Abelian Ramanujan Graphs

Chen Kejun*
Department of Mathematics, Suzhou University
Suzhou 215006, China

Abstract

Some special sum graphs and difference graphs, based on abelian groups, are discussed. In addition to Li's result on character sum estimates, Weil's character sum estimates are also used to show that these are indeed Ramanujan graphs.

Keywords: Ramanujan graph, eigenvalue, character sum.

1 Introduction

Roughly speaking, a Ramanujan graph is a connected regular graph whose nontrivial eigenvalues are relatively small in absolute value. The interest in Ramanujan graphs arise from their use in communication networks, extremal graph theory and computational complexity (see [10], [2], [3]). The detailed definition of a Ramanujan graph is as follows.

Let G be a finite (directed) graph. The *adjacency matrix* of G, denoted by A, is a square matrix with entry at (x, y) equal to the number of edges from x to y in G. If G is *k-regular*, that is, the out-degree and the in-degree at each vertices of G are equal to k, then k is an eigenvalue of A with multiplicity equal to the number of the connected components of G. If, in addition, G is *r-partite*, i.e., the vertices of G can be partitioned into r disjoint sets $V_0, \cdots, V_{r-1}$ such that the outedges from vertices in V_i end in V_{i+1} for $i \in Z_r$, then ζk is also an eigenvalue of A for all r-th roots ζ of unity. Call ζk's the trivial eigenvalues of A, and the remaining eigenvalues nontrivial. For a k-regular graph G, let $\lambda(G)$ be the maximal nontrivial

*Permanent address: Department of Mathematics, Yancheng Teachers College, Jiangsu 224002, China

eigenvalue of A in absolute value. If G is k-regular with A diagonalizable by a unitary matrix, then the trace of AA^t is nk and the eigenvalues of AA^t are the absolute value of the eigenvalues of A squared. Thus if G is also r-partite, then $n \geq rk$ and $rk^2 + (n-r)\lambda(G)^2 \geq nk$, which yields the following trivial lower bound of $\lambda(G)$:

$$\lambda(G) \geq \left(\frac{n-rk}{n-r}\right)^{1/2} \sqrt{k}.$$

A nontrivial lower bound for undirected graphs is given by Alon and Boppana.

Lemma 1.1 ([10]) *For k-regular undirected graphs G, we have*

$$lim\ inf\ \lambda(G) \geq 2\sqrt{k-1}$$

as $|G| \longrightarrow \infty$.

The same lower bound also holds for k-regular directed graphs G with A diagonalizable by unitary matrices. This is because the bipartite undirected graph with adjacency matrix

$$\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$$

has eigenvalues $\pm|\lambda|$, where λ runs through eigenvalues of A. In view of the above lemma, we say that a k-regular graph G has small eigenvalues if $\lambda(G) \leq 2\sqrt{k-1}$. Following Lubotzky-Phillips-Sarnak [10], we call a graph G *Ramanujan graph* if

(1) G is k-regular;

(2) $\lambda(G) \leq 2\sqrt{k-1}$;

(3) A is diagonalizable by a unitary matrix.

Since a directed k-regular r-partite Ramanujan graph will give rise to r bipartite undirected Ramanujan graphs, therefore we include also directed graphs in the definition above.

Despite the fact that a random k-regular graph G has a high probability to have $\lambda(G)$, if not already $\leq 2\sqrt{k-1}$, not much bigger than $2\sqrt{k-1}$, it is difficult to verify if a given graph is indeed Ramanujan. Hence it is desirable to have explicit constructions of Ramanujan graphs. Up to date, there are some known systematic methods. For details, we refer the reader to [1], [3]-[7].

In this article, we review some known results on sum graph and difference graph in Section 2. Some special sum graphs and difference graphs, based on abelian groups, are discussed in Section 3. In addition to Li's result on character sum estimates, Weil's character sum estimates will also be used to show that these are indeed Ramanujan graphs.

2 Known Results

In [5], certain graphs, namely sum graph and difference graph, are proven to be Ramanujan. Let G be a finite abelian group and let S be a k-subset of G. Two k-regular graphs, called *sum graph* $X_s(G,S)$ and *difference graph* $X_d(G,S)$ on G, are defined as follows. For $x \in G$, the out-neighbors of x in $X_s(G,S)$ (resp. $X_d(G,S)$) are those $y \in G$ such that $x+y \in S$ (resp. $y-x \in S$). It follows from the definition that the sum graph is undirected and the difference graph is usually directed; it is undirected if and only if S is symmetric, that is, $S=-S$. The sum graphs and difference graphs have some nice properties (see [5] or [8]). Li provided in [5] the following.

Lemma 2.1 ([5]) *Let G be a finite abelian group and let S be a k-subset of G. If*

$$\left|\sum_{s\in S}\psi(s)\right| \leq 2\sqrt{k-1}$$

for all nontrivial characters ψ of G, then $X_s(G,S)$ and $X_d(G,S)$ are Ramanujan graphs.

This reduces a combinatorial problem to the problem of character sum estimates. It is well known that for any nontrivial character ψ of G, $\sum_{s\in G}\psi(s)=0$. So, we have the following two trivial results:

(i) $X_s(G,S)$ and $X_d(G,S)$ are Ramanujan graphs when $|G|-2\sqrt{|G|}+2 \leq |S| < |G|$ since

$$\left|\sum_{s\in S}\psi(s)\right| = \left|-\sum_{s\in G\setminus S}\psi(s)\right| \leq |G\setminus S| \leq 2\sqrt{|S|-1}.$$

(ii) If $X_s(G,S)$ and $X_d(G,S)$ are Ramanujan graphs with $S\subseteq G$ and $|S| \leq \lfloor\frac{|G|}{2}\rfloor$, then $X_s(G,G\setminus S)$ and $X_d(G,G\setminus S)$ are also Ramanujan. The reason is that,

$$\left|\sum_{s\in G\setminus S}\psi(s)\right| = \left|-\sum_{s\in S}\psi(s)\right| \le 2\sqrt{|S|-1} \le 2\sqrt{|G\setminus S|-1}.$$

Let $F = GF(q)$ and $F_n = GF(q^n)$. For $\alpha \in F_n$, the *norm* $N_{F_n/F}(\alpha)$ of α over F is defined by

$$N_{F_n/F}(\alpha) = \alpha\cdot\alpha^q\cdots\alpha^{q^{n-1}} = \alpha^{(q^n-1)/(q-1)}.$$

Let N_n be the kernel of the norm map $N_{F_n/F}$, i.e., $N_n = \{\alpha \in F_n : N_{F_n/F}(\alpha) = 1\}$. Then N_n is a multiplicative subgroup of order $\frac{q^n-1}{q-1}$ in $F_n^* = F_n \setminus \{0\}$. Let t be a primitive element of F_n. Assume $n \ge 2$. Put

$$S_n = \left\{\frac{t^q+a}{t+a} :\ a \in F \cup \{\infty\}\right\}.$$

Here, when $a = \infty$, ∞/∞ is interpreted as 1. It is easy to see that $|S_n| = q+1$ and S_n is contained in N_n. The following two lemmas were shown by Li in [5].

Lemma 2.2 ([5]) *For each nontrivial character χ of N_n, we have*

$$\left|\sum_{s\in S_n}\chi(s)\right| \le (n-2)\sqrt{q}.$$

Lemma 2.3 ([5]) *For all nontrivial additive character ψ of F_n, we have*

$$\left|\sum_{s\in S_n}\psi(s)\right| \le (2n-2)\sqrt{q}.$$

It follows from Lemma 2.2 that $X_s(N_n, S_n)$ and $X_d(N_n, S_n)$ are Ramanujan graphs for $n = 3, 4$. In fact, when $n = 3$, the bound is $\sqrt{q}$, so we may enlarge S by randomly joining in up to $\sqrt{q}$ more elements in N_3 and still get Ramanujan graphs. As a consequence of Lemma 2.3, $X_s(F_2, S_2)$ and $X_d(F_2, S_2)$ are Ramanujan graphs. It can also be shown (see [5]) that $X_s(N_2 \times F_2, S_2)$ and $X_d(N_2 \times F_2, S_2)$ are Ramanujan graphs.

3 Main Results

In this section, we shall show that some special sum graphs and difference graphs on N_3 or F are Ramanujan graphs by using Lemma 2.2 or Weil's character sum estimates respectively.

For $c \in F \cup \{\infty\}$, let $T_c = \frac{t+c}{t^q+c} S_n$, i.e.,

$$T_c = \left\{ \frac{t+c}{t^q+c} \cdot \frac{t^q+a}{t+a} : \ a \in F \cup \{\infty\} \right\}.$$

Obviously, $T_\infty = S_n$ and $T_c \subset N_n$ for all $c \in F \cup \{\infty\}$.

Lemma 3.1 *If $n \geq 3$, then $T_c \cap T_d = \{1\}$ for all $c, d \in F$, $c \neq d$.*

Proof. Clearly, $1 \in T_c \cap T_d$ for $c, d \in F$, $c \neq d$. So, we need only to prove that $|T_c \cap T_d| = 1$. Otherwise, there must exist $a, b \in F$, $a \neq b$, such that

$$\frac{t+c}{t^q+c} \cdot \frac{t^q+a}{t+a} = \frac{t+d}{t^q+d} \cdot \frac{t^q+b}{t+b} \neq 1. \tag{1}$$

It follows that

$$\begin{aligned} & (t^{q+1} + ct^q + at + ac)(t^{q+1} + bt^q + dt + bd) \\ - \ & (t^{q+1} + at^q + ct + ac)(t^{q+1} + dt^q + bt + bd) = 0, \end{aligned}$$

that is,

$$[(b+c-a-d)t^{q+1} + (bc-ad)(t^q+t) + abc + bcd - abd - acd](t^q - t) = 0.$$

Since $t^q - t \neq 0$ we have

$$(b+c-a-d)t^{q+1} + (bc-ad)(t^q+t) + abc + bcd - abd - acd = 0. \tag{2}$$

Since $x^q = x$ for any $x \in F$ and $(y+z)^q = y^q + z^q$ for all $y, z \in F_n$, we have from (2)

$$(b+c-a-d)t^{q(q+1)} + (bc-ad)(t^{q^2} + t^q) + abc + bcd - abd - acd = 0. \tag{3}$$

From (3) and (2), we get

$$[(b+c-a-d)t^q + bc - ad](t^{q^2} - t) = 0.$$

Since $n \geq 3$, we have $t^{q^2} - t \neq 0$ and thus

$$(b+c-a-d)t^q + bc - ad = 0. \tag{4}$$

If $b+c-a-d \neq 0$ then $bc - ad \neq 0$ from (4) and

$$t^{q(q-1)} = \left(\frac{ad-bc}{b+c-a-d} \right)^{q-1} = 1,$$

which contradicts the fact that t is a primitive element of F_n ($n \geq 3$). This forces that $b + c - a - d = 0$ and $bc - ad = 0$. So, we may assume that $a = ub$, $c = ud$, where $u \neq 1$ noting that $c \neq d$. Thus we have $b + ud - ub - d = 0$, i.e., $(b - d)(1 - u) = 0$. It follows that $b = d$. Therefore,

$$\frac{t+d}{t^q+d} \cdot \frac{t^q+b}{t+b} = 1,$$

which is a contradiction to (1). □

Similarly, we have the following.

Lemma 3.2 *If $n \geq 3$, then $T_c \cap T_\infty = \{1\}$ for all $c \in F$.*

Remark: By Lemma 3.1 and Lemma 3.2, we know that $\left|\bigcup_{c \in F \cup \{\infty\}} T_c\right| = q(q+1) + 1 = |N_3|$. On the other hand, $\bigcup_{c \in F \cup \{\infty\}} T_c \subseteq N_3$. So, we have $N_3 = \bigcup_{c \in F \cup \{\infty\}} T_c$.

Lemma 3.3 *Let $c \in F \cup \{\infty\}$, then for each nontrivial multiplicative character χ of N_n, we have*

$$\left|\sum_{s \in T_c} \chi(s)\right| \leq (n-2)\sqrt{q}.$$

Proof. Let $y_c = \frac{t+c}{t^q+c}$, then $T_c = y_c S_n$. For any nontrivial multiplicative character χ of N_n, by Lemma 2.2, we have

$$\left|\sum_{s \in T_c} \chi(s)\right| = \left|\sum_{s \in S_n} \chi(y_c)\chi(s)\right| = \left|\sum_{s \in S_n} \chi(s)\right| \leq (n-2)\sqrt{q}.$$

□

By Lemma 2.1 and Lemma 3.3, we have the following immediately.

Theorem 3.4 *Let $c \in F \cup \{\infty\}$, then $X_s(N_n, T_c)$ and $X_d(N_n, T_c)$ are Ramanujan graphs for $n = 3, 4$.*

When $n = 3$, we may not only enlarge T_c by randomly joining in some elements in N_3 but also compress T_c by removing some elements from T_c and still get Ramanujan graphs.

Theorem 3.5 *Let $c \in F \cup \{\infty\}$. Then*

(i) *$X_s(N_3, T_c \cup P)$ and $X_d(N_3, T_c \cup P)$ are Ramanujan graphs for any u-subset P of $N_3 \setminus T_c$, where $u \le \lfloor 2\sqrt{q - \sqrt{q}+1} - \sqrt{q} + 2 \rfloor$.*

(ii) *$X_s(N_3, T_c \setminus M)$ and $X_d(N_3, T_c \setminus M)$ are Ramanujan graphs for any v-subset M of T_c, where $v \le \lfloor 2\sqrt{q + \sqrt{q}+1} - \sqrt{q} - 2 \rfloor$.*

Proof. For any nontrivial multiplicative character χ of N_3, by Lemma 3.3, we have

$$\left|\sum_{s \in T_c \cup P} \chi(s)\right| = \left|\sum_{s \in T_c} \chi(s) + \sum_{s \in P} \chi(s)\right| \le \sqrt{q} + u,$$

and

$$\left|\sum_{s \in T_c \setminus M} \chi(s)\right| = \left|\sum_{s \in T_c} \chi(s) - \sum_{s \in M} \chi(s)\right| \le \sqrt{q} + v.$$

Now $|T_c \cup P| = q + 1 + u$. When $u \le \lfloor 2\sqrt{q - \sqrt{q}+1} - \sqrt{q} + 2 \rfloor$, we have $\sqrt{q} + u \le 2\sqrt{q+u}$. Hence

$$\left|\sum_{s \in T_c \cup P} \chi(s)\right| \le 2\sqrt{|T_c \cup P| - 1}.$$

Similarly, we have

$$\left|\sum_{s \in T_c \setminus M} \chi(s)\right| \le 2\sqrt{|T_c \setminus M| - 1}$$

if $v \le \lfloor 2\sqrt{q + \sqrt{q}+1} - \sqrt{q} - 2 \rfloor$. The conclusions (i) and (ii) are obtained from Lemma 2.1. ∎

Theorem 3.6 *$X_s(N_3, T_{c_1} \cup T_{c_2})$ and $X_d(N_3, T_{c_1} \cup T_{c_2})$ are Ramanujan graphs for any $c_1, c_2 \in F \cup \{\infty\}$, $c_1 \ne c_2$.*

Proof. Let χ be a nontrivial multiplicative character of N_3. By Lemmas 3.1-3.3, we have

$$\left|\sum_{s \in T_{c_1} \cup T_{c_2}} \chi(s)\right| = \left|\sum_{s \in T_{c_1}} \chi(s) + \sum_{s \in T_{c_1}} \chi(s) - \chi(1)\right| \le 2\sqrt{q} + 1.$$

Since $|T_{c_1} \cup T_{c_2}| = 2q+1$ and $2\sqrt{q}+1 \le 2\sqrt{2q}$, we have

$$\left| \sum_{s \in T_{c_1} \cup T_{c_2}} \chi(s) \right| \le 2\sqrt{|T_{c_1} \cup T_{c_2}| - 1}.$$

By Lemma 2.1, we get the result. □

Similar to Theorem 3.5 we may enlarge $T_{c_1} \cup T_{c_2}$ by randomly joining in some elements in N_3 or compress $T_{c_1} \cup T_{c_2}$ by removing some elements from $T_{c_1} \cup T_{c_2}$ and still get Ramanujan graphs. We have the following.

Theorem 3.7 *Let $c_1, c_2 \in F \cup \{\infty\}$, $c_1 \ne c_2$. Then*

(i) *$X_s(N_3, T_{c_1} \cup T_{c_2} \cup P)$ and $X_d(N_3, T_{c_1} \cup T_{c_1} \cup P)$ are Ramanujan graphs for any u-subset P of $N_3 \setminus (T_{c_1} \cup T_{c_2})$, where $u \le \lfloor 2\sqrt{2}\sqrt{q-\sqrt{q}} - 2\sqrt{q}+1 \rfloor$.*

(ii) *$X_s(N_3, (T_{c_1} \cup T_{c_2}) \setminus M)$ and $X_d(N_3, (T_{c_1} \cup T_{c_2}) \setminus M)$ are Ramanujan graphs for any v-subset M of $T_{c_1} \cup T_{c_2}$, where $v \le \lfloor 2\sqrt{2}\sqrt{q+\sqrt{q}+1} - 2\sqrt{q} - 3 \rfloor$.*

Proof. For any nontrivial multiplicative character χ of N_3, by Lemmas 3.1-3.3, we have

$$\begin{aligned} \left| \sum_{s \in T_{c_1} \cup T_{c_2} \cup P} \chi(s) \right| &= \left| \sum_{s \in T_{c_1}} \chi(s) + \sum_{s \in T_{c_1}} \chi(s) + \sum_{s \in P} \chi(s) - \chi(1) \right| \\ &\le 2\sqrt{q} + u + 1. \end{aligned}$$

Now $|T_{c_1} \cup T_{c_2} \cup P| = 2q+u+1$ and $2\sqrt{q}+u+1 \le 2\sqrt{2q+u}$ when $u \le \lfloor 2\sqrt{2}\sqrt{q-\sqrt{q}} - 2\sqrt{q}+1 \rfloor$. So, we have

$$\left| \sum_{s \in T_{c_1} \cup T_{c_2} \cup P} \chi(s) \right| \le 2\sqrt{|T_{c_1} \cup T_{c_2} \cup P| - 1}.$$

By Lemma 2.1 we obtain conclusion (i). And conclusion (ii) can be proved similarly. □

Theorem 3.8 *If $q \ge 19$, then $X_s(N_3, \bigcup_{i=1}^{3} T_{c_i})$ and $X_d(N_3, \bigcup_{i=1}^{3} T_{c_i})$ are Ramanujan graphs for any three distinct elements c_1, c_2 and c_3 in $F \cup \{\infty\}$.*

Proof. Let χ be a nontrivial multiplicative character of N_3. By Lemmas 3.1-3.3, we have

$$\left|\sum_{s\in T_{c_1}\cup T_{c_2}\cup T_{c_3}} \chi(s)\right| = \left|\sum_{i=1}^{3}\sum_{s\in T_{c_i}} \chi(s) - 2\chi(1)\right| \le 3\sqrt{q}+2.$$

Obviously, $3\sqrt{q}+2 \le 2\sqrt{3q}$ if $q \ge 19$. Since $|T_{c_1} \cup T_{c_2} \cup T_{c_3}| = 3q+1$, we have

$$\left|\sum_{s\in T_{c_1}\cup T_{c_2}\cup T_{c_3}} \chi(s)\right| \le 2\sqrt{|T_{c_1} \cup T_{c_2} \cup T_{c_3}| - 1}.$$

The desired result is obtained from Lemma 2.1. □

Now we use Weil's character sum estimate to show that some sum graphs and difference graphs on $F = GF(q)$ are Ramanujan graphs. Weil's theorem on additive character sums can be found in Lidl and Niederreiter [9, Theorem 5.38].

Theorem 3.9 *(Weil's theorem [9]) Let $f \in F[x]$ be of degree $n \ge 1$ with $gcd(n,q) = 1$ and let χ be a nontrivial additive character of F. Then*

$$\left|\sum_{c\in F} \chi(f(c))\right| \le (n-1)\sqrt{q}.$$

Let $q = ne+1$ be a prime power and let ξ be a primitive element of $F = GF(q)$. Let D be the set of all e-th roots of unity in F, i.e., $D = \{\xi^{jn} : j = 0,1,\cdots,e-1\}$. Then we have the following.

Lemma 3.10 *For any nontrivial additive character χ of F, we have*

$$\left|\sum_{c\in D} \chi(c)\right| \le \frac{(n-1)\sqrt{q}+1}{n}.$$

Proof. Take $f(x) = x^n \in F[x]$. It is easy to see that $gcd(n,q) = 1$. By Theorem 3.9, we have

$$\left|\sum_{c\in F} \chi(f(c))\right| \le (n-1)\sqrt{q}. \tag{5}$$

Let

$$D_i = \{\xi^{ei+j} : j = 0,1,\cdots,e-1\}$$

and

$$f(D_i) = \{f(c) : \ c \in D_i\},$$

$i = 0, 1, \cdots, n-1$. It is easy to see that $D_i \cap D_j = \emptyset$, $0 \le i < j \le n-1$ and $f(D_0) = f(D_1) = \cdots = f(D_{n-1}) = D$. Note that $F \setminus \{0\} = \bigcup_{i=0}^{n-1} D_i$, we have from (5)

$$n\left|\sum_{c\in D}\chi(c)\right| - 1 \le \left|\sum_{c\in F\setminus\{0\}} \chi(f(c)) + \chi(0)\right| \le (n-1)\sqrt{q}.$$

This implies the required result. □

Theorem 3.11 *$X_s(F, D)$ and $X_d(F, D)$ are Ramanujan graphs if $q = ne + 1 \ge q(n)$, $2 \le n \le 5$, where $q(2) = 5$, $q(3) = 8$, $q(4) = 16$ and $q(5) = 47$.*

Proof. Let χ be a nontrivial additive character of F. By Lemma 3.10, we know that

$$\left|\sum_{c\in D}\chi(c)\right| \le \frac{(n-1)\sqrt{q}+1}{n}.$$

It is readily checked that

$$\frac{(n-1)\sqrt{q}+1}{n} \le 2\sqrt{\frac{q-1}{n} - 1}$$

when $q \ge q(n)$, $2 \le n \le 5$. Since $|D| = \frac{q-1}{n}$, we have

$$\left|\sum_{c\in D}\chi(c)\right| \le 2\sqrt{|D|-1}.$$

The proof is completed by Lemma 2.1. □

We may enlarge D by randomly joining in some elements in F and still get Ramanujan graphs.

Theorem 3.12 *$X_s(F, D \cup P)$ and $X_d(F, D \cup P)$ are Ramanujan graphs for any u-subset P of $F \setminus D$, where $u \le 2\sqrt{\frac{q-1}{n} - A_n} - A_n + 2$ and $A_n = \frac{(n-1)\sqrt{q}+1}{n}$.*

Proof. For any nontrivial multiplicative character χ of F, by Lemma 3.10, we know that

$$\left|\sum_{c\in D\cup P}\chi(c)\right| = \left|\sum_{c\in D}\chi(c)+\sum_{c\in P}\chi(c)\right| \le A_n + u.$$

It is readily checked that

$$A_n + u \le 2\sqrt{\frac{q-1}{n}+u-1},$$

when $u \le 2\sqrt{\frac{q-1}{n} - A_n} - A_n + 2$. Since $|D\cup P| = \frac{q-1}{n} + u$, we have

$$\left|\sum_{c\in D\cup P}\chi(c)\right| \le 2\sqrt{|D\cup P|-1}.$$

By Lemma 2.1, we get our conclusion. □

Similarly, we may compress D by removing some elements from D and still get Ramanujan graphs. We have the following.

Theorem 3.13 *$X_s(F, D\backslash M)$ and $X_d(F, D\backslash M)$ are Ramanujan graphs for any v-subset M of D, where $v \le 2\sqrt{\frac{q-1}{n} + A_n} - A_n - 2$ and $A_n = \frac{(n-1)\sqrt{q}+1}{n}$.*

Acknowledgement The author would like to thank Professor L. Zhu for his help during the preparing of this paper.

References

[1] J. Angel, N. Celniker, S. Poulos, A. Terras, C. Trimble, and E. Velasquez, *Special functions on finite upper half planes*, Contemp. Math. **138** (1992), 1-26.

[2] F. Bien, *Constructions of telephone networks by group representations*, Notices Amer. Math. Soc. **336** (1989), 5-22.

[3] P. Chiu, *Cubic Ramanujan graphs*, Combinatorica, to appear.

[4] F. R. K. Chung, *Diameters and eigenvalues*, J. Amer. Math. Soc. **2** (1989), 187-186.

[5] W.-C. W. Li, *Character sums and abelian Ramanujan graphs*, J. Number Theory **41** (1992), 199-217.

[6] W.-C. W. Li, *Number theoretic construction of Ramanujan graphs*, Astérisque, French Math. Soc. No. 228 **3-4** (1995), 101-120.

[7] W.-C. W. Li, *A survey of Ramanujan graphs*, Arithmetic, geometry and coding theory, (Luminy, 1993), 127-143. de Gruyter, Berlin, 1996.

[8] W.-C. W. Li, *Number Theory with Applications*, World Scientific Publishing Co Pte Ltd, 1996.

[9] R. Lidl and H. Niederreiter, *Finite Fields*, Encyclopedia of Mathematics and Its Applications, Vol. **20** (Cambridge University Press, Cambridge, 1983).

[10] A. Lubotzky, R. Phillips and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), 261-277.