists a $p-3$-regular graph which has the integrity $p-1$. If p is even, then $\overline{G}(p, \frac{p(p-2)}{2})$ is union of K_2's which has girt $\infty > 5$. Result follows from Theorem 2.2 part (c). □

Before we state theorems we recall some of definitions on graph theory. For the positive integers $r \geq 2$ and $n \geq 3$, determine the smallest positive integer $p = f(r,n)$ for which there exists an r-regular graph with girth n having order $p = f(r,n)$. Such graphs are called $(r,n)-$*cages*. $[r,n]-$*graph* is being used to indicate r-regular graph with girth n. Thus (r,n)-cage is an $[r,n]$-graph; indeed, it is one of minimum order. For more information about (r,n)-cages see for example [7]. That $f(r,n)$ always exists has been shown by Erdös and Sachs [9].

Theorem 3.3 *[9] For every pair of integers $r,n \geq 3$, the number $f(r,n)$ exists and in fact*

$$f(r,n) \leq \left(\frac{r-1}{r-2}\right)\left[(r-1)^{n-1} + (r-1)^{n-2} + (r-4)\right].$$

It was shown in [4] that $f(r,5) \geq r^2+1$ for all $r \geq 2$. Hoffman and Singleton [12] proved that equality holds for $r = 2,3,$ and 7, and perhaps 57, since it is not known whether there is a $[57,5]$-graph of order 57^2+1.

Theorem 3.4 *[12] For $r \geq 2$, $f(r,5) \geq r^2+1$. Furthermore, for $r \neq 57$, equality holds if and only if $r = 2,3,$ or 7.*

When $r = s^m + 1$ for some prime s and positive integer m, then

$$f(r,6) = \tfrac{2(r-1)^3-2}{r-2}$$

and the $(r,6)$-cage is unique (see [5], Chap. 23).

Once we state the above theorems, the followings are immediate consequences.

Theorem 3.5 *For integers $r \geq 2$ and $p \geq r^2+1$ there exists $(p-r-1)$-regular graph G of order p such that $I(G) = p-1$.*

Proof. Use Theorem 3.4 to construct $(r,5)$-cage on $p = f(r,5) \geq r^2+1$ vertices. Take complement of $(r,5)$-cage as $(p-r-1)$-regular graph on p vertices. Use Theorem 2.2 part (c). ▯

With a similar proof as above and use the fact about $(r,6)$-cage we can state the following theorem.

Theorem 3.6 *If $r = s^m + 1$ for some prime s and positive integer m, then there exists $(p-r-1)$-regular graph $G(p,q)$ with the integrity $p-1$, where $p = \frac{2(r-1)^3-2}{r-2}$ and $q = \frac{(r-1)^3-1}{r-2} - \frac{(r+1)}{2}$.*

For certain p and r positive integers the following theorems show that among all $G(p, \frac{pr}{2})$ graphs there exists an r-regular graph which has the maximum integrity.

Theorem 3.7 *Let $p \geq 5$ be a positive integer. Then among all $G(p, \frac{p(p-3)}{2})$ graphs there exist a $p-3$-regular graph that has the maximum integrity.*

Proof.
There exists a $p-3$-regular graph which has the integrity $p-1$ by Theorem 3.2. By Theorem 2.2 part (d) $I(G(p,q)) = p > p-1$ if and only if $q = \frac{p(p-1)}{2} > \frac{p(p-3)}{2}$. Therefore $p-3$-regular graph has the maximum integrity among all $G(p, \frac{p(p-3)}{2})$ graphs. □

Similarly

Theorem 3.8 *For integers $r \geq 2$ and $p \geq r^2 + 1$ there exists $(p-r-1)$-regular graph G of order p such that $I(G)$ is maximum among all graphs $G(p, \frac{(p-r-1)p}{2})$.*

Theorem 3.9 *If $r = s^m + 1$ for some prime s and positive integer m, then there exists $(p-r-1)$-regular graph G which has the maximum integrity among all graphs $G(p,q)$ where $p = \frac{2(r-1)^3-2}{r-2}$ and $q = \frac{(r-1)^3-1}{r-2} - \frac{(r+1)}{2}$.*

In the following examples we answer the same question asked in Section 1 for $G(6,9)$ and $G(6,12)$ graphs.

Example 3.1 We compute the integrity of all $G(6,9)$ and $G(6,12)$ graphs. These graphs are taken from [11] and depicted in Appendix. We have the following values

$$I(G_1) = 3,$$
$$I(G_2) = I(G_5) = 5,$$

and for remaining k's

$$I(G_k) = 4.$$

and

$$I(H_2) = 4,$$
$$I(H_1) = I(H_3) = I(H_4) = I(H_5) = 5.$$

From these values we make the following observations:

1. There exist a 3-regular graph, for example G_2, such that $I(G(6,9)) \leq I(G_2)$.
2. There exist a 4-regular graph, for example H_5, such that $I(G(6,12))$ $I(H_5)$.

□

In general the integrity of r-regular is not known. For special cases we know the integrity of r-regular graphs. These are K_{r+1}, $K_{r,r}$, and r-cube. It was shown [6] that the integrity of the r-cube is $O(2^r logr/\sqrt{r})$. Since we have partial answer for the question stated in Section 1 the following is a still open question.

Question : *Let r and p be two positive integers such that $\frac{pr}{2}$ is also an integer. Then among all $G(p, \frac{pr}{2})$ graphs is there a r-regular graph, say G^*, such that*

$$I(G(p, \tfrac{pr}{2})) \leq I(G^*) \ ?$$

4 Integrity Graphs

As we know no formula exists for the integrity of a graph in general. This, however, is not our interest; instead, it is the following problem: For positive integers $n, p \geq 1$, determine the smallest positive integer $q[p, n]$ for which there exists a connected graph with the integrity n

and order p having size $q[p,n]$. We call such a graph *integrity graph* and denote by $IG(p,n)$.

Since $IG(p,n)$ is a connected graph $IG(p,n)$ is at least a tree and $IG(p,n)$ can be a complete graph. Therefore we have

$$p-1 \leq q[p,n] \leq \frac{p(p-1)}{2}.$$

By Theorem 2.13 in [1] $I(K_{m-1}+\overline{K}_{p-m+1}) = m$ where $1 \leq m \leq p$. Then we have the following theorem for lower and upper bound of $q[p,n]$.

Theorem 4.1 *Let p and n be two positive integers such that $n \leq p$. Then*

$$p-1 \leq q[p,n] \leq \frac{(n-1)(2p-n)}{2}.$$

The following is an easy observation from Theorem 2.2 and Theorem 2.4.

Theorem 4.2 *Let n and p be two positive integers grader than 3. Then*
(a) $q[p,p] = \frac{p(p-1)}{2}$.
(b) $q[p,n] = p-1$ *for all p such that* $\lceil 2\sqrt{p+1}\rceil - 2 \geq n$.
(c) $q[p,p-1] \leq \frac{p(p-3)}{2}$.

By Theorem 2.2 $IG(p,1)$ does not exist and $IG(p,2)$ is the star $K_{1,p-1}$. We try to determine $q[p,n]$ for small values of p and $n \geq 3$. Before this we state the following lemma.

Lemma 4.1 *Let G be a connected graph with $p=9$ and $q=10$, that is $G = G(9,10)$. Then $I(G) = I(G(9,10)) \leq 5$.*

Proof. By Corollary 3.2 $I(G(9,10)) \leq I(C_9) = 5$ if two cycles with no chord have a common edge. If these two cycles do not have a common edge, then by Corollary 3.2 $I(G(9,10)) \leq \lceil 2\sqrt{8}\rceil = 6$. But observe that if these two cycles C_n and C_m do not have an common edge, then we have the following cases.

Case 1. If these two cycles have common vertex v, then $G(9,10)-v$ is one of the following

$$P_2 \cup P_6,\ P_3 \cup P_5,\ P_4 \cup P_4.$$

And

$$I(P_2 \cup P_6) = I(P_3 \cup P_5) = I(P_4 \cup P_4) = 4.$$

Hence $I(G(9,10)) \leq 5$.

Case 2. If these two cycles do not have common vertex, then then $G(9,10)$ is one of the following

$$G(3,3) \cup G(6,6) \cup \{e\},\ G(4,4) \cup G(5,5) \cup \{f\}.$$

That is $G(3,3)$ and $G(6,6)$ are connected with an edge e or $G(4,4)$ and $G(5,5)$ are connected with an edge f.

Suppose $G(9,10)$ is the following: $G(4,4)$ and $G(5,5)$ are connected with an edge $f = (u,v)$ where $u \in V(G(4,4))$ and $v \in V(G(5,5))$. Define $S = \{v\}$ then

$$I(G(9,10)) \leq |S| + m(G(9,10) - v) = 5.$$

Suppose $G(9,10)$ is the following: $G(3,3)$ and $G(6,6)$ are connected with an edge $e = (u,v)$ where $u \in V(G(3,3))$ and $v \in V(G(6,6))$. Since $G(6,6)$ has a cycle then there exist a vertex $w \in V(G(6,6))$ such that $G(9,10) - \{v,w\}$ has three component and the maximum number of vertices of component is at most three. Hence

$$I(G(9,10)) \leq |\{v,w\}| + m(G(9,10) - \{v,w\}) = 5.$$

Similarly one can show that $I(G(8,9)) \leq 5$. □

$q[p,n]$ for small n and p

- $n = 3$
  - $q[3,3] = 3$ and $q[p,3] = p - 1$ for all $p \geq 4$ by Theorem 4.2.
- $n = 4$
  - $q[4,4] = 6$ by Theorem 4.2.
  - $q[5,4] = 5$ since $I(C_5) = 4$.

- $q[p,4] = p-1$ for all $p > 5$ by Theorem 4.2.

- $n = 5$
  - $q[5,5] = 10$ by Theorem 4.2.
  - $q[6,5] > 6$ by Corollary 3.3 and $q[6,5] = 9$ by computing the integrity of all $G(6,k)$'s for $k = 7,8,9$.
  - $q[7,5] = 7$ and $q[8,5] = 8$ since $I(C_7) = I(C_8) = 5$.
  - $q[p,5] = p-1$ for all $p > 8$ by Theorem 4.2.

- $n = 6$
  - $q[6,6] = 15$ by Theorem 4.2.
  - $q[7,6] \geq 9$ by Corollary 3.2 and $q[7,6] \leq 13$ by Figure 5.
  - $q[8,6] \geq 10$ by Lemma 4.1 and $q[8,6] \leq 12$ by Figure 5.
  - $q[9,6] = 11$ by Lemma 4.1 and Figure 4.
  - $q[10,6] = 10$ and $q[11,6] = 11$ since $I(C_{10}) = I(C_{11}) = 6$
  - $q[p,6] = p-1$ for all $p \geq 12$ by Theorem 4.2.

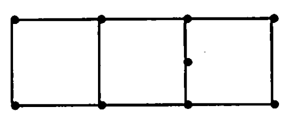

Figure 4

$G(7,8)$ depicted in Figure 5 has girt 5 so $\overline{G}(7,8)$ is $G(7,13)$ graph with the integrity 6 by Theorem 2.2. There are 558 $G(7,k)$ graphs for $k = 9,10,11,12$ and 1643 $G(8,k)$ graphs for $k = 10,11$. One can compute the integrity of all these graphs to determine exact value of $q[7,6]$ and $q[8,6]$.

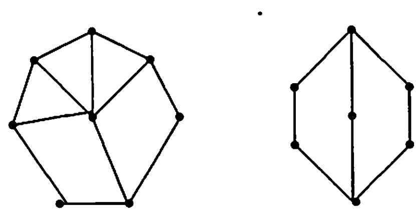

Figure 5

We summarize the numbers computed above in a table as follows:

n	p	q[p,n]	
3	3	3	Theorem 4.2 (a)
	4	3	Theorem 4.2 (b)
4	4	6	Theorem 4.2 (a)
	5	5	$I(C_5) = 5$
	6	5	Theorem 4.2 (b)
5	5	10	Theorem 4.2 (a)
	6	9	Exhaustive search
	7	7	$I(C_7) = 5$
	8	8	$I(C_8) = 5$
	9	8	Theorem 4.2 (b)
6	6	15	Theorem 4.2 (a)
	7	$9 \leq, \leq 13$	Corollary 3.2 and Figure 5
	8	$10 \leq, \leq 12$	Lemma 4.1 and Figure 5
	9	11	Lemma 4.1 and Figure 4
	10	10	$I(C_{10} = 6$
	11	11	$I(C_{11} = 6$
	12	11	Theorem 4.2 (b)

Obviously integrity graph $IG(p, n)$ is not unique for given parameters n and p. For example there are two $IG(6,5)$ graphs(G_2, G_5 depicted in Appendix). They are not isomorphic graphs.

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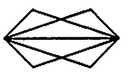
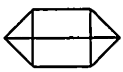
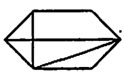
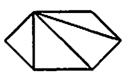
Appendix

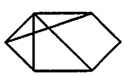
All the $G(6,9)$ graphs.

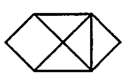
G_1 

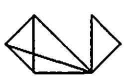
G_2 

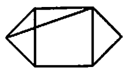
G_3 

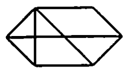
G_4 

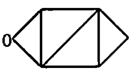
G_5 

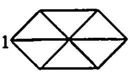
G_6 

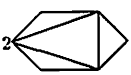
G_7 

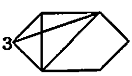
G_8 

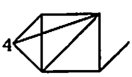
G_9 

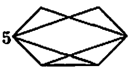
G_{10} 

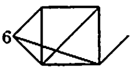
G_{11} 

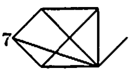
G_{12} 

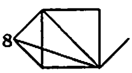
G_{13} 

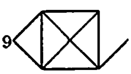
G_{14} 

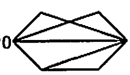
G_{15} 

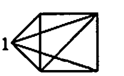
G_{16} 

G_{17} 

G_{18} 

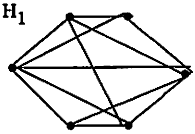
G_{19} 

G_{20} 

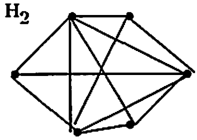
G_{21} 

All the $G(6,12)$ graphs.

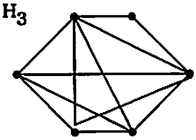
H_1



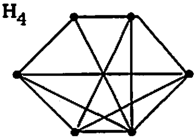
H_2



H_3



H_4



H_5